

# Soliton dynamics in complex potentials

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**Abstract.** Soliton propagation dynamics under the presence of a complex potential are investigated. Cases of both symmetric and non-symmetric potentials are studied in terms of their effect on soliton dynamics. The existence of an invariant of soliton propagation under specific symmetry conditions for the real and the imaginary part of the potential is shown. The rich set of dynamical features of soliton propagation include dynamical trapping, periodic and non-periodic soliton mass variation and non-reciprocal dynamics. These features are systematically investigated with the utilization of an effective particle phase space approach which is shown in remarkable agreement with direct numerical simulations. The generality of the results enables the consideration of potential applications where the inhomogeneity of the gain and loss is appropriately engineered in order to provide desirable soliton dynamics.

## 1. Introduction

Soliton formation and dynamics in spatially inhomogeneous structures is a subject of intense research interest with applications to many branches of physics, including optical waves in nonlinear photonic structures [1] and matter waves in Bose-Einstein Condensates (BEC), [2]. Spatial modulations of the linear or the nonlinear refractive index of an optical medium have been shown to result in the formation of self-localized waves that have no counterpart in homogeneous systems. [3, 4, 5] Lattice solitons have been shown to exist in a large variety of configurations. [6, 7] In the case of strong spatial modulations, soliton profiles can be interestingly complex but wave dynamics are rather restricted due to the deep soliton trapping and the resulting transverse immobility. Contrarily, in the case of rather weak modulations, soliton profiles remain simple but soliton dynamics can be quite rich and have interesting features with great potential for applications. In such cases solitons move actually as effective particles in a potential, with the form of the latter depending strongly on the characteristics of the soliton. Therefore, different solitons may undergo qualitatively different dynamics in the same inhomogeneous structure. [6, 7]

The consideration of spatial modulation of the material gain and losses appears naturally as a next step for engineering the soliton formation and dynamics and opens new possibilities for applications. The formation of gap solitons has been investigated in periodic lattices with inhomogeneous [8, 9, 10, 11] gain and loss properties. Even for cases of homogeneous gain/loss it has been shown that the interplay between the dynamical soliton power variation and the refractive index modulation results in a rich set of soliton dynamical features. [12, 13, 14]

In this work we study soliton dynamics under the presence of relatively weak symmetric and non-symmetric complex potentials. The presence of gain and loss not only affects the soliton



mass (power) but also the effective potential under which the soliton is moving due to the spatial modulation of the refractive index. Soliton dynamics are studied in the three-dimensional phase space of an effective particle of varying mass and the role of spatial symmetries as well as deviations from symmetry is investigated.

## 2. Model

Soliton propagation in the presence of a complex potential is described by the inhomogeneous NLS equation:

$$iu_z + u_{xx} + [V(x) + iW(x)]u + 2|u|^2u = 0 \quad (1)$$

where  $u$  is the wave field envelope,  $z$  the normalized propagation distance, and  $x$  the scaled transverse coordinate.  $V(x)$  and  $W(x)$  are the real and imaginary parts of the complex potential. The soliton can be treated as an effective particle [15] of variable mass  $m = \int |u|^2 dx$  and momentum  $p = i \int (uu_x^* - u_x u^*) dx = mv$  at a position  $x_0$ , corresponding to soliton's center, moving with velocity  $v$  in an effective potential  $U_{eff}$  due to the actual complex potential, according to the equations

$$\frac{dm}{dz} = 2 \int_{-\infty}^{+\infty} |u|^2 W(x) dx \quad (2)$$

$$m \frac{dv}{dz} = -\frac{\partial}{\partial x_0} \left[ 2 \int_{-\infty}^{+\infty} |u|^2 V(x) dx \right] \equiv -\frac{\partial U_{eff}}{\partial x_0} \quad (3)$$

$$\frac{dx_0}{dz} = v \quad (4)$$

The dynamical system defined by Eqs. (2)-(4) determines soliton dynamics. In the case of real potential ( $W = 0$ ), soliton moves with a constant mass, the system has fixed points at the extrema of the effective potential ( $U_{eff}$ ) and the total energy of the effective particle  $H = mv^2/2 + U_{eff}(x_0)$  is conserved. The presence of a nonzero part of the potential ( $W \neq 0$ ) introduces an additional degree of freedom related to the particle mass variation and causes the destruction of the conserved quantity of total energy. These features result in drastic qualitative changes of soliton dynamics in comparison to cases of real potentials.

