

Relativistic solitons in pulsar wind nebulae

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Abstract. The conversion mechanisms responsible for the transformation of the relativistic stellar wind to the observed pulsar nebula electromagnetic radiation is one of the most interesting problems of the high energy astrophysics. In particular, the existence of a connection between the giant flares of the gamma-radiation from the Crab nebula (which were discovered in 2011 by Fermi and AGILE telescopes) and observed dynamical structures remains being a question. To address the origin of the observed dynamical structures in the shocked wind we consider here a model of a weak nonlinear perturbation propagating transverse to the mean quasi-stationary magnetic field in the pulsar nebula plasma. The plasma is supposed to be electron-positron and relativistic. It is shown that in the regime of the strong scattering of pairs by the stochastic magnetic field fluctuation the propagation of magnetic field perturbation can be described by well-known Korteweg — de Vries (KdV) equation. One of the KdV solutions is a soliton — a long lived solitary wave, propagating without changing of its shape. The width of plasma magnetic soliton in the presence of ultra-relativistic pairs is discussed.

1. Introduction

The relativistic pulsar wind pulsar nebulae are unique sources of electromagnetic radiation detected in all parts of the electromagnetic spectrum. The pulsar wind nebulae (PWN) are able to transform the energy of a rotating neutron star into electromagnetic energy of their emission with a high efficiency. The processes, responsible for this are now the hot issues of the high energy astrophysics. High angular resolution radio, optical and X-ray observations revealed the presence of dynamical structures in the Crab nebulae[1].

The study of the origin of the dynamical structures like jets and wisps may help to understand the physics of the energy conversion mechanism in the relativistic pair plasma of the pulsar nebula and the nature of the observed gamma-ray variability[1–3]. To interpret the observations one need a theoretical model for dynamical magnetic structures, propagating in the relativistic pair plasma of the PWNe.

Non-linear equations for magnetic perturbations in a collisionless pair plasma with infinite particle mean free path were derived in [4]. The authors come up with the Korteweg- de Vries type of equation. The model allows the soliton solutions - a long lived solitary wave, propagating without changing of its shape. The condition of collisionless might not be satisfied in the case of the PWN plasma where pair scattering by stochastic magnetic fluctuations may be important. Also, the energetic distribution of particles in the PWN plasma might be strongly non-equilibrium. The nonthermal case with specified above scatterings is considered.



2. The model

For our problem we use the kinetic approach. The kinetic equation:

$$\partial_t f + \mathbf{v} \partial_{\mathbf{r}} f + \dot{\mathbf{p}} \partial_{\mathbf{p}} f = I[f] \quad (1)$$

Here t , \mathbf{r} and \mathbf{p} are the time, the radius vector of the particle and the momentum of a particle (in the laboratory frame), respectively, $I[f]$ is a collisional operator, and $\dot{\mathbf{p}} = e \left(\mathbf{E} + \frac{1}{c} [\mathbf{v} \mathbf{b}] \right)$ - the Lorentz force. The distribution function f is as usually sought in the form of $f = f_0 + \tilde{f}$, where f_0 is its isotropic part.

We suppose that the perturbation of the magnetic field propagates along x -axis, transverse to mean quasi-stationary magnetic field $\mathbf{B}_0 || z$. Our perturbation could be considered as a linearly-polarized wave, whose electric field $\mathbf{E} || y$, and magnetic field $\mathbf{b} || z$.

