# Percolation of interdependent networks with degree-correlated inter-connections

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**Abstract.** In interdependent networks, failures of nodes in one constituent network lead nodes in another network to fail. This happens recursively and leads to a cascade of failures. It is known that the interdependent networks with random inter-connections have weaker robustness than the individual networks. However, if the interdependent networks have degree correlations between the networks constructing them as in the actual cases, the robustness of the interdependent networks may be changed. In this paper, we perform numerical simulations on interdependent networks and obtain the giant cluster sizes after the cascade of failures in order to evaluate the robustness. We show that when a interdependent network has a positive degree inter-correlation, it has the stronger robustness than that for the networks with no degree correlation. We show not only the numerical simulation results but theoretical ones for the robustness of the interdependent networks.

## 1. Introduction

Most modern systems are not simple. They are complex and construct large networks. Then, it is important that the networks are robust, in other words whether the networks are functional after random removal of a fraction [1]. Furthermore, several modern systems are more complex. They are not isolated networks but connected to one another. They are called interdependent networks. For example, the infrastructures such as water supply, transportation, and fuel and power stations are coupled together mutually. Even if systems are well robust individually, the whole coupled system may have weaker robustness than the individual systems [2]. Random removal of a small fraction of nodes from a constituent network can produce an iterative cascade of failures in total interdependent networks. Electrical blackouts frequently result from a cascade of failures in interdependent networks with the power grids and their communication support systems. For example, the blackout that affected much of Italy on 28 September 2003 occurred by the cascade in the electric network system constituted by the power and communication networks.

In this paper, we consider correlations between the networks that construct interdependent networks. The purpose of this paper is to find how the correlation or the degree distribution affects the robustness of the whole network. We perform numerical simulations for Erdős-Rényi (ER) networks and Barabási-Albert (BA) networks. ER networks have Poissonian degree distributions, and BA networks have power-law degree distributions. Therefore, BA networks have nodes with very large degree that are called hubs. Both of the ER networks and the BA networks are robuster when they have degree inter-correlations, and the difference between the

robustness with no correlation and strong correlations in the network constructed by the BA networks is larger than that in the network constructed by the ER networks. We show not only numerical simulation results but theoretical ones. Although several workers [3][4][5] theoretically investigate the robustness of interdependent networks with degree inter-correlations, there have not been any theoretical research for the robustness of interdependent networks consisting of random and scale-free networks with arbitrary inter-correlation. The theory can be used for networks with any degree distributions and any correlations between two constituent networks. The theoretical results fit well with those of numerical simulation results.

## 2. Model

### 2.1. Interdependent networks

As an interdependent network, we consider a network constructed from connected two networks, A and B with the same number of nodes. Each node $A_i$ in network A depends on a node $B_i$ in network B and vice versa. That is, $A_i$ and $B_i$ are mutually connected. If node $A_i$ stops functioning owing to an attack or a failure, node $B_i$ also stops functioning. Similarly, if node $B_i$ stops functioning then node $A_i$ stops functioning, since each node in one network can depend only on one node in the other network. In the interdependent networks, we begin by randomly removing a fraction $(1 - p)$ of nodes in network A, and failures cascade as in Figure 1. A group that nodes are mutually connected with links is called a cluster. Therefore, if the number of the nodes that the largest or giant cluster contains is large, the network is robust.
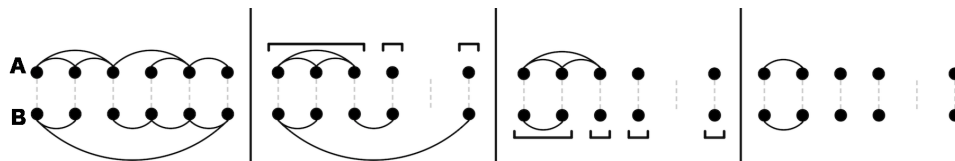


**Figure 1.** A model of a cascade of failures

1. Remove failures. We remove a fraction $(1 - p)$ of A-nodes, where $p$ is the percentage of remaining nodes. where A-nodes mean the nodes in the network A. Then we remove the nodes in the network B, B-nodes, that are connected to the removed A-nodes, as shown in the second figure of Figure 1.

2. Remove the links in the network B across the different clusters in the network A. For example, the network A is divided into three clusters in the second figure of Figure 1, and we remove two links in the network B that connect nodes which are, respectively, in the different clusters in the network A. As the result, we obtain the third figure of Figure 1.

3. Remove the links in the network A across the different clusters in the network B. It is the similar procedure to step 2, as shown in the fourth figure of Figure 1.

4. Repeat steps 2 and 3 until the convergence.

After the convergence, if the percentage $\mu_\infty$ of nodes in the giant cluster is high, the network is robust because many nodes are connected and the network is functional. This model shows a percolation phase transition where $p$ equals a threshold $p_c$. That is, $p$ is smaller than $p_c$, then $\mu_\infty = 0$. Therefore the network has no cluster. Otherwise, $p$ is larger than $p_c$, then $\mu_\infty > 0$. That is, the network has a cluster. The smaller $p_c$ is, the robuster the network is.