From Eqs. (2),(3) we can obtain

$$m \frac{dv}{dm} = \frac{-\frac{\partial}{\partial x_0} \left[ \int_{-\infty}^{+\infty} |u|^2 V(x) dx \right]}{\int_{-\infty}^{+\infty} |u|^2 W(x) dx} \equiv I(m, v, x_0) \quad (5)$$

with the quantity  $I(m, v, x_0)$  depending, in general, on all soliton parameters. However, it is readily seen that under the condition

$$\frac{\partial V(x)}{\partial x} = CW(x) \quad (6)$$

with  $C$  being a constant, we have  $I(m, v, x_0) = -C$  resulting in a conserved quantity of the effective particle motion given by

$$K(m, v) = C \ln m + v = \text{const.} \quad (7)$$

that restricts soliton dynamics in two-dimensional submanifolds of the phase space  $(x_0, v, m)$ . Moreover, this condition implies the existence of a stable/unstable fixed point at the minima/maxima of the real part of the potential.

The above conditions and discussion are generic with respect to the amplitude and the profile of the complex potential, since the equations (2) and (3) are exact equations for the soliton mass and velocity variation under propagation when  $u$  is an exact solution of eq. (1). However, in this work we are mostly interested in soliton dynamics that occur in relatively weak potentials where the solitons are quite mobile. In this case the equations (2) and (3) can be treated perturbatively and provide analytical results by utilizing in the respective integrals the well known soliton solution of the homogeneous NLS equation ( $V = W \equiv 0$ ) that is given by  $u = \eta \operatorname{sech}[\eta(x - x_0)] \exp[i(vx/2 + 2\phi)]$  with  $x_0$  and  $v = dx_0/dz$  being the position and the velocity of the soliton center and  $d\phi/dz = \eta^2/2 - v^2/8$ . The soliton mass is  $m = 2\eta$ .

In the following, we focus on periodic potentials and we investigate soliton dynamics for cases where symmetry conditions or the the condition (6) are either fulfilled or violated. In all cases, the amplitude of the various potentials are of the order of  $10^{-2}$  so that the perturbative approach is valid.

### 3. Results

A characteristic periodic profile of the complex potential is the sinusoidal profile

$$V(x) = V_0 \cos(K_0x + \Delta x), \quad W(x) = W_0 \sin(L_0x) \quad (8)$$

with  $V_0, W_0$  being the amplitudes and  $K_0, L_0$  the wavenumbers of the real and imaginary parts of the potential. The real part of the potential is an even function for  $\Delta x = 0$ . The complex potential is known to have a purely real spectrum under the additional condition  $W_0 < V_0$  for  $K_0 = L_0$ . [16] The condition (6) for the existence of the invariant quantity (7) requires both  $\Delta x = 0$  and  $K_0 = L_0$  but does not restrict the relative amplitude of the real and imaginary parts. For the potential (8), Eqs. (2), (3) provide

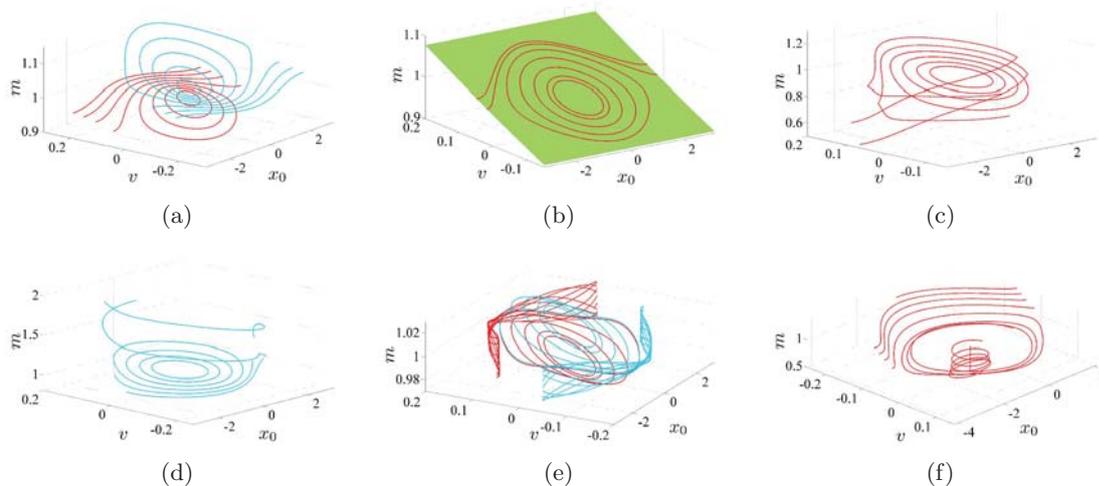
$$\frac{dm}{dz} = -\frac{2\pi W_0 L_0}{\sinh(L_0\pi/m)} \sin(L_0x), \quad m \frac{dv}{dz} = -\frac{\partial U_{eff}}{\partial x_0} \quad (9)$$