As we say in the introduction, we consider a 'collisional' case, where we mean the regime of the strong scattering of pairs by the stochastic magnetic field fluctuation. As usual, a typical frequency of collisions ν can be introduced. We make an assumption that the typical frequencies of the considered processes related with propagation of the perturbation are small in comparison with the typical frequency of collisions ν . Also, we should introduce the gyrofrequency

$$\Omega = \frac{eB_0}{m_e c \gamma}$$

Here e - the elementary charge, m_e - the mass of a particle, B_0 - the induction of the mean quasi-stationary magnetic field, c - the light velocity, γ - the Lorentz-factor. It is known that the frequency of collisions might not be larger than the gyrofrequency. So, we can write:

$$\omega \ll \nu \leq \Omega \quad (2)$$

We use the relaxation time approximation for a collisional operator:

$$-\nu (f - f_0) \equiv -\nu \tilde{f} \quad (3)$$

Here f_0 is the isotropic part of the distribution function. The system of the kinetic equations takes the form:

$$\partial_t f_p + v_x \partial_x f_p - \Omega \cdot \hat{\mathbf{O}} f_p + e \left(E_y - \frac{1}{c} v_x b \right) \partial_{p_y} f_p + \frac{e}{c} v_y b \partial_{p_x} f_p = -\nu_1 \tilde{f}_p + \nu_2 \tilde{f}_e \quad (4)$$

$$\partial_t f_e + v_x \partial_x f_e + \Omega \cdot \hat{\mathbf{O}} f_e - e \left(E_y - \frac{1}{c} v_x b \right) \partial_{p_y} f_p - \frac{e}{c} v_y b \partial_{p_x} f_p = -\nu_2 \tilde{f}_e + \nu_1 \tilde{f}_p \quad (5)$$

Here $\hat{\mathbf{O}} = [\mathbf{p} \times \partial_{\mathbf{p}}]$ is the angular momentum operator. Subscripts 'p' and 'e' refer to positrons and electrons, respectively, and \tilde{f}_p , \tilde{f}_e are the anisotropic parts of the distribution functions. Here we write the collisional operators in a symmetric to the sorts of the particles form.

3. The derivation of an evolution equation

To get an evolution equation we need to derive a current response at the field perturbation. The only non-zero component j_y is calculated as a moment of the distribution function:

$$j_y = \sum e_i \int v_y f_i d\mathbf{p} \quad (6)$$

Here the summation is done by the sorts of the particles. Thus, we need to derive the anisotropic part of the distribution function \tilde{f} . We seek \tilde{f} as an expansion in the spherical functions.

3.1. Linear current response

Firstly, we derive a linear part of the current response. We leave in the kinetic equations only linear in the fields terms.

$$\partial_t f_p + v_x \partial_x f_p - \Omega_z \partial_\phi f_p + e E_y \partial_p f_0 \frac{p_y}{p} = -\nu_1 \tilde{f}_p + \nu_2 \tilde{f}_e \quad (7)$$

$$\partial_t f_e + v_x \partial_x f_e + \Omega_z \partial_\phi f_e - e E_y \partial_p f_0 \frac{p_y}{p} = \nu_1 \tilde{f}_p - \nu_2 \tilde{f}_e \quad (8)$$

Here the relation $\hat{O}_z = \partial_\phi$ is used. The distribution function is sought in the form of

$$\tilde{f} = f_x \sin\theta \cos\phi + f_y \sin\theta \sin\phi + f_z \cos\theta$$

We take the Fourier transform, rewrite the kinetic equations (7),(8) in the spherical coordinates, multiply them at $\sin\theta \cos\phi$ or $\sin\theta \sin\phi$ and integrate over the angles. Thus, we get an algebraic system

$$\begin{aligned} (-i\omega + \nu_1) f_{px,k\omega} - \Omega f_{py,k\omega} - \nu_2 f_{ex,k\omega} &= 0 \\ \Omega f_{px,k\omega} + (-i\omega + \nu_1) f_{py,k\omega} - \nu_2 f_{ey,k\omega} &= -e E_{y,k\omega} \partial_p f_0 \\ -\nu_1 f_{px,k\omega} + (-i\omega + \nu_2) f_{ex,k\omega} + \Omega f_{ey,k\omega} &= 0 \\ -\nu_1 f_{py,k\omega} - \Omega f_{ex,k\omega} + (-i\omega + \nu_2) f_{ey,k\omega} &= e E_{y,k\omega} \partial_p f_0 \end{aligned} \quad (9)$$

