

Nonlinear waves on the surface of non-isothermal fluid film

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Abstract. A mathematical model of non-isothermal flow of a fluid film is presented. In conditions of Marangoni instability wave characteristics are calculated, identify areas of instability of a fluid film. It shows the influence of thermocapillary forces and forces of surface viscosity on the form of waves. Nonlinear development of perturbations belonging to a continuous band of wave numbers on the surface of a thin fluid layer is investigated within the framework of a nonlinear parabolic equation.

1. Introduction

The investigation of a thin layer of a viscous fluid (a fluid film) that combines thin and large contacted surfaces connected with the implementation of the trends in heat and mass transfer apparatus, for example, thermal power, chemical, food, pharmaceutical industry. The study of the wave flow of a fluid film started by P. Kapitsa [1, 2] and further developed in numerous works of scientists, for example, L. Kholpanov [3], V. Shkadov [4], V. Levich [5], V. Nakoryakov, S. Alekseenko [6], L. Jones, S. Whitaker [7], F. Stainthorp [8] C. Massot, F. Irani, E. Lightfoot [9], V. Penev [10], B. Gjjevik [11], C. Yih [12] from different countries. On regimes of the flow of a fluid film are influenced of various physical and chemical factors. The use of additives of surface-active substances qualitatively changes the character of the flow of a fluid film. The gradients of temperature and concentration cause the heterogeneity of the surface tension and the appearance of the surface tension gradients, resulting in the interphase surface arise force (thermocapillary), the nature of which is determined by the real physical process.

Various heat-mass exchange processes occur in conditions of interfacial instability of the interface, caused by a variety of effects, factors. Special attention attracts researchers interfacial instability associated with the Marangoni effect. The Marangoni effect related to the change at the interface of surface tension from point to point. Emerging Marangoni instability manifests itself in changes of the wave characteristics [13].

2. Mathematical model of non-isothermal liquid film

Consider for a thin layer of viscous incompressible fluid (fluid film) thick under the action of gravity on solid inclined plane in a rectangular coordinate system OXYZ.



The mathematical model of the flow of a three-dimensional non-isothermal fluid film of thickness δ is a system of Navier–Stokes equations and continuity equations with boundary conditions, taking into account the effect of heat – mass – transfer processes [13]:

$$\begin{cases} \frac{\partial u_+}{\partial t_+} + u_+ \frac{\partial u_+}{\partial x_+} + v_+ \frac{\partial u_+}{\partial y_+} + w_+ \frac{\partial u_+}{\partial z_+} = -\frac{\partial P_+}{\partial x_+} + F_x + \frac{1}{\text{Re}} \left(\frac{\partial^2 u_+}{\partial x_+^2} + \frac{\partial^2 u_+}{\partial y_+^2} + \frac{\partial^2 u_+}{\partial z_+^2} \right) \\ \frac{\partial v_+}{\partial t_+} + u_+ \frac{\partial v_+}{\partial x_+} + v_+ \frac{\partial v_+}{\partial y_+} + w_+ \frac{\partial v_+}{\partial z_+} = -\frac{\partial P_+}{\partial y_+} + F_y + \frac{1}{\text{Re}} \left(\frac{\partial^2 v_+}{\partial x_+^2} + \frac{\partial^2 v_+}{\partial y_+^2} + \frac{\partial^2 v_+}{\partial z_+^2} \right) \\ \frac{\partial w_+}{\partial t_+} + u_+ \frac{\partial w_+}{\partial x_+} + v_+ \frac{\partial w_+}{\partial y_+} + w_+ \frac{\partial w_+}{\partial z_+} = -\frac{\partial P_+}{\partial z_+} + F_z + \frac{1}{\text{Re}} \left(\frac{\partial^2 w_+}{\partial x_+^2} + \frac{\partial^2 w_+}{\partial y_+^2} + \frac{\partial^2 w_+}{\partial z_+^2} \right) \\ \frac{\partial u_+}{\partial x_+} + \frac{\partial v_+}{\partial y_+} + \frac{\partial w_+}{\partial z_+} = 0, \end{cases} \quad (1)$$

$$y = 0: \quad u_+ = w_+ = 0, \quad v_+ = V_0 \quad (2)$$

$$\begin{aligned} y = \delta: \quad & \frac{1}{\text{Re}} \left[2 \frac{\partial u_+}{\partial x_+} \frac{\partial \delta_+}{\partial x_+} - 2 \frac{\partial v_+}{\partial y_+} \frac{\partial \delta_+}{\partial x_+} - \left(\frac{\partial v_+}{\partial x_+} + \frac{\partial u_+}{\partial y_+} \right) + \frac{\partial \delta_+}{\partial z_+} \left(\frac{\partial u_+}{\partial z_+} + \frac{\partial w_+}{\partial x_+} \right) \right] \\ & + M \frac{\partial \delta_+}{\partial x_+} + N \left(\frac{\partial^2 u_+}{\partial x_+^2} + \frac{\partial^2 w_+}{\partial x_+ \partial z_+} \right) + \tau_x = 0 \end{aligned} \quad (3)$$

