An enhanced approach to actuator fault estimation design for linear continuous-time systems

Dušan Krokavec, Anna Filasová, Vladimír Serbák, and Pavol Liščinský

Department of Cybernetics and Artificial Intelligence, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia

E-mail: dusan.krokavec@tuke.sk; anna.filasova@tuke.sk; vladimir.serbak@tuke.sk; pavol.liscinsky@tuke.sk

Abstract. An enhanced approach to fault estimation systems design, adjusted for linear continuous-time systems, is proposed in the paper. Based on LMI principle the method exploits state-space observer principle in an adaptive fault estimation scheme for single actuator faults. A simulation example, subject to different type of failures, demonstrates the effectiveness of the proposed form of the fault estimation technique.

1. Introduction

Operating conditions in modern engineering systems are still exposed to possibility of system failure. Any failure of the sensors, actuators or other system components can drastically change the system behavior. Fault tolerant control (FTC) allows a strategy to improve reliability of the whole system and so many techniques have been proposed especially for sensor and actuator failures with application to a wide range of engineering fields. In some cases, fault estimation strategies only are needed to carry on controlling the faulty system and, respecting this fact, many sophisticated modifications have been developed, e.g., sliding mode observers, neural network based approaches and adaptive observer technique.

To estimate actuator faults for the linear time invariant systems without external disturbance the principles based on adaptive observers are frequently used, which make estimation of actuator faults by integrating the system output errors. First introduced in [11], this principle was applied also for descriptor systems [10], linear systems with time delays [3], [13], system with nonlinear dynamics [1], a class of nonlinear systems described by Takagi-Sugeno fuzzy models [6], [7] and linear stochastic Markovian jumping systems [5]. Some generalizations can be found in [15].

Following the results presented in [12], two methods for actuator fault estimation based on the adaptive observer technique are proposed in the paper. Following examination of the model-based fault estimation schemes, an enhanced algorithm using H$_\infty$ approach is provided. Applied enhanced conditions in the scheme increase rapidity and develop a general framework for fast fault estimation in adaptive observer structures for linear deterministic systems. The approach utilizes the measurable input and output vector variables, design conditions are based on linear matrix inequality (LMI) technique giving an effective way to calculate the observer parameters.

The paper is organized as follows. Ensuing the introduction given in Sec. 1, Sec. 2 presents the problem formulation focusing on assumptions about the system and actuator fault properties.

In Sec 3 a short description of the main properties of methods exploiting the adaptive observer technique for actuator faults estimation in linear systems is presented and an enhanced condition of the H$_\infty$ observer existence are analyzed and proven in Sec. 4. In Sec. 5, simulation results are presented and, finally, some concluding remarks are reached in Sec. 6.

Throughout the paper, the following notation was used: $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and the matrix $\boldsymbol{X}$, respectively, diag$[\,\cdot\,]$ enters up a block diagonal matrix, rank$(\,\cdot\,)$ remits the rank of a matrix, for a square matrix $\boldsymbol{X} < 0$ means that $\boldsymbol{X}$ is a symmetric negative definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $I\!\!R$ notes the set of real numbers, and $I\!\!R^n$, $I\!\!R^{n\times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively.

2. Problem formulation

A linear dynamic multi-input, multi-output (MIMO) system in presence of an unknown fault can be described by the state-space equations in the following form

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{f}(t), \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t), \tag{2}$$

where $\boldsymbol{q}(t) \in I\!\!R^n$, $\boldsymbol{u}(t) \in I\!\!R^r$, and $\boldsymbol{y}(t) \in I\!\!R^p$ are vectors of the system, input and output variables, respectively, $\boldsymbol{f}(t) \in I\!\!R^s$ is the unknown fault vector, $\boldsymbol{A} \in I\!\!R^{n\times n}$ is the system dynamic matrix, $\boldsymbol{E} \in I\!\!R^{n\times s}$ is the fault input matrix and $\boldsymbol{B} \in I\!\!R^{n\times r}$ and $\boldsymbol{C} \in I\!\!R^{p\times n}$ are the system input and output matrices.

To estimate actuator faults and the system states simultaneously, the following adaptive state estimator is proposed [2], [8]

$$\dot{\boldsymbol{q}}_e(t) = \boldsymbol{A}\boldsymbol{q}_e(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{f}_e(t) + \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)), \tag{3}$$

$$\boldsymbol{y}_e(t) = \boldsymbol{C}\boldsymbol{q}_e(t), \tag{4}$$

where $\boldsymbol{q}_e(t) \in I\!\!R^n$ is the state observer vector, $\boldsymbol{f}_e(t)$ is an estimation of the fault $\boldsymbol{f}(t)$, $\boldsymbol{y}_e(t) \in I\!\!R^m$ is the vector of estimated output variables, $\boldsymbol{J} \in I\!\!R^{n\times p}$ is the estimator gain matrix while $n > p$.