with

$$U_{eff} = -\frac{2\pi V_0 K_0}{\sinh(K_0\pi/m)} \cos(K_0x + \Delta x) \quad (10)$$

Soliton moves as a particle of varying mass in a potential having a constant spatial period but dynamically varying amplitude due to its strong dependence on the particle mass. The topology of the orbits in the three-dimensional phase space  $(x_0, v, m)$  depend strongly on the parameters of the potential as shown in Fig. 1. The case of an even real part and an odd imaginary part with equal periods ( $L_0 = K_0$ ) is shown in Fig. 1(a) for soliton initial conditions corresponding to  $m = 1$ , positive and negative velocities ( $v$ ) and various positions ( $x_0$ ). It is obvious that, in contrast to the conservative case  $W_0 = 0$ , initial conditions with  $x_0$  and  $v$  of opposite sign do not follow the same orbit. Moreover, all orbits with the same initial mass and velocity are restricted on the two-dimensional invariant manifold (7), due to the fulfillment of the condition (6) as shown in Fig. 1(b). Characteristic cases of trapped and traveling soliton propagation are shown in Fig. 2(a) and (b). It is worth emphasizing that in the case of a conservative potential the soliton amplitude and width oscillate in such a way that the soliton mass remain constant, whereas in the dissipative case the soliton mass undergoes oscillations.

Phase space orbits for the case of a potential with an even real and an odd imaginary part, but with different periods of a rational ratio, are depicted in Figs. 1(c) and (d) for positive and negative initial velocities, respectively. In this case the condition (6) is not fulfilled and orbits are not restricted in a two-dimensional manifold. Moreover, as shown in Fig. 1(d), the soliton mass variation can be nonperiodic. Soliton propagation for such a characteristic case of continuous mass increasing is depicted in Fig. 2(c). The case of real and imaginary parts with



**Figure 1.** Phase space orbits of the effective particle model for a soliton with initial mass  $m(0) = 1$  in the periodic potential (8) with  $V_0 = 0.01$ ,  $W_0 = V_0/2$  and  $K_0 = 1$ . (a) Potential:  $L_0 = 1$ ,  $\Delta x = 0$ , Initial conditions:  $x_0(0) = 0$ ,  $v(0) > 0$  (red / dark gray),  $v(0) < 0$ , (cyan / light gray); (b) Potential:  $L_0 = 1$ ,  $\Delta x = 0$ , Initial conditions:  $v(0) = 0.05$  (the two-dimensional surface (7) is also shown); (c) Potential:  $L_0 = 1/3$ ,  $\Delta x = 0$ , Initial conditions:  $v(0) = 0.05$ ; (d) Potential:  $L_0 = 1/3$ ,  $\Delta x = 0$ , Initial conditions:  $v(0) = -0.05$ ; (e) Potential:  $L_0 = \sqrt{2}$ ,  $\Delta x = 0$ , Initial conditions:  $v(0) > 0$  (red / dark gray),  $v(0) < 0$ , (cyan / light gray); (f) Potential:  $L_0 = 1$ ,  $\Delta x = -\pi/3$ , Initial conditions:  $v(0) = 0$ ,  $x_0(0) = \pi/3, 1.5\pi/3, 2.8\pi/3$ .

spatial periods of an irrational ratio is depicted in Fig. 1(e). It is shown that in addition to trapped orbits, we also have orbits corresponding to traveling solitons with quasiperiodic mass oscillations, each one densely filling a two-dimensional surface. Soliton propagation for such a characteristic case is shown in Fig. 2(d), where the inset shows the details of the quasiperiodic mass and amplitude oscillations.

