

Poisson structure on the phase space associated to the hamiltonian dynamics of coupled Korteweg-de Vries type equations

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Abstract. We present the hamiltonian structures for a wide class of coupled Korteweg-de Vries systems, including the Gear and Grimshaw system that models the strong interaction of internal waves in a stratified liquid and the system of Lou, Tong, Hu and Tang that describes a two layer fluid model. Among the hamiltonian structures of these systems we found new Poisson brackets which define consistent algebras of observables.

1. Introduction

Since Hirota-Satsuma [1], coupled Korteweg-de Vries (KdV) systems have been object of a extensive study in view of their intrinsically relevant mathematical properties and of its relationship with several physics models. In particular it was derived in [2] a system which describes linearly stable internal waves in a density stratified fluid and in [3] it was analyzed a system which under some physical reductions can be used in the description of the atmospheric and oceanic phenomena. An interesting study of coupled KdV systems related with its stability properties was presented in [4].

Among the properties of such systems one has the existence of associated Lax pairs and Bäcklund transformations as well as the existence of multi-solitonic solutions and the Painlevé property [5].

In this work we present a lagrangian approach to the Gear and Grimshaw coupled KdV system [2, 6], which includes two different lagrangians with two associated hamiltonians describing the coupled system. Through that description is possible to obtain the complete hamiltonian structure of the system with his inherited Poisson mathematical description. We notice that this procedure give rise in a natural way to a pencil of Poisson structures for the system.

2. Coupled KdV systems

The system introduced by Gear and Grimshaw [2] and analyzed in [6] is

$$\begin{aligned} u_t + uu_x + u_{xxx} + a_3v_{xxx} + a_1vv_x + a_2(uv)_x &= 0 \\ b_1v_t + rv_x + vv_x + v_{xxx} + b_2a_3u_{xxx} + b_2a_2uu_x + b_2a_1(uv)_x &= 0, \end{aligned} \quad (1)$$



where a_1, a_2, a_3, b_1, b_2 and r are real constants with b_1 and b_2 positive. u and v are real valued functions, $u = u(x, t), v = v(x, t)$.

In [6] the Cauchy problem was shown to be well posed. A hamiltonian functional for this system was also obtained. The hamiltonian density is

$$\mathcal{H} = b_2(u_x)^2 + (v_x)^2 + 2b_2a_3u_xv_x - b_2\frac{u^3}{3} - b_2a_2u^2v - b_2a_1uv^2 - \frac{v^3}{3} - rv^2. \quad (2)$$

If we consider x to be of dimension 1, then u and v must have dimension -2 in order that the different terms on the above expression have the same dimension. From the field equation (1) we conclude that t must have dimension -3. We then notice that the term rv^2 has different dimension unless the constant r has dimension -2. This feature distinguish this term from the others where the constant coefficients a_a, a_2, a_3, b_1, b_2 are dimensionless.

The hamiltonian function introduced in [4] has the same terms as $\mathcal{H}$ in (2) together with $-r_1v^2 - r_2u^2 - kvu$ which have coefficients with dimension -2.

Similar terms appear in the field equations in [3] where in order to consider a hamiltonian system we must take $\alpha_1 = \delta_1 = 0$ (in their notation).

The hamiltonian functional in all these cases contains terms of weight -6, essentially they contain all possible terms of that weight.

3. Hamiltonian structure of the coupled KdV system

The field equations (1) and the hamiltonian density $\mathcal{H}$ in (2) arise from a lagrangian density $\mathcal{L}$ formulated in terms of the fields w, y defined as

$$\begin{aligned} u &= w_x \\ v &= y_x. \end{aligned} \quad (3)$$

The lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}w_xw_t - \frac{1}{2}y_xy_t - \mathcal{H}. \quad (4)$$

By taking independent variations of $L = \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} \mathcal{L} dx$ with respect to w and y one obtains the field equations (1).

The interesting point of the coupled system is that there exists a second lagrangian from which we can deduce, by taking independent variations with respect to w and y , the same field equations. The new lagrangian density gives rise, via a Legendre transformation, to a new hamiltonian structure of the system.

Let us consider the first basic lagrangian density

$$\hat{\mathcal{L}} = \frac{1}{2}w_xw_t - \frac{1}{2}y_xy_t - \hat{\mathcal{H}}, \quad (5)$$

where

$$\begin{aligned} \hat{\mathcal{H}} &= \alpha_1(w_{xx})^2 + \alpha_2(y_{xx})^2 + \alpha_3w_{xx}y_{xx} + \alpha_4(w_x)^3 + \\ &+ \alpha_5(w_x)^2y_x + \alpha_6(y_x)^2w_x + \alpha_7(y_x)^3 + \alpha_8(w_x)^2 + \alpha_9w_xy_x + \alpha_{10}(y_x)^2. \end{aligned} \quad (6)$$

The corresponding fields equations are

$$\begin{aligned} w_{xt} &= 2\alpha_1w_{xxxx} + \alpha_3y_{xxxx} - 3\alpha_4\left((w_x)^2\right)_x - 2\alpha_5(w_xy_x)_x - \alpha_6\left((y_x)^2\right)_x - 2\alpha_8w_{xx} - \alpha_9y_{xx} \equiv \frac{\delta H}{\delta w} \\ y_{xt} &= 2\alpha_2y_{xxxx} + \alpha_3w_{xxxx} - \alpha_5\left((w_x)^2\right)_x - 2\alpha_6(y_xw_x)_x - 3\alpha_7\left((y_x)^2\right)_x - \alpha_9w_{xx} - 2\alpha_{10}y_{xx} \equiv \frac{\delta H}{\delta y}. \end{aligned} \quad (7)$$

Now we introduce the second basic lagrangian density,

$$\tilde{\mathcal{L}} = -\frac{1}{2}w_x y_t - \frac{1}{2}y_x w_t - \tilde{\mathcal{H}}. \quad (8)$$

The corresponding field equations are

$$\begin{aligned} w_{xt} &= \frac{\delta \tilde{\mathcal{H}}}{\delta y} \\ y_{xt} &= \frac{\delta \tilde{\mathcal{H}}}{\delta w} \end{aligned} \quad (9)$$

where $\tilde{\mathcal{H}} = \hat{\mathcal{H}}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_{10}) = \int_{-\infty}^{+\infty} \tilde{\mathcal{H}} dx$, $\tilde{\mathcal{H}}$ has the same general form (6) but now in terms of the coefficients $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_{10}$. In order to obtain the same field equations we get a relation between the coefficients. For example: $\tilde{\alpha}_3 = 2\alpha_1, \tilde{\alpha}_2 = \frac{1}{2}\alpha_3$ and so on.

We thus have two different lagrangians and two different hamiltonian densities which give rise to the same field equations. The two lagrangians provides two different basic Poisson bracket structures. From the two basic Poisson structures one may construct a pencil of Poisson brackets with an associated hamiltonian for each value of the parameter of the pencil.

The field equations for suitable values of the constants $\alpha_i, i = 1, \dots, 10$ describe the Grimshaw equations in [4].

4. Conclusions

We presented the most general hamiltonian approach for the Gear and Grimshaw coupled KdV system which contains several physical models, in particular the one considered in [3]. We introduce two lagrangians and associated hamiltonians for the systems and discuss the Poisson pencil structure generated by them.

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