The task is to design the matrix $\boldsymbol{J}$ in such a way that the observer dynamics matrix $\boldsymbol{A}_e = \boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}$ is stable and $\boldsymbol{f}_e(t)$ approximates a slowly varying actuator fault $\boldsymbol{f}(t)$. Actuator faults are represented as a exchange of the matrix $\boldsymbol{E}$ to $\boldsymbol{B}$ for $s = r$.

The state observer (3), (4) is combined with the law for the fault estimation updating of the form [12]

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{G}\boldsymbol{H}^T \boldsymbol{e}_y(t), \tag{5}$$

where

$$\boldsymbol{e}_y(t) = \boldsymbol{y}(t) - \boldsymbol{y}_e(t), \tag{6}$$

$\boldsymbol{H} \in I\!\!R^{p\times s}$ is the gain matrix and $\boldsymbol{G} = \boldsymbol{G}^T > 0$, $\boldsymbol{G} \in I\!\!R^{s\times s}$ is a learning weight matrix that has to be set interactively in the design step.

It has to be noted that a modifications of (5) were proposed in [14] for time varying $\boldsymbol{f}(t)$ in the form

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{G}\boldsymbol{H}^T(\sigma\boldsymbol{e}_y(t) + \dot{\boldsymbol{e}}_y(t)), \tag{7}$$

where $\sigma \in I\!\!R$ is a positive scalar, determining together with $\boldsymbol{G}$ the learning rate. A generalization of the adaptive state estimator is given in [10], where

$$\dot{\boldsymbol{q}}_e(t) = \boldsymbol{A}\boldsymbol{q}_e(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{f}_e(t) + \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)) + \boldsymbol{J}_d(\dot{\boldsymbol{y}}(t) - \dot{\boldsymbol{y}}_e(t)), \tag{8}$$

$$\boldsymbol{y}_e(t) = \boldsymbol{C}\boldsymbol{q}_e(t), \tag{9}$$

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{L}\boldsymbol{e}_y(t) + \boldsymbol{L}_d\dot{\boldsymbol{e}}_y(t) \tag{10}$$

It is obvious that if $\boldsymbol{J}_d = \boldsymbol{0}$ and if it can be set $\boldsymbol{L}$ as $\sigma\boldsymbol{L}_d$ with $\boldsymbol{L}_d = \boldsymbol{G}\boldsymbol{H}^T$ then this estimator reduces to the so called fast adaptive fault estimator (3), (4), (7). Analogously, if $\boldsymbol{J}_d = \boldsymbol{0}$, $\boldsymbol{L}_d = \boldsymbol{0}$ and $\boldsymbol{L} = \boldsymbol{G}\boldsymbol{H}^T$ the estimator reduces to the zero-integral estimator (3), (4), (5). In this work the zero-integral estimator is used for slowly-varying faults.

Assumption 1 *The couple $(\boldsymbol{A}, \boldsymbol{C})$ is observable and $rank(\boldsymbol{C}\boldsymbol{E}) = rank(\boldsymbol{E})$.*

Assumption 2 *The unknown fault vector, changing unexpectedly when a fault occurs, is piecewise constant, differentiable and bounded, i.e., $|\boldsymbol{f}(t)| \leq \boldsymbol{f}_{max} < \infty$, the upper bound $\boldsymbol{f}_{max}$ of the fault magnitude is known, and the value of $\boldsymbol{f}(t)$ is set to zero until a fault occurs.*

Under these assumptions design of the matrix parameters of the observers has to ensure asymptotic convergence of the estimation errors (6) and

$$\boldsymbol{e}_f(t) = \boldsymbol{f}(t) - \boldsymbol{f}_e(t) \tag{11}$$

to zero values. Note, Assumption 2 implies that the derivative $\boldsymbol{e}_f(t)$ with respect to time can be considered as

$$\dot{\boldsymbol{e}}_f(t) = -\dot{\boldsymbol{f}}_e(t). \tag{12}$$

3. The adaptive observer principle in actuator faults estimation

If single actuator faults influence the system through different input vectors (columns of the matrix $\boldsymbol{B}$), it is possible to avoid design of estimators with the tuning matrix parameter $\boldsymbol{G} > 0$ and formulate the design task through the set of LMIs and a linear matrix equality.