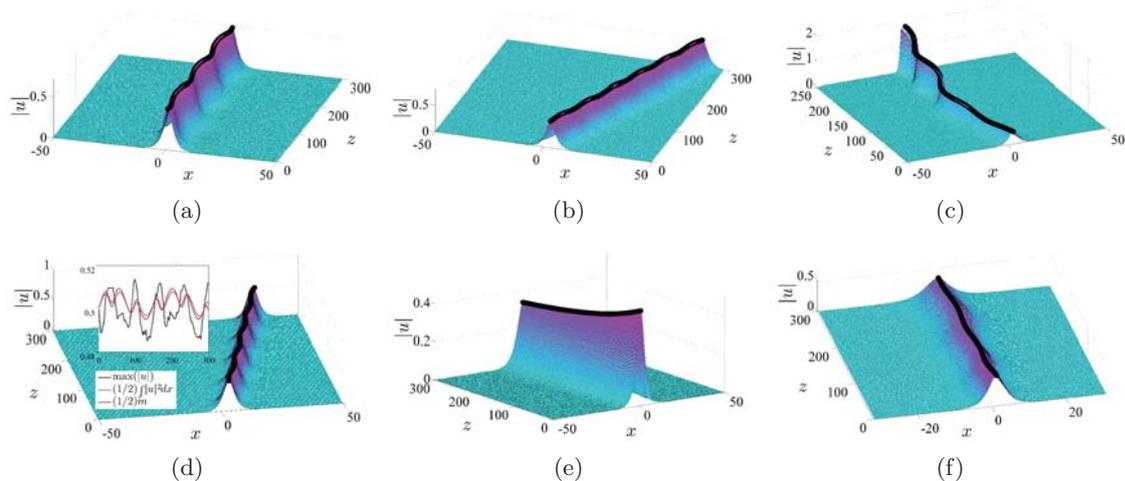
Finally, a case where neither a spatial symmetry exist nor the condition (6) is fulfilled is shown in Fig. 1(e), where the real part is not an even function whereas the imaginary part is an odd function. In this case, there exist an initial condition for which  $x_0 = -\Delta x$  and  $v = 0$  remain constant but the local loss is nonzero, resulting to a soliton evolution where the soliton mass continuously decreases and no transverse soliton motion takes place. Such a characteristic case is depicted in Fig. 2(e). Other initial conditions can result to traveling solitons with increasing mass or trapped solitons with decreasing mass, as also shown in Fig. 2(f).

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**Figure 2.** Soliton evolution under propagation as obtained from numerical simulations of the NLS equation (1) and the effective particle model. The thick black line depicts the effective particle orbit  $(x_0, v, m/2)$ . (a) Potential: as in Fig. 1(a),(b), Initial conditions:  $m(0) = 1$ ,  $x_0(0) = 1$ ,  $v(0) = 0.02$ ; (b) Potential: as in Fig. 1(a),(b), Initial conditions:  $m(0) = 1$ ,  $x_0(0) = \pi$ ,  $v(0) = 0.08$ ; (c) Potential: as in Fig. 1(c),(d), Initial conditions:  $m(0) = 1$ ,  $x_0(0) = \pi$ ,  $v(0) = -0.05$ ; (d) Potential: as in Fig. 1(e), Initial conditions:  $m(0) = 1$ ,  $x_0(0) = \pi$ ,  $v(0) = 0.05$ , The inset depicts  $\max_x(|u|)$  (black),  $(1/2) \int |u|^2 dx$  (red) and  $m/2$  (blue); (e) Potential: as in Fig. 1(f), Initial conditions:  $m(0) = 1$ ,  $x_0(0) = \pi/3$ ,  $v(0) = 0$ ; (f) Potential: as in Fig. 1(f), Initial conditions:  $m(0) = 1$ ,  $x_0(0) = 1.3\pi/3$ ,  $v(0) = 0$ .

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