$$\begin{aligned} & \frac{1}{\text{Re}} \left[2 \frac{\partial w_+}{\partial z_+} \frac{\partial \delta_+}{\partial z_+} - 2 \frac{\partial v_+}{\partial y_+} \frac{\partial \delta_+}{\partial z_+} - \left(\frac{\partial w_+}{\partial y_+} + \frac{\partial v_+}{\partial z_+} \right) + \frac{\partial \delta_+}{\partial x_+} \left(\frac{\partial u_+}{\partial z_+} + \frac{\partial w_+}{\partial x_+} \right) \right] \\ & + M \frac{\partial \delta_+}{\partial z_+} + N \left(\frac{\partial^2 w_+}{\partial z_+^2} + \frac{\partial^2 u_+}{\partial x_+ \partial z_+} \right) + \tau_z = 0 \end{aligned} \quad (4)$$

$$P_+ = \frac{2}{\text{Re}} \left[\frac{\partial v_+}{\partial y_+} - \frac{\partial \delta_+}{\partial x_+} \left(\frac{\partial u_+}{\partial y_+} + \frac{\partial v_+}{\partial x_+} \right) - \frac{\partial \delta_+}{\partial z_+} \left(\frac{\partial w_+}{\partial y_+} + \frac{\partial v_+}{\partial z_+} \right) \right] - \sigma_+ \left(\frac{\partial^2 \delta_+}{\partial x_+^2} + \frac{\partial^2 \delta_+}{\partial z_+^2} \right)$$

$$+ P_0 - \text{sgn} \Delta T \frac{\frac{\rho_1}{\rho_2} - 1}{(\text{Re} \cdot \text{Pr} \cdot \text{Ku})^2} \frac{1}{\delta^2} \quad (5)$$

$$\frac{\partial \delta}{\partial t_+} = v_+ - u_+ \frac{\partial \delta}{\partial x_+} - w_+ \frac{\partial \delta}{\partial z_+} + \frac{1}{\text{Re} \cdot \text{Pr} \cdot \text{Ku}} \frac{1}{\delta}. \quad (6)$$

In formulas (1–6), dimensionless quantities: $u_+ = \frac{u}{u_0}$, $v_+ = \frac{v}{u_0}$, $w_+ = \frac{w}{u_0}$ are projections of velocity onto the corresponding coordinate axes; $t_+ = \frac{tu_0}{\delta_0}$ is time; $x_+ = \frac{x}{\delta_0}$, $y_+ = \frac{y}{\delta_0}$, $z_+ = \frac{z}{\delta_0}$ are variables;

$\delta_+ = \frac{\delta}{\delta_0}$ is the thickness of the fluid film; $\text{Re} = \frac{u_0 \delta_0 \rho}{\mu}$ is the Reynolds number; $F_x = \frac{g_x \delta_0}{u_0^2}$, $F_y = -\frac{g_y \delta_0}{u_0^2}$, $F_z = \frac{g_z \delta_0}{u_0^2}$ are the projections of the Froude number onto the corresponding coordinate axes; $P_+ = \frac{P}{\rho u_0^2}$ is pressure; $N = \frac{k+e}{\rho \delta_0^2 u_0}$ is the surface viscosity parameter; $\bar{\tau}_x = \frac{\tau_x}{\rho u_0^2}$, $\bar{\tau}_z = \frac{\tau_z}{\rho u_0^2}$ are the projections of tangential stress; $M = M_T + M_K$ is the Marangoni parameter, where $M_T = \frac{\partial \sigma}{\partial T^0} \left(\frac{\partial T^0}{\partial y} \right)_{y=\delta} \frac{1}{\rho u_0^2}$, $M_K = \frac{\partial \sigma}{\partial K} \frac{\partial K}{\partial \delta} \frac{1}{\rho u_0^2}$; $\text{Pr} = \frac{\rho c_p \nu}{\lambda}$ is the Prandtl number; $\text{Ku} = \frac{r'}{c_p \Delta T}$ is the number of a phase transition; $\sigma_+ = \frac{\sigma}{\rho u_0^2 \delta_0}$ is the surface tension parameter, where u_0 is the average velocity of the main flow of the fluid film, δ_0 is the thickness of the film in the unperturbed state. During condensation, $\text{sgn} \Delta T = 1$, whereas, during evaporation, $\text{sgn} \Delta T = -1$.