Proposition 1 *The actuator fault estimator is stable if there exist symmetric positive definite matrix $\boldsymbol{P} \in I\!\!R^{n \times n}$ and matrices $\boldsymbol{H} \in I\!\!R^{p \times s}$, $\boldsymbol{Y} \in I\!\!R^{n \times p}$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \tag{13}$$

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{Y}\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{Y}^T < 0, \tag{14}$$

$$\boldsymbol{P}\boldsymbol{E} = \boldsymbol{C}^T\boldsymbol{H}. \tag{15}$$

When the above conditions hold, the observer gain matrix is given by

$$\boldsymbol{J} = \boldsymbol{P}^{-1}\boldsymbol{Y} \tag{16}$$

and the adaptive fault estimation algorithm is

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{G}\boldsymbol{H}^T\boldsymbol{C}\boldsymbol{e}_q(t). \tag{17}$$

where

$$\boldsymbol{e}_q(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t). \tag{18}$$

matrix $\boldsymbol{H}$ is obtained as an LMI variable and $\boldsymbol{G} \in I\!\!R^{s \times s}$ is an interactive setting symmetric positive definite matrix.

Proof: (compare, e.g., [3], [14]) From the system model (1), (2) and the observer model (3), (4) it can be seen that

$$
\begin{aligned}
\dot{\boldsymbol{e}}_q(t) &= \dot{\boldsymbol{q}}(t) - \dot{\boldsymbol{q}}_e(t) \\
&= \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{f}(t) - \boldsymbol{A}\boldsymbol{q}_e(t) - \boldsymbol{B}\boldsymbol{u}(t) - \boldsymbol{E}\boldsymbol{f}_e(t) - \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)) \\
&= (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{E}\boldsymbol{e}_f(t),
\end{aligned} \tag{19}
$$

where the observer error is

$$\boldsymbol{e}_q(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t). \tag{20}$$

Since $\boldsymbol{e}_q(t)$ is linear with respect to system parameters, it is possible to consider the Lyapunov function candidate in the following form

$$v(\boldsymbol{e}_q(t)) = \boldsymbol{e}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) \tag{21}$$

where $\boldsymbol{P} > 0$, $\boldsymbol{G} > 0$ are symmetric positive definite matrices and an actuator fault satisfied the condition

$$\dot{\boldsymbol{f}}_e(t) = \boldsymbol{G}\boldsymbol{H}^T\boldsymbol{e}_y(t) = \boldsymbol{G}\boldsymbol{H}^T\boldsymbol{C}\boldsymbol{e}_q(t), \tag{22}$$

as well as

$$\boldsymbol{e}_f(t) = \boldsymbol{f}_0 - \boldsymbol{f}_e(t) \ \Rightarrow \ \dot{\boldsymbol{e}}_f(t) = -\dot{\boldsymbol{f}}_e(t). \tag{23}$$

Then, the derivative of $v(\boldsymbol{e}_q(t))$ with respect to t is

$$\dot{v}(\boldsymbol{e}_q(t)) = \dot{v}_0(\boldsymbol{e}_q(t)) + \dot{v}_1(\boldsymbol{e}_q(t)) < 0, \tag{24}$$

where

$$\begin{aligned}
\dot{v}_0(\boldsymbol{e}_q(t)) &= \dot{\boldsymbol{e}}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\dot{\boldsymbol{e}}_q(t) \\
&= ((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{E}\boldsymbol{e}_f(t))^T \boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\left((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{E}\boldsymbol{e}_f(t)\right) \\
&= \boldsymbol{e}_q^T(t)\left((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\right)\boldsymbol{e}_q(t) \\
&\quad + \boldsymbol{e}_q^T(t)\boldsymbol{P}\boldsymbol{E}\boldsymbol{e}_f(t) + \boldsymbol{e}_f^T(t)\boldsymbol{E}^T\boldsymbol{P}\boldsymbol{e}_q(t)
\end{aligned} \tag{25}$$

and

$$\dot{v}_1(\boldsymbol{e}_q(t)) = \dot{\boldsymbol{e}}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) + \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\dot{\boldsymbol{e}}_f(t) = -\dot{\boldsymbol{f}}_e^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) - \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\dot{\boldsymbol{f}}_e(t). \tag{26}$$

Substituting (22) into (26) leads to

$$\begin{aligned}
\dot{v}_1(\boldsymbol{e}_q(t)) &= -\boldsymbol{e}_q^T(t)\boldsymbol{C}^T\boldsymbol{H}\boldsymbol{G}\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) - \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{G}\boldsymbol{H}^T\boldsymbol{C}\boldsymbol{e}_q(t) \\
&= -\left(\boldsymbol{e}_q^T(t)\boldsymbol{C}^T\boldsymbol{H}\boldsymbol{e}_f(t) + \boldsymbol{e}_f^T(t)\boldsymbol{H}^T\boldsymbol{C}\boldsymbol{e}_q(t)\right).
\end{aligned} \tag{27}$$

