Hierarchy of solutions to the NLS equation and multi-rogue waves

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Abstract. The solutions to the one dimensional focusing nonlinear Schrödinger equation (NLS) are given in terms of determinants. The orders of these determinants are arbitrarily equal to $2N$ for any nonnegative integer N and generate a hierarchy of solutions which can be written as a product of an exponential depending on t by a quotient of two polynomials of degree $N(N+1)$ in x and t. These solutions depend on $2N-2$ parameters and can be seen as deformations with $2N-2$ parameters of the Peregrine breather P_N : when all these parameters are equal to 0, we recover the P_N breather whose the maximum of the module is equal to $2N+1$. Several conjectures about the structure of the solutions are given.

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1. Introduction

Here, we consider the one dimensional focusing nonlinear Schrödinger equation (NLS) to describe the phenomena of rogue waves. Recently, many works about NLS equation have been published using different methods. Rational solutions to the NLS equation were written in 2010 as a quotient of two wronskians [1]; the present author constructed in [2] another representation of the solutions to the NLS equation in terms of a ratio of two wronskians of even order $2N$ composed of elementary functions using truncated Riemann theta functions in 2011; Guo, Ling and Liu found in 2012 another representation of the solutions as a ratio of two determinants [3] using generalized Darboux transformation; a new approach was proposed by Ohta and Yang in [4] using Hirota bilinear method; finally, the present author has obtained in 2013 rational solutions in terms of determinants which do not involve limits in [5].

The present paper presents multi-parametric families of quasi rational solutions to NLS of order N in terms of determinants (determinants of order $2N$) depending on $2N-2$ real parameters. With this method, a hierarchy of solutions to the NLS equation is obtained. With this representation, at the same time, the well-known ring structure, but also the triangular shapes also given by Ohta and Yang [4], Akhmediev et al. [6], Matveev and Dubard [7] are found.

The aim of this paper is to summarize results on solutions to the NLS equation depending on $2N-2$ parameters and try to classify them. Solutions depending on $2N-2$ parameters give the (analogue) Peregrine breather P_N of order N as a particular case when all the parameters are equal to 0 : for this reason, these solutions will be called $2N-2$ parameters deformations of the Peregrine P_N.

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The paper is organized as follows. First, we recall the representation of the solutions to the NLS equation in terms of wronskians. This representation allows to obtain quasi rational solutions to NLS equation, when some parameter tends towards 0. Quasi rational solutions depending a priori on $2N-2$ parameters at the order N are constructed.
Moreover we give a theorem which states the structure of the quasi-rational solutions to the NLS equation. Families depending on $2N-2$ parameters for the N-th order as a ratio of two polynomials of degree $N(N+1)$ of x and t multiplied by an exponential depending on t are obtained. We state that the highest amplitude of modulus of the Peregrine breather of order N is equal to $2N+1$. Then we try to classify the hierarchy of solutions to the NLS equation in function of the order N and the parameters $\tilde{a}_j$ and $\tilde{b}_j$.

2. Families of multi-parametric solutions to NLS equation in terms of a ratio of two determinants depending on $2N-2$ parameters

We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

We recall the main result obtained in [2]. In the following, we need to define some notations. The parameters $-1 < \lambda_\nu < 1$, $\nu = 1, \ldots, 2N$, are real numbers such that

$$\begin{array}{l} -1 < \lambda_{N+1} < \lambda_{N+2} < \ldots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1 \\ \lambda_{N+j} = -\lambda_j, \quad j = 1, \ldots, N. \end{array} \tag{2}$$

The terms κ_ν, δ_ν and γ_ν are defined by

$$\begin{array}{l} \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \\ \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \ldots N. \end{array} \tag{3}$$

We choose the parameters a_j and b_j in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a}_k j^{2k+1} \epsilon^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b}_k j^{2k+1} \epsilon^{2k+1}, \quad 1 \leq j \leq N. \tag{4}$$