Only f_{py} gives a non-trivial contribution to the j_y . Substituting it to (6), we get

$$j_{y,k\omega} = \int \frac{2 [\Omega^2 i\omega + (\nu_1 + \nu_2) \omega^2 - i\omega^3] \cdot \sin\theta \sin\phi \cdot e^2 E_{y,k\omega} \partial_p f_0 v_y d^3 p}{\omega^4 + 2(\nu_1 + \nu_2) i\omega^3 - ((\nu_1 + \nu_2)^2 + 2\Omega^2) \omega^2 - 2(\nu_1 + \nu_2) \Omega^2 i\omega + (\nu_1 - \nu_2)^2 \Omega^2 + \Omega^4} \quad (10)$$

Due to relations (2), the denominator can be expanded in the series in the small ratios of ω to the gyrofrequency and the frequency of collisions. Thus,

$$j_{y,k\omega} = \int \frac{2[\Omega^2 i\omega + (\nu_1 + \nu_2) \omega^2 - i\omega^3]}{(\nu_1 - \nu_2)^2 \Omega^2 + \Omega^4} \left(1 + \frac{((\nu_1 + \nu_2)^2 + 2\Omega^2) \omega^2 + 2(\nu_1 + \nu_2) \Omega^2 i\omega}{(\nu_1 - \nu_2)^2 \Omega^2 + \Omega^4} \right) \sin\theta \sin\phi \cdot e^2 E_{y,k\omega} \partial_p f_0 v_y d^3 p \quad (11)$$

Then we write a Fourier transform for the Maxwell equation,

$$-ik b_{k\omega} = \frac{4\pi}{c} j_{y,k\omega} - \frac{1}{c} i\omega E_{y,k\omega} \quad (12)$$

and substitute here the result for $j_{y,k\omega}$. Thus, we can write a dispersion equation

$$\omega = V k \left(1 - a_D^2 k^2 \right) - i\chi k^2 \quad (13)$$

Here we introduce new designations: the phase velocity V , the dispersion length a_D and the damping χ , which are defined as

$$V = \frac{c}{\sqrt{1 + \omega_p^2 A_1}} \quad (14)$$

$$a_D^2 = \frac{V^4}{c^2} \omega_p^2 B_1 \quad (15)$$

$$\chi = \frac{V^4}{c^2} \omega_p^2 B_2 \quad (16)$$

And A_1 , B_1 , B_2 are the integrals over momentum

$$\omega_p^2 A_1 = \int \frac{2}{(\nu_1 - \nu_2)^2 + \Omega^2} \cdot \sin\theta \sin\phi \cdot e^2 \partial_p f_0 v_y d^3 p \quad (17)$$

$$\begin{aligned} \omega_p^2 B_1 = & \int \left(-\frac{2}{((\nu_1 - \nu_2)^2 + \Omega^2)\Omega^2} \right) \cdot \sin\theta \sin\phi \cdot e^2 \partial_p f_0 v_y d^3 p + \\ & + \int \left(\frac{4(\nu_1 + \nu_2)^2}{((\nu_1 - \nu_2)^2 + \Omega^2)^2 \Omega^2} + \frac{2((\nu_1 + \nu_2)^2 + 2\Omega^2)}{((\nu_1 - \nu_2)^2 + \Omega^2)^2 \Omega^2} \right) \cdot \sin\theta \sin\phi \cdot e^2 \partial_p f_0 v_y d^3 p \end{aligned} \quad (18)$$

$$\begin{aligned} \omega_p^2 B_2 = & - \int \left(\frac{2(\nu_1 + \nu_2)}{((\nu_1 - \nu_2)^2 + \Omega^2)\Omega^2} \right) \cdot \sin\theta \sin\phi \cdot e^2 \partial_p f_0 v_y d^3 p + \\ & + \int \frac{4(\nu_1 + \nu_2)}{((\nu_1 - \nu_2)^2 + \Omega^2)^2} \cdot \sin\theta \sin\phi \cdot e^2 \partial_p f_0 v_y d^3 p \end{aligned} \quad (19)$$

Here $\omega_p = \left(\frac{4\pi e^2 n_0}{m_e} \right)^{1/2}$ is the plasma frequency.