The projections of the velocity of (1)–(5) we substitute in equation (6) and get a nonlinear differential equation for the deviation $\psi(x, t)$ of the two-dimensional free surface of the fluid film in the form:

$$\begin{aligned} & \left(\frac{1}{\text{Re Pr Ku}} - \frac{1}{\text{Re Pr Ku}} \psi \right) + \left(a_7 \frac{\partial}{\partial x} + a_{13} \right) \frac{\partial \psi}{\partial t} + a_1 \frac{\partial^4 \psi}{\partial x^4} + a_4 \frac{\partial^3 \psi}{\partial x^3} + a_6 \frac{\partial^2 \psi}{\partial x^2} \\ & + a_{11} \frac{\partial \psi}{\partial x} + a_{14} \psi \frac{\partial \psi}{\partial x} + a_{16} \psi \frac{\partial^2 \psi}{\partial x^2} + a_{17} \psi \frac{\partial^2 \psi}{\partial x \partial t} + a_{21} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial t} + a_{22} \left(\frac{\partial \psi}{\partial x} \right)^2 \\ & + a_{26} \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x^2} + a_{28} \psi \frac{\partial^3 \psi}{\partial x^3} + a_{30} \frac{\partial \psi}{\partial x} \frac{\partial^3 \psi}{\partial x^3} + a_{34} \psi \frac{\partial^4 \psi}{\partial x^4} + a_{37} \psi^2 \frac{\partial \psi}{\partial x} + a_{39} \psi^2 \frac{\partial^2 \psi}{\partial x^2} + a_{40} \psi^2 \frac{\partial^2 \psi}{\partial x \partial t} \\ & + a_{44} \psi \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial t} + a_{45} \psi \left(\frac{\partial \psi}{\partial x} \right)^2 + a_{49} \psi \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x^2} + a_{51} \psi \frac{\partial \psi}{\partial x} \frac{\partial^3 \psi}{\partial x^3} + a_{55} \psi^2 \frac{\partial^4 \psi}{\partial x^4} + a_{58} \psi^2 \frac{\partial^3 \psi}{\partial x^3} = 0. \end{aligned} \quad (7)$$

The numbering of the coefficients of the equation (7) is more complete his mind [14].

The coefficients of Eq. (7) are given by

$$a_1 = -\frac{\text{Re} \sigma}{3}, \quad a_4 = -\frac{\text{Re}^2 F_x N}{2}, \quad a_6 = -\frac{\text{Re} F_y}{3} - \frac{\text{Re} M}{2} + \frac{3}{40} \text{Re}^3 F_x (\tau_x + F_x) + \frac{2}{3} \text{sgn} \Delta T \frac{\text{Re}}{(\text{Re Pr Ku})^2},$$

$$a_7 = \frac{5}{24} \text{Re}^2 F_x, \quad a_{11} = -\text{Re} F_x - \text{Re} \tau_x, \quad a_{13} = -1, \quad a_{14} = -2 \text{Re} F_x - \text{Re} \tau_x, \quad a_{21} = a_{17}, \quad a_{22} = a_{16},$$

$$a_{16} = -\text{Re} F_y - \text{Re} M + \frac{3}{8} \text{Re}^3 F_x \tau_x + \frac{9}{20} \text{Re}^3 F_x^2, \quad a_{17} = 4a_7, \quad a_{18} = \frac{3}{8} \text{Re}^3 (F_z \tau_x + F_x \tau_z) + \frac{9}{10} \text{Re}^3 F_x^2,$$

3. The surface condition of non-isothermal liquid film

Negative temperature gradients occur in the flow of a fluid film on a heated surface. In this case significance $M > 0$ is for the Marangoni parameter.

Under large temperature gradients thermocapillary forces can destroy a fluid film that can cause local overheating cooled film wall and create an emergency situation when working heat and mass– transfer apparatus. The process of development of disturbances in time simulating the destruction of a film (with significant temperature gradients) presented in Figure 1.

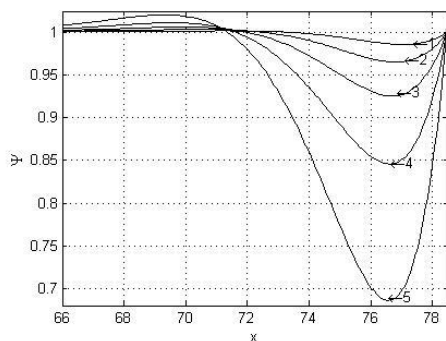


Figure 1. Surface conditions of a vertical film for $Re=10$, $M=1$: 1 - $t=0$; 2 - $t=0.0125$; 3 - $t=0.025$; 4 - $t=0.0375$; 5 - $t=0.05$