Complex numbers e_ν $1 \leq \nu \leq 2N$ are defined in the following way :

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \quad a, b \in \mathbf{R}. \tag{5}$$

The terms $x_{r,\nu}$ ($r = 3$, 1) are defined by

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \tag{6}$$

We consider X_ν and Y_ν defined by

$$\begin{array}{l} X_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ Y_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2, \end{array}$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ defined in (3), (6); parameters e_ν are defined by (5).
We define the functions $\varphi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ by

$$\begin{array}{l} \varphi_{4j+1,k} = \gamma_k^{4j-1} \sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} = -\gamma_k^{4j+1} \sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2} \cos X_k, \end{array} \tag{7}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} \varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin X_{N+k}, \end{array} \tag{8}$$

for $1 \leq k \leq N$.
We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k.

$$\begin{array}{ll} \psi_{4j+1,k} = \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} = \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} = -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} = -\gamma_k^{4j+2} \cos Y_k, \end{array} \tag{9}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} \psi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{array} \tag{10}$$

for $1 \leq k \leq N$.
All the functions $\varphi_{j,k}$ and $\psi_{j,k}$ and their derivatives depend on ϵ and can all be prolonged by continuity when $\epsilon = 0$.
Then we get the following result [5] :

Theorem 2.1 *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{11}$$

is a quasi-rational solution to the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

quotient of two polynomials $N(x,t)$ and $D(x,t)$ depending on $2N-2$ real parameters $\tilde{a_j}$ and $\tilde{b_j}$, $1 \leq j \leq N-1$.
N and D are polynomials of degrees $N(N+1)$ in x and t, where

$$\begin{array}{ll} n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \leq j \leq 2N & n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N & n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ d_{j1} = \psi_{j,1}(x,t,0),\ 1 \leq j \leq 2N & d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N & d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ 2 \leq k \leq N,\ 1 \leq j \leq 2N & \end{array}$$

The functions φ and ψ are defined in (7),(8), (9), (10).

Moreover, we have the following result which gives the highest amplitude of the module of the Peregrine breather of order N :

Theorem 2.2 *The function v_0 defined by*

$$v_0(x,t) = \exp(2it - i\varphi) \times \left(\frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \right)_{(\tilde{a_j}=\tilde{b_j}=0,\ 1\leq j\leq N-1)} \tag{12}$$

is the Peregrine breather of order N solution to the NLS equation (1) whose highest amplitude in module is equal to $2N+1$.

The two previous results has been proven by the author. We postpone to give their demonstrations to another publication in order not to weigh down the text of this article.

3. Hierarchy of solutions to NLS equation depending on $2N-2$ parameters

For each nonnegative integer N, we have constructed solutions to NLS equation depending on $2N-2$ parameters.
The case $N=1$ correspond to the classical breather, first constructed by Peregrine [8] in 1983; the analogue breather in the case $N=2$ was built for the first time by Akhmediev, Eleonski and Kulagin [9, 10] in 1986. Other analogues of the Peregrine breathers of order 3 and 4 were constructed in a series of articles by Akhmediev et al. [11, 12] using Darboux transformations. It is only in 2012 that solutions of order 3 and 4 with respectively 4 and 6 parameters were first explicitly found by Matveev using another method based on results of [13], but only published in 2013 in [7].
The solutions for orders 3 and 4 have also been explicitly found by the present author [14, 15]. The equivalence between with two types of solutions was made in [7] for the order 3; the equivalence between these solutions for the order 4 was made by the present author in [15].
In a series of articles, we have studied other higher orders. We have also explicitly found the solutions at order 5 with 8 parameters [16] : these expressions are too extensive to be presented : it takes 14049 pages! For orders 6, 7 the solutions are also explicitly found but are more complicated and cannot be published in any review; the analysis has been done in all these cases in respectively [17, 18]. The solutions for order 8, with 14 parameters are also found and submitted to a review.
The cases of orders 9 and 10 have just been finished and the solutions with respectively 16 and 18 parameters are explicitly found.
From these various studies, it arises that the solutions have quite particular structures depending on the parameters $\tilde{a}_j$ and $\tilde{b}_j$. The parameters $\tilde{a}_j$ and $\tilde{b}_j$ play a similar role in obtaining the structures of the solutions. One can thus establish a certain number of conjectures about these solutions at the order N.