Making the inverse Fourier transform for $j_{y,k\omega}$, we can write for the linear current response in new designations:

$$j_y = \frac{1}{4\pi} \frac{c}{V} \left[\left(1 - \frac{V^2}{c^2} \right) \partial_t b + V a_D^2 \partial_x^3 b - \chi \partial_x^2 b \right] \quad (20)$$

Here we used a useful relation $E_y = \frac{V}{c} b$, which could be easily got by taking the Fourier transform of the Maxwell equation $\partial_x E_y = -\frac{1}{c} \partial_t b$.

3.2. Nonlinear current response

We also have to derive the nonlinear current response. We are going to solve the kinetic equations (4), (5) in the coordinate space (ct, \mathbf{r}) taking into account the non-linear in the fields terms. We can use the approximation method for this part of the problem.

In the first approximation only linear in the fields terms should be left. We seek the first approximation of the anisotropic distribution function in the form of

$$f_{(p,e)1} = \sum C_{1m}^{(p,e)} Y_{1m}(\theta, \phi) \quad (21)$$

The kinetic equations (4),(5) are rewritten in the spherical coordinates, multiplied at $\sin\theta \cos\phi$ or $\sin\theta \sin\phi$ and integrated over the angles. Thus, we get a system of equations

$$\partial_t C_{1m}^p - i\Omega m C_{1m}^p + \nu_1 C_{1m}^p = im \sqrt{\frac{2\pi}{3}} e E_y \partial_p f_0 + \nu_2 C_{1m}^e \quad (22)$$

$$\partial_t C_{1m}^e + i\Omega m C_{1m}^e + \nu_2 C_{1m}^e = -im \sqrt{\frac{2\pi}{3}} e E_y \partial_p f_0 + \nu_1 C_{1m}^p \quad (23)$$

We can easily build a Green function for each of the equations (22),(23): it will take the form $G_{1m}^{(p,e)}(t - \tilde{t}) = e^{-\nu_{(1,2)}(t - \tilde{t}) \pm i\Omega m(t - \tilde{t})} \Theta(t - \tilde{t})$, where sign + and frequency ν_1 refer to positrons, and Θ - the Heaviside function. Then, according to the Green function definition, we can build a formal solution. For positrons:

$$C_{1m}^p = im \sqrt{\frac{2\pi}{3}} \partial_p f_0 \int_0^{+\infty} e E_y(t - \tilde{t}, x) e^{-\nu_1 \tilde{t} + i\Omega m \tilde{t}} d\tilde{t} + \nu_2 \int_0^{+\infty} C_{1m}^e e^{-\nu_1 \tilde{t} + i\Omega m \tilde{t}} d\tilde{t} \quad (24)$$

To get the solutions for electrons one should do the substitutions: $e \rightarrow -e$, $\Omega \rightarrow -\Omega$, $\nu_1 \rightarrow \nu_2$, $\nu_2 \rightarrow \nu_1$.