Thus, substituting (25) and (27) into (19), the following inequality is obtained

$$\begin{aligned}
\dot{v}(\boldsymbol{e}_q(t)) &= \boldsymbol{e}_q^T(t)\left((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\right)\boldsymbol{e}_q(t) \\
&\quad + \boldsymbol{e}_q^T(t)\left(\boldsymbol{P}\boldsymbol{E} - \boldsymbol{C}^T\boldsymbol{H}\right)\boldsymbol{e}_f(t) + \boldsymbol{e}_f^T(t)\left(\boldsymbol{E}^T\boldsymbol{P} - \boldsymbol{H}^T\boldsymbol{C}\right)\boldsymbol{e}_q(t) < 0.
\end{aligned} \tag{28}$$

If there is set the condition

$$\boldsymbol{e}_q^T(t)\left(\boldsymbol{P}\boldsymbol{E} - \boldsymbol{C}^T\boldsymbol{H}\right)\boldsymbol{e}_f(t) + \boldsymbol{e}_f^T(t)\left(\boldsymbol{E}^T\boldsymbol{P} - \boldsymbol{H}^T\boldsymbol{C}\right)\boldsymbol{e}_q(t) = 0, \tag{29}$$

then the last equality gives

$$\boldsymbol{P}\boldsymbol{E} - \boldsymbol{C}^T\boldsymbol{H} = 0 \tag{30}$$

and, evidently, (30) implies (15).

Using the above given condition, the resulting formula for $\dot{v}(\boldsymbol{e}_q(t))$ takes the form

$$\dot{v}(\boldsymbol{e}_q(t)) = \boldsymbol{e}_q^T(t)\left((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\right)\boldsymbol{e}_q(t) < 0, \tag{31}$$

where

$$\boldsymbol{P}(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}) + (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})^T\boldsymbol{P} < 0. \tag{32}$$

Since the inequality (32) can be written as

$$\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{J}\boldsymbol{C} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{C}^T\boldsymbol{J}^T\boldsymbol{P} < 0, \tag{33}$$

by introducing the substitution

$$\boldsymbol{P}\boldsymbol{J} = \boldsymbol{Y}, \tag{34}$$

it is possible to express (33) as (14). This concludes the proof. ∎

4. Enhanced design conditions

Analyzing (19), i.e., the differential equation of the form

$$\dot{\boldsymbol{e}}_q(t) = (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{E}\boldsymbol{e}_f(t), \qquad (35)$$

it is evident that $\boldsymbol{e}_f(t)$ acts on the state error dynamics as an unknown disturbance and, evidently, this differential equation is so not autonomous after a fault occurrence. Reflecting this fact, the enhanced approach is proposed to decouple Lyapunov matrix $\boldsymbol{P}$ from the system matrices $\boldsymbol{A}$, $\boldsymbol{C}$ by introducing a slack matrix $\boldsymbol{Q}$ in the observer stability condition, as well as to decouple the tuning parameter δ from the matrix $\boldsymbol{G}$ in the learning rate setting and to use it to tune the observer dynamic properties. Since the design principle for unknown input observer can not be used in the case if a fault is to be estimated, the impact of faults on observer dynamics is moreover minimized with respect to the H$_\infty$ norm of the transfer functions matrix of $\boldsymbol{e}_y$ and $\boldsymbol{e}_f$, while a reduce in the fault amplitude estimate is easily countervailable using the matrix $\boldsymbol{G}$. In this sense it is proposed to formulate the enhanced design conditions as follows:

Theorem 1 *The actuator fault estimator is stable if for given positive $\delta \in \mathbb{R}$ there exist symmetric positive definite matrices $\boldsymbol{P} \in \mathbb{R}^{n \times n}$, $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, matrices $\boldsymbol{H} \in \mathbb{R}^{p \times s}$, $\boldsymbol{Y} \in \mathbb{R}^{n \times p}$ and a positive scalar $\gamma \in \mathbb{R}$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \quad \boldsymbol{Q} = \boldsymbol{Q}^T > 0, \quad \gamma > 0, \qquad (36)$$

$$\begin{bmatrix} \boldsymbol{QA} + \boldsymbol{A}^T\boldsymbol{Q} - \boldsymbol{YC} - \boldsymbol{C}^T\boldsymbol{Y}^T + \boldsymbol{C}^T\boldsymbol{C} & * & * \\ \boldsymbol{P} - \boldsymbol{Q} + \delta\boldsymbol{QA} - \delta\boldsymbol{YC} & -2\delta\boldsymbol{Q} & * \\ \boldsymbol{0} & \delta\boldsymbol{E}^T\boldsymbol{Q} & -\gamma\boldsymbol{I}_{rf} \end{bmatrix} < 0, \qquad (37)$$

$$\boldsymbol{C}^T\boldsymbol{H} = \boldsymbol{QE}. \qquad (38)$$

When the above conditions are affirmative the estimator gain matrix is given by the relation

$$\boldsymbol{J} = \boldsymbol{Q}^{-1}\boldsymbol{Y} \qquad (39)$$

and the adaptive fault estimation algorithm is given by (17).