### 2.2. Correlation between networks

There is a inter-correlation between two networks in the real world. Therefore it is important that we consider the networks with not only no degree inter-correlation but a degree inter-

correlation. We make a interdependent network with a degree correlation between constituent networks by connecting a node in a network to one in another network sampled according to its degree. Actually a node is sampled with a probability proportional to $k^\alpha$ from a network and pairs with a similarly sampled node from another network, where $k$ is the degree of the node and $\alpha$ is a parameter for the correlation. When $\alpha$ is 0, the networks have no inter-correlation because the connected nodes are sampled at random. On the other hand, when $\alpha$ is $\infty$, the networks have a very strong correlation. This procedure makes a positive correlation or no correlation. It is, however, difficult to make a negative correlation.

## 3. Simulation

We perform numerical simulations for interdependent networks constructed by two Erdős-Rényi (ER) networks and ones constructed by two Barabási-Albert (BA) networks with the total nodes of each constituent networks $N = 25600$ and the average degree $\langle k \rangle = 4$ for various inter-correlation coefficient of degrees. For both ER and BA networks, we find that the stronger the correlation is, the robuster the networks are. Threshold $p_c$ is approximately a linear function of the correlation coefficient $r$. In individual networks, large degree nodes are hard to be removed from a cluster because they have many links. However, when the interdependent networks have no correlation, large degree nodes easily remove from clusters because the nodes in the other network inter-connected to the large degree nodes may have small degrees and be easily removed. When the networks have strong positive correlations, large degree nodes usually connect to large degree nodes. Therefore both nodes are hard to be removed from the clusters. Then, the networks that have strong correlations are robuster than those with no correlation. Comparing BA networks with ER networks, we find that the difference between robustness with no correlation and the strong correlations for BA networks is larger than one for ER networks, because BA networks have more hubs than ER networks. If the hubs are removed, they greatly affect the cascade. Moreover, for no inter-correlation networks constructed by BA networks, such an iterative process of cascading failures leads to a first-order phase transition. When the correlation is strong, a second-order phase transition is, however, shown, although a first-order phase transition is observed for non- and inter-correlated networks with ER networks.

## 4. Theory

As in [2], we use the generating function method [8] for evaluating robustness of interdependent networks. Although the degree distribution of each network is unchanged in the cascading processes for randomly inter-connected interdependent networks, one for interdependent networks with the degree inter-correlation is varying in the cascading process. We take into account the variation of the distribution theoretically. In the theory, we use the results of [6] and [7] for the degree distribution of nodes which is not in the giant cluster and one for the constituent network in which some nodes which are linked with failed nodes in another network are removed. The robustness of the networks are theoretically calculated and compared with simulation results in the next section.

## 5. Comparison between theoretical and simulation results

Figure 2 shows the comparison between the theoretical and numerical simulation results for the networks with BA networks where degree distributions are $P(k) \propto k^{-3}$. The difference between the theoretical and simulation results is small for $\mu_\infty > 0.2$, but it is large for $\mu_\infty < 0.2$. However, the value for $\mu_\infty > 0.2$ is more important than the value for $\mu_\infty < 0.2$ because when $\mu_\infty < 0.2$, the network is almost broken and the functional network needs a certain value of $\mu_\infty$. The theoretical results for ER networks and BA networks where the degree distribution are $P(k) \propto k^{-2.7}$ agree well with the numerical simulation results similarly. Therefore the theoretical

results reproduce simulation results very well for both ER networks and BA networks, with any correlations.
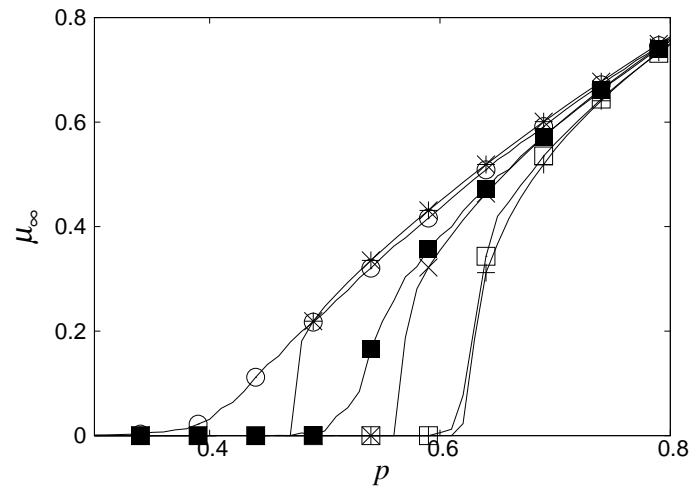


**Figure 2.** Theoretical and simulation results for $\mu_\infty$ as a function of $p$ for BA networks. Circles, filled squares and open squares are simulation results for r=0.986, 0.365 and 0 and asterisks, crosses and pluses are theoretical ones for r=0.986, 0.365 and 0, respectively.

## 6. Conclusion

In this paper, we study interdependent networks with degree-correlated inter-connections. Although interdependent networks generally have weak robustness, they are robuster if the networks have the strong degree correlations between constituted networks. The robustness with the strong correlation is higher when the network has large hubs. Specifically BA networks is robuster than ER networks, and BA networks with degree distribution $P(k) \propto k^{-2.7}$ is robuster than BA networks with degree distribution $P(k) \propto k^{-3.0}$. Furthermore we propose a simple theoretical method for this model. Then, the theoretical results almost fit to the numerical simulation ones and we can use the theory for networks with any degree distribution and any correlation. It is very useful to treat modern complex systems with the theory.

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