3.1. Case $a_1 \neq 0$ (or $b_1 \neq 0$)

For $\tilde{a}_1 \neq 0$ or $\tilde{b}_1 \neq 0$ and other parameters equal to 0, one obtains a triangle with $\frac{N(N+1)}{2}$ peaks. It is important to note that we obtain triangle only in this case; in all the other cases for only one parameter non equal to 0, we obtain rings.
We do not have the space to give the figures in the present text (see the figures on line).

3.2. Case $a_{N-1} \neq 0$ (or $b_{N-1} \neq 0$), $N \geq 3$

For $\tilde{a}_{N-1} \neq 0$ or $\tilde{b}_{N-1} \neq 0$ and other parameters equal to 0, one obtains only one ring of $2N-1$ peaks with in the center Peregrine P_{N-2} of order $N-2$ [1]; here, $N \geq 3$.
Again, we cannot give the figures in the present text; it can be seen on line.

3.3. Case $a_{N-2} \neq 0$ (or $b_{N-2} \neq 0$), $N \geq 5$

For $\tilde{a}_{N-2} \neq 0$ or $\tilde{b}_{N-2} \neq 0$ and other parameters equal to 0, one obtains two concentric rings of $2N-3$ peaks with in the center Peregrine P_{N-4} of order $N-4$; here $N \geq 5$.
We also have not the space to give the figures here (see the corresponding figures on line).

3.4. Case $a_{N-3} \neq 0$ (or $b_{N-3} \neq 0$), $N \geq 7$

We have the same type of figures in this case, but we don't have the space to give in this publication : namely, at order N, we get 3 rings of $2N-5$ peaks with the P_{N-6} breather in the center (the figures can be seen on line).

[1] This conjecture was already formulated by different authors, in particular by Akhmediev et al. [19]

3.5. General case

In general, we can conjecture that :
For $\tilde{a}_{N-k} \neq 0$ or $\tilde{b}_{N-k} \neq 0$ and other parameters equal to 0, for $N > 0$ and $2k \leq N-1$, one obtains k concentric rings of $2N-2k+1$ peaks with in the center Peregrine P_{N-2k} of order $N-2k$.
It would be relevant to study the cases for the integers k such that $k > \frac{N}{2}$ and the parameters $a_{N-k} \neq 0$ or $b_{N-k} \neq 0$; the structure seems to be more complicated and would be clarified.

4. Conclusion

Here we have given the structure of quasi-rational solutions to the one dimensional focusing NLS equation at the order N. They can be expressed as a product of an exponential depending on t by a ratio of two polynomials of degree $N(N+1)$ in x and t.
If we choose $\tilde{a}_i = \tilde{b}_i = 0$ for $1 \leq i \leq N$, we obtain the classical (analogue) Peregrine breather. Thus these solutions appear as $2N-2$-parameters deformations of the Peregrine breather P_N of order N; this P_N breather has an highest amplitude in module equal to $2N+1$.
These results have been recently proven by the author and their proofs will be given in another publication.
The method described in the present paper provides a powerful tool to get explicit solutions to the NLS equation and to understand the behavior of rogue waves. There are currently many applications in these fields as recent works by Akhmediev et al. [20] or Kibler et al. [21] attest it in particular.
This study leads to a better understanding of the phenomenon of rogue waves, and it would be relevant to go on with the higher orders.
A beginning of classification of the solutions to NLS equation was started with Akhmediev et al. [22]. It would be important in the future to prove the conjectures given in this paper and to give a complete classification for the order N of the quasi rational solutions to the NLS equation.

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