Here and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof: Using Krasovskii theorem [4], the Lyapunov function candidate can be considered as follows

$$v(\boldsymbol{e}_q(t)) = \boldsymbol{e}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) + \int\limits_0^t \left(\boldsymbol{e}_y^T(r)\boldsymbol{e}_y(r) - \gamma\boldsymbol{e}_f^T(r)\boldsymbol{e}_f(r) \right) \mathrm{d}r, \qquad (40)$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{G} = \boldsymbol{G}^T > 0$, $\gamma > 0$ and γ is an upper bound of square of H$_\infty$ norm of the transfer function matrix $\boldsymbol{e}_f \asymp \boldsymbol{e}_y$. Then the derivative of $v(\boldsymbol{e}_q(t))$ with respect to t has to be negative, i.e.,

$$\begin{aligned} \dot{v}(\boldsymbol{e}_q(t)) \ &= \dot{\boldsymbol{e}}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\dot{\boldsymbol{e}}_q(t) + \dot{\boldsymbol{e}}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) \\ &+ \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\dot{\boldsymbol{e}}_f(t) + \boldsymbol{e}_y^T(t)\boldsymbol{e}_y(t) - \gamma\boldsymbol{e}_f^T(t)\boldsymbol{e}_f(t) < 0. \end{aligned} \qquad (41)$$

If it is assumed that the statements (22), (23) hold, then substitution of (22) into (41) leads to

$$\begin{aligned} \dot{v}(\boldsymbol{e}_q(t)) \ &= \dot{\boldsymbol{e}}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\dot{\boldsymbol{e}}_q(t) \\ &- \boldsymbol{e}_q^T(t)\boldsymbol{C}^T\boldsymbol{H}\boldsymbol{G}\boldsymbol{G}^{-1}\boldsymbol{e}_f(t) - \boldsymbol{e}_f^T(t)\boldsymbol{G}^{-1}\boldsymbol{G}\boldsymbol{H}^T\boldsymbol{C}\boldsymbol{e}_q(t) \\ &+ \boldsymbol{e}_y^T(t)\boldsymbol{e}_y(t) - \gamma\boldsymbol{e}_f^T(t)\boldsymbol{e}_f(t) < 0, \end{aligned} \qquad (42)$$

$$
\begin{aligned}
\dot{v}(\boldsymbol{e}_q(t)) \quad &= \dot{\boldsymbol{e}}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\dot{\boldsymbol{e}}_q(t) - \boldsymbol{e}_q^T(t)\boldsymbol{C}^T\boldsymbol{H}\boldsymbol{e}_f(t) - \boldsymbol{e}_f^T(t)\boldsymbol{H}^T\boldsymbol{C}\boldsymbol{e}_q(t) \\
&\quad + \boldsymbol{e}_y^T(t)\boldsymbol{e}_y(t) - \gamma\boldsymbol{e}_f^T(t)\boldsymbol{e}_f(t) < 0,
\end{aligned}
\tag{43}
$$

respectively.

Since (35) implies

$$
(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{E}\boldsymbol{e}_f(t) - \dot{\boldsymbol{e}}_q(t) = 0,
\tag{44}
$$

it is possible to define the following condition based on the equality (44)

$$
\left(\boldsymbol{e}_q^T(t)\boldsymbol{Q} + \dot{\boldsymbol{e}}_q^T(t)\delta\boldsymbol{Q}\right)\left((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) + \boldsymbol{E}\boldsymbol{e}_f(t) - \dot{\boldsymbol{e}}_q(t)\right) = 0,
\tag{45}
$$

where $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix and $\delta \in \mathbb{R}$ is a positive scalar. Then, inserting (45) and its transposition into (43), the following has to be satisfied