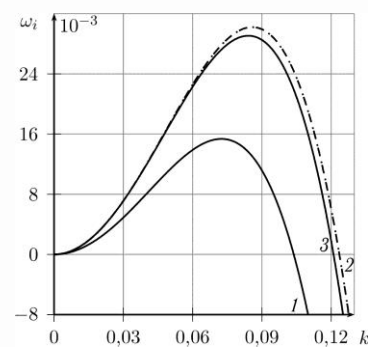


Figure 2. Increment as a function of wave number:

1- $M=0$, 2- $M=1$, 3- $M=1$, $N=1$

4. Nonlinear parabolic equation

On the surface of a fluid film developing disturbances belonging continuous band of wave numbers (wave packet), under the following assumptions: the wave number band is $\Delta k_\Sigma = o(\varepsilon)$, and the increment is $\omega_i = \varepsilon^2 \bar{\omega}_i = o(\varepsilon^2)$, where ε is a small parameter. The equation for the amplitude A envelope of a wave packet is a nonlinear parabolic equation of the form [15, 16]:

$$\begin{aligned} \frac{\partial A}{\partial t_2} + \frac{i}{\varepsilon} \frac{\partial \omega_i}{\partial k_x} \frac{\partial A}{\partial x_1} + \frac{i}{\varepsilon} \frac{\partial \omega_i}{\partial k_z} \frac{\partial A}{\partial z_1} - \frac{i}{2} \left(\frac{\partial^2 \omega_r}{\partial k_x^2} + i \frac{\partial^2 \omega_i}{\partial k_x^2} \right) \frac{\partial^2 A}{\partial x_1^2} \\ - \frac{i}{2} \left(\frac{\partial^2 \omega_r}{\partial k_z^2} + i \frac{\partial^2 \omega_i}{\partial k_z^2} \right) \frac{\partial^2 A}{\partial z_1^2} - i \left(\frac{\partial^2 \omega_r}{\partial k_x \partial k_z} + i \frac{\partial^2 \omega_i}{\partial k_x \partial k_z} \right) \frac{\partial^2 A}{\partial x_1 \partial z_1} = \bar{\omega}_i A - (\beta_1 + i\beta_2) |A|^2 A. \end{aligned} \quad (8)$$

The coefficients of the nonlinear term in equation (8) characterize the nonlinear damping of perturbations (β_1) and the dependence of phase (frequency) on the amplitude (β_2). The first- and second-order derivatives in Eq. (8) are calculated using the dispersion equation

$$\begin{aligned} \omega(a_7 k_x + a_9 k_z + i) + a_1 k_x^4 + a_2 k_x^2 k_z^2 + a_3 k_z^4 - a_4 i k_x^3 - a_5 i k_z^3 - a_6 k_x^2 - a_8 k_x k_z \\ - a_{10} k_z^2 + a_{11} i k_x + a_{12} i k_z - \frac{1}{Re \cdot Pr \cdot Ku} = 0, \end{aligned} \quad (9)$$

where $\omega = \omega_r + i\omega_i$, ω_r – frequency, ω_i – increment.

The work presents the results of computational experiments for vertical film of water for Reynolds number $Re = 5$ as in free dripping wet film, and considering thermocapillary forces and forces of surface viscosity: increment (Figure 2).

The waves are wave numbers $k_{\omega_{t\max}}$ with the highest probability are realized in experiments. Wave characteristics are shown in Table 1, where:

$c_r = \frac{\omega_r}{k}$ is the phase velocity, $V_{gr} = \frac{d\omega_r}{dk}$ is the group velocity.

Table 1. Wave characteristics

| Re=5 | $k_{\omega_{t\max}}$ | $\omega_{t\max}$ | ω_r | c_r | V_{gr} | β_1 | β_2 |
|----------|----------------------|------------------|------------|--------|----------|-----------|-----------|
| M=0, N=0 | 0,0725 | 0,0154 | 0,2139 | 2,9520 | 2,9520 | 1,6896 | -0,0333 |
| M=1, N=0 | 0,0858 | 0,0302 | 0,2493 | 2,9055 | 2,9055 | 1,3955 | -0,2014 |
| M=1, N=1 | 0,0842 | 0,0291 | 0,2405 | 2,8560 | 2,7503 | 1,2897 | -0,1159 |

5. Conclusions

1. We have presented a nonlinear mathematical model of a nonisothermal three-dimensional fluid film for Reynolds number of $Re \leq 20$.
2. We have presented a nonlinear mathematical model for the state of the free surface of a nonisothermal fluid film.
3. Using a dispersion equation, we have calculated the wave characteristics: the frequency, the increment and the phase velocity.
4. We have calculated the forms of waves on the surface of a fluid film under inhomogeneous surface tension.
5. We have derived a nonlinear parabolic equation for the amplitude of the envelope of a wave packet, which is one of the basic models for nonlinear media (a Ginzburg–Landau –type model).

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