In the second approximation of the distribution function we have to consider nonlinear in the fields terms in the equations (4),(5). The second order of the distribution function is sought as

$$f_{(p,e)2} = \sum C_{2m}^{(p,e)} Y_{2m}(\theta, \phi) \quad (25)$$

After the same to the first order derivation transformations, we get for positrons:

$$C_{2m}^p = -\sqrt{\frac{2\pi}{3}} m^2 \partial_p f_0 \frac{e\Omega}{B_0} \int_0^{+\infty} dt_1 b(t-t_1, x) e^{-\nu_1 t_1 + i\Omega m t_1} \int_0^{+\infty} dt_2 E_y(t-t_1-t_2, x) e^{-\nu_1 t_2 + i\Omega m t_2} + \\ + i m \nu_2 \frac{\Omega}{B_0} \int_0^{+\infty} dt_1 b(t-t_1, x) e^{-\nu_1 t_1 + i\Omega m t_1} \int_0^{+\infty} dt_2 C_{1m}^e e^{-\nu_1 t_2 + i\Omega m t_2} + \\ + \nu_2 \int_0^{+\infty} dt_1 C_{2m}^e e^{-\nu_1 t_1 + i\Omega m t_1} \quad (26)$$

To get the solutions for electrons one again should do the substitutions: $e \rightarrow -e$, $\Omega \rightarrow -\Omega$, $\nu_1 \rightarrow \nu_2$, $\nu_2 \rightarrow \nu_1$.

The equations (24), (26) and the corresponding pair for electrons could be considered as a system of coupled integral equations. This system could be reduced to an algebraic system: one should rewrite the fields and the coefficients C as the inverse Fourier transformation integrals and rearrange the integrals (appearing fractions of the form $1/(\nu_a - i(\omega \pm m\Omega))$ should be expanded in the series in the small ratio of ω to other frequencies).

Finally, the solution of the system gives us a nonlinear contribution in the current response:

$$j_y^{NL} = \frac{V^2}{c} \lambda_1 \frac{b}{B_0} \partial_x b \quad (27)$$

Here V - introduced earlier phase velocity, λ_1 — the coefficient, defined by integral over momentum.

Thus, we substitute the sum of (27) and (20) to the Maxwell equation and after simple transformations get the evolution equation:

$$\partial_t b + V \partial_x b + V a_D^2 \partial_x^3 b + \frac{V^3}{c^2} \lambda_1 \frac{b}{B_0} \partial_x b = \chi \partial_x^2 b \quad (28)$$

Since this moment, we assume $\nu_1 = \nu_2 \equiv \nu$. We also take ν in the frequently used form of $\nu = a\Omega$. Further evaluations can show that if $a \ll 1$, the small damping $\chi \partial_x^2 b$ can be neglected in quite a wide range of the parameters. We introduce designations $\mu = V a_D^2$, $\lambda = \frac{V^3}{c^2} \lambda_1$. Thus, for the dimensionless function $h = b/B_0$ we can write

$$\partial_t h + V \partial_x h + \mu \partial_x^3 h + \lambda h \partial_x h = 0 \quad (29)$$

This is a well-known Korteweg—de Vries equation (KdV). One of the KdV solutions is a soliton — a solitary wave, propagating without changing of its shape.

4. Evaluations

A practically important problem is the evaluation of the soliton width w , which is defined as $w = \sqrt{\frac{12\mu}{\lambda h_0}}$ (here h_0 — the dimensionless soliton amplitude). The angular resolution of the Hubble Space Telescope (HST) lets us to resolve objects with the size larger than $\sim 10^{15}$ cm in the Crab nebula. The evolution equation, which was obtained under the conditions, relevant to the PWNe, has the soliton solutions in a wide range of the parameters. The propagating perturbation of the magnetic field in the Crab nebula can be observed as moving structures like wisps in the high resolution images of synchrotron emission. Thus, if the solitons can propagate in the conditions of the Crab nebula, and their width is $\sim 10^{15}$ cm or larger, they could be resolved by the HST.

In the evaluations the induction B_0 is taken, according to the observations, at $200\mu G$. The concentration is estimated from the pressure, which could be, from the one hand, evaluated as a distribution function moment $P = \frac{1}{3} \int p v f d^3 p$, and, from another hand, estimated as a magnetic pressure $\frac{B_0^2}{8\pi}$. The evaluations are done for the temperatures from the lower part of the

relativistic range. The frequency ν is taken in the form of $\nu = a\Omega$, where $a \ll 1$. The soliton amplitude is taken equal to 0.1, because we consider a weakly-nonlinear perturbation.