$$
\begin{aligned}
\dot{v}(\boldsymbol{e}_q(t)) \quad &= \dot{\boldsymbol{e}}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\dot{\boldsymbol{e}}_q(t) - \boldsymbol{e}_q^T(t)\boldsymbol{C}^T\boldsymbol{H}\boldsymbol{e}_f(t) - \boldsymbol{e}_f^T(t)\boldsymbol{H}^T\boldsymbol{C}\boldsymbol{e}_q(t) \\
&\quad + (\boldsymbol{e}_q^T(t)\boldsymbol{Q} + \dot{\boldsymbol{e}}_q^T(t)\delta\boldsymbol{Q})((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) - \dot{\boldsymbol{e}}_q(t)) \\
&\quad + ((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) - \dot{\boldsymbol{e}}_q(t))^T(\boldsymbol{Q}\boldsymbol{e}_q(t) + \delta\boldsymbol{Q}\dot{\boldsymbol{e}}_q(t)) \\
&\quad + (\boldsymbol{e}_q^T(t)\boldsymbol{Q} + \dot{\boldsymbol{e}}_q^T(t)\delta\boldsymbol{Q})\boldsymbol{E}\boldsymbol{e}_f(t) + \boldsymbol{e}_f^T(t)\boldsymbol{E}^T(\boldsymbol{Q}\boldsymbol{e}_q(t) + \delta\boldsymbol{Q}\dot{\boldsymbol{e}}_q(t)) \\
&\quad + \boldsymbol{e}_y^T(t)\boldsymbol{e}_y(t) - \gamma\boldsymbol{e}_f^T(t)\boldsymbol{e}_f(t) < 0.
\end{aligned}
\tag{46}
$$

If the following condition is introduced

$$
\boldsymbol{e}_f^T(t)\left(\boldsymbol{E}^T\boldsymbol{Q} - \boldsymbol{H}^T\boldsymbol{C}\right)\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\left(\boldsymbol{Q}\boldsymbol{E} - \boldsymbol{C}^T\boldsymbol{H}\right)\boldsymbol{e}_f(t) = 0,
\tag{47}
$$

this gives the equality

$$
\boldsymbol{Q}\boldsymbol{E} - \boldsymbol{C}^T\boldsymbol{H} = 0,
\tag{48}
$$

which implies (38).

The condition (47) allows to write (46) as

$$
\begin{aligned}
\dot{v}(\boldsymbol{e}_q(t)) \quad &= \dot{\boldsymbol{e}}_q^T(t)\boldsymbol{P}\boldsymbol{e}_q(t) + \boldsymbol{e}_q^T(t)\boldsymbol{P}\dot{\boldsymbol{e}}_q(t) \\
&\quad + (\boldsymbol{e}_q^T(t)\boldsymbol{Q} + \dot{\boldsymbol{e}}_q^T(t)\delta\boldsymbol{Q})((\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})\boldsymbol{e}_q(t) - \dot{\boldsymbol{e}}_q(t)) \\
&\quad + (\boldsymbol{e}_q^T(t)(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})^T - \dot{\boldsymbol{e}}_q^T(t))(\boldsymbol{Q}\boldsymbol{e}_q(t) + \delta\boldsymbol{Q}\dot{\boldsymbol{e}}_q(t)) \\
&\quad + \delta\dot{\boldsymbol{e}}_q^T(t)\boldsymbol{Q}\boldsymbol{E}\boldsymbol{e}_f(t) + \delta\boldsymbol{e}_f^T(t)\boldsymbol{E}^T\boldsymbol{Q}\dot{\boldsymbol{e}}_q(t) + \boldsymbol{e}_y^T(t)\boldsymbol{e}_y(t) - \gamma\boldsymbol{e}_f^T(t)\boldsymbol{e}_f(t)
\end{aligned}
\tag{49}
$$

and to prescribe the stability condition as

$$
\dot{v}(\boldsymbol{e}_q(t)) = \boldsymbol{e}_q^{\bullet T}(t)\boldsymbol{P}^{\bullet}\boldsymbol{e}_q^{\bullet}(t) < 0,
\tag{50}
$$

where

$$
\boldsymbol{P}^{\bullet} = \begin{bmatrix} \boldsymbol{Q}(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}) + \boldsymbol{C}\boldsymbol{C}^T + (\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})^T\boldsymbol{Q} & \boldsymbol{P} - \boldsymbol{Q} + \delta(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C})^T\boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{P} - \boldsymbol{Q} + \delta\boldsymbol{Q}(\boldsymbol{A} - \boldsymbol{J}\boldsymbol{C}) & -2\delta\boldsymbol{Q} & \delta\boldsymbol{Q}\boldsymbol{E} \\ \boldsymbol{0} & \delta\boldsymbol{E}^T\boldsymbol{Q} & -\gamma\boldsymbol{I}_{rf} \end{bmatrix} < 0. \tag{51}
$$

Introducing the notation

$$
\boldsymbol{Q}\boldsymbol{J} = \boldsymbol{Y}
\tag{52}
$$

then (51) implies (37). This concludes the proof. ∎

5. Illustrative example

In the example, there is considered the system (1), (2) in the state-space representation, where the system matrices are

$$
A = \begin{bmatrix} -3.2341 & -0.0356 & 0.0200 & 0.0267 \\ -0.0356 & -3.1883 & -0.0502 & 0.0189 \\ 0.0200 & -0.0502 & -3.2575 & 0.0100 \\ 0.0267 & 0.0189 & 0.0100 & -3.1537 \end{bmatrix}, \quad B = E = \begin{bmatrix} -0.3802 & 1.4370 \\ 0.6959 & -0.5393 \\ 2.3171 & -0.4301 \\ 0.0000 & 0.6507 \end{bmatrix},
$$

$$
C = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix}.
$$

Solving (13)-(15) with respect to the LMI matrix variables P, H, and Y using Self-Dual-Minimization (SeDuMi) package [9] for Matlab, the estimator parameter design problem was solved as feasible and

$$
P = \begin{bmatrix} 0.1770 & 0.2107 & 0.0321 & 0.0457 \\ 0.2107 & 0.6888 & 0.0678 & 0.2060 \\ 0.0321 & 0.0678 & 0.1586 & -0.0955 \\ 0.0457 & 0.2060 & -0.0955 & 0.6740 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.3691 & 0.3090 \\ -0.7451 & -0.1267 \\ 0.0251 & 0.1647 \\ 0.2357 & -0.8632 \end{bmatrix},
$$

$$
H = \begin{bmatrix} 0.4026 & -0.1207 \\ -0.2489 & 0.2774 \end{bmatrix}.
$$

The estimator gain matrix was computed using (16) as follows

$$
J = \begin{bmatrix} -1.2534 & 2.8937 \\ -1.1442 & -0.6872 \\ 1.5003 & -0.0182 \\ 0.9967 & -1.2693 \end{bmatrix},
$$

and guaranties the stable actuator fault observer, where the system matrix eigenvalues spectrum is

$$
\rho(A_e) = \rho(A - JC) = \{-2.1745 \ -2.6067 \ -3.3377 \pm 0.1566\,\mathrm{i}\}.
$$

Setting the tuning parameter G as follows

$$
G = \begin{bmatrix} 1.7 & 1.0 \\ 1.0 & 5.5 \end{bmatrix}
$$

the observer fault response is given in Fig. 1. This figure presents the fault signal, as well as its estimation, reflecting at first a single actuator fault in the the second actuator. This actuator fault starts at the time instant $t = 30s$ and is applied for $40s$. The learning parameter G has been set experimentally, considering the maximal value of the fault signal amplitude. Then, at the time instant $t = 100s$ a fault of the first actuator is applied for $40s$.

From the simulation results in Fig. 1 it can be observed that the differences between the signals reflecting single actuator faults and the observer approximated ones tends to zero.

Setting the tuning parameters $\delta = 5$ and solving (36)-(38) with respect the LMI matrix variables P, Q, H, and Y, γ, the estimator parameter design problem was solved as feasible and the design parameters were

$$
P = \begin{bmatrix} 25.7515 & 18.1103 & 3.3556 & -4.3147 \\ 18.1103 & 61.2780 & 4.9193 & 4.9418 \\ 3.3556 & 4.9193 & 20.7111 & -7.5051 \\ -4.3147 & 4.9418 & -7.5051 & 62.0896 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.6530 & -0.7548 \\ -1.6303 & -1.6135 \\ -0.2333 & 0.5191 \\ 0.4721 & -4.5613 \end{bmatrix},
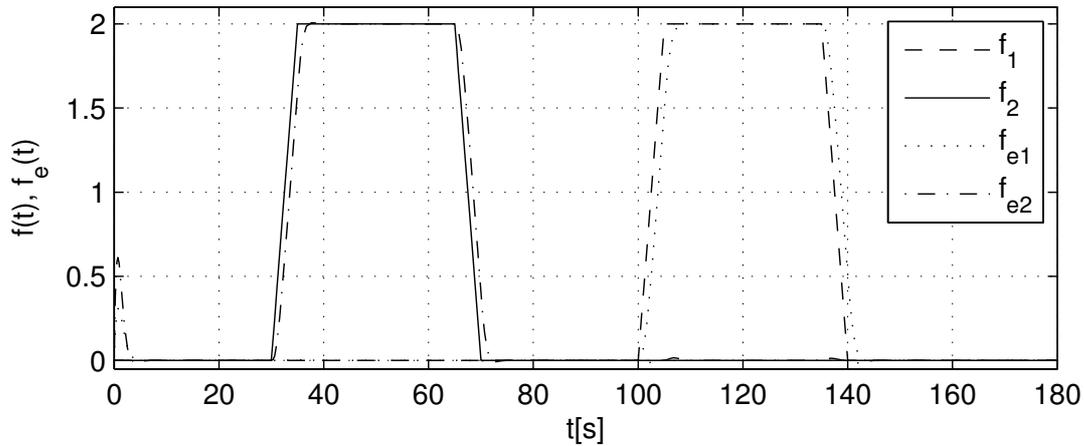$$