The soliton width is evaluated in both cases of the equilibrium Maxwell-Jüttner (relativistic Maxwell) energy distribution, and of the non-equilibrium one. In the second case we use the Hybrid distribution, which is the matching of the Maxwell-Jüttner distribution at relatively low energies and the power-law energy distribution with the index 2 and the cutoff Lorentz-factor 10^9 at high energies (these values are well corresponding to the observations). The results for

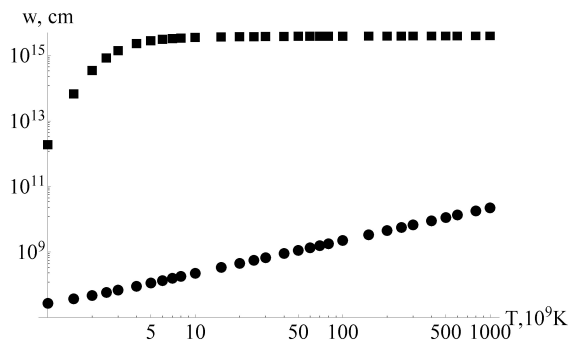


Figure 1. Soliton widths w for: ● - Maxwell-Jüttner, ■ - Hybrid distributions; $a = 0.1$

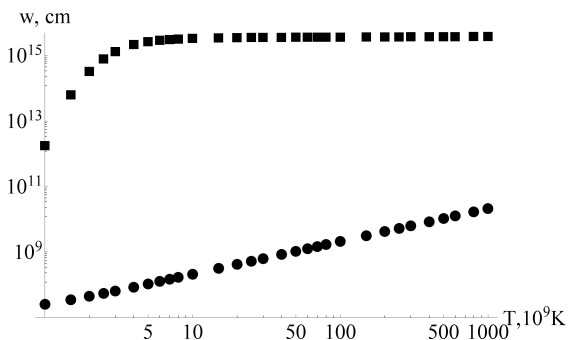


Figure 2. Soliton widths w for: ● - Maxwell-Jüttner, ■ - Hybrid distributions; $a = 10^{-3}$

different parameter values are presented in the Figures (1),(2) (they are not identical). We derived the widths $w(T)$ for various pair plasma temperatures T just to illustrate the range of the parameters space, the detailed analysis of this will be done elsewhere. The results show, that the soliton width w substantially depend on the energy distribution. In the case of quasi-equilibrium pair distribution w is in the range of 10^7 — 10^{10} cm, while in the presence of the power-law tails of pair distribution its values almost always exceed 10^{15} cm — which is about the resolution limit for the HST in the Crab nebula. Thus, the observed widths of the dynamical wisps in the Crab nebula may constrain the pair distribution function. This fact opens up a prospective of an observational study of the relativistic pairs gas properties (e.g., the pressure).

5. Conclusions

In this paper we discuss a theoretical model of a propagation of a weakly-nonlinear perturbation of the magnetic field transverse to the mean quasi-stationary magnetic field in the relativistic non-thermal electron-positron plasma of the pulsar wind nebula. By kinetic approach we derive an evolution equation, and show that it takes the form of well-known Korteweg—de Vries equation for the wide range of the parameters. This equation has a class of stable solutions — the solitons. The soliton widths are derived for quasi-equilibrium and for strongly non-thermal pair energy distributions. For the non-equilibrium case, when the pressure is dominated by the ultrarelativistic non-thermal synchrotron emitting pairs, the obtained soliton width exceeds the resolution limit of the Hubble Space Telescope for the Crab nebula. It means, that the solitons can be responsible for the moving structures like wisps, observed in the high resolution images of the synchrotron emission. Our results make it possible to constrain the pair distribution function (as a consequence, e.g., the relativistic pair gas pressure), by observational results.

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