Figure 1. The actuator faults and its estimation using fault observer designed pursuant to Proposition 1

$$Q = \begin{bmatrix} 1.9347 & 2.1134 & 0.1955 & 0.5689 \\ 2.1134 & 5.9730 & 0.3846 & 1.9950 \\ 0.1955 & 0.3846 & 1.2356 & -0.7994 \\ 0.5689 & 1.9950 & -0.7994 & 7.2932 \end{bmatrix}, \quad H = \begin{bmatrix} 3.0563 & -0.9781 \\ -1.8683 & 2.9047 \end{bmatrix}.$$

The obtained estimator gain matrix J from (39) is

$$J = \begin{bmatrix} -0.0057 & -0.0353 \\ -0.0256 & -0.0104 \\ -0.0010 & 0.0063 \\ 0.0091 & -0.0743 \end{bmatrix}.$$

which gives that all eigenvalues of the estimator dynamic matrix spectrum $\rho(A_e)$ are stable, where

$$\rho(A_e) = \rho(A - JC) = \{ \ -2.9469 \ -3.1321 \ -3.2411 \ -3.2704 \ \}.$$

Setting the learning parameter G as follows

$$G = \begin{bmatrix} 1.3 & 0.5 \\ 0.5 & 2.1 \end{bmatrix}$$

the observed actuator faults are illustrated in Fig. 2. This figure presents the fault signal, as well as its estimation, reflecting at first a single actuator fault in the the second actuator, starting at the time instant $t = 30s$ and applied for $40s$, then the fault of the first actuator is introduced in the time instant $t = 100s$ and lasts for $40s$. The learning parameter G and the tuning parameter δ were set experimentally considering the maximal value of fault signal amplitude. Similarly to the previous case, the faults considered in simulation do not cause closed-loop instability. In the simulation results in Fig. 2 it can be seen that the exists very small differences between the signals reflecting single actuator faults and the observer approximate ones for piecewise constant actuator faults.

It has to be noted that now the matrix norms of G and J are substantially smaller then the matrix norms of these matrices computed using the standard design conditions (13)-(15). Moreover, since in this case matrix norm of H is significantly higher, LMI solution was carried out with higher precision.
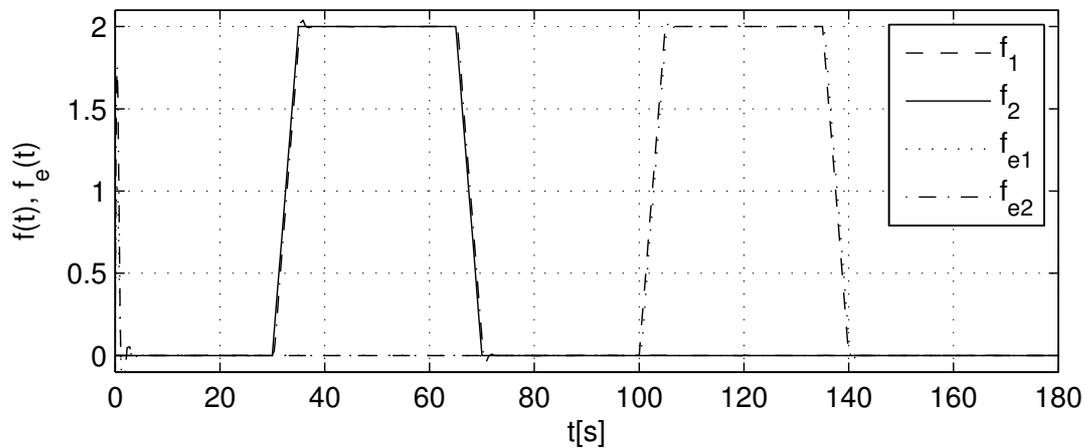
Figure 2. The actuator faults and its estimation with enhanced condition

6. Concluding remarks

Presented fault estimation method for linear continuous-time systems provides useful and easily implementable structure in process of fault detection, isolation and identification. Proposed approach to fault estimation design utilizing enhanced design conditions allows even better results, where the occurred actuator faults are estimated sooner and more precise as can be seen in simulation results. Tuning parameters $\boldsymbol{G}$ and δ were set interactively, incorrect values of these parameters would result in unstable or noisy response of the fault estimation signals. A simulation example, subject to given type of failures, demonstrates the effectiveness of the proposed form of the fault estimation design technique.

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