State control of discrete-time linear systems to be bound in state variables by equality constraints

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Abstract. The paper is concerned with the problem of designing the discrete-time equivalent PI controller to control the discrete-time linear systems in such a way that the closed-loop state variables satisfy the prescribed equality constraints. Since the problem is generally singular, using standard form of the Lyapunov function and a symmetric positive definite slack matrix, the design conditions are proposed in the form of the enhanced Lyapunov inequality. The results, offering the conditions of the control existence and the optimal performance with respect to the prescribed equality constraints for square discrete-time linear systems, are illustrated with the numerical example to note effectiveness and applicability of the considered approach.

1. Introduction

The control of the plant operates in such a way that some variables reflect certain physical limits in the prescribed demarcation occurs in practice very often. The underlying stimulus relies on the fact that the industrial systems may operate at acting conditions which are defined not only by given amplitude limitations but also by the ratios of certain process variables. With regard to this fact, the constraint control conceptions in such a way generalized have widely theoretical significance and practical sense.

Since it is possible to design the controllers that stabilize systems and concurrently force the closed-loops some kind of bounds [1], [13], to cope this problem efficiently exploiting the technique based on linear matrix inequalities (LMI), the control design task has to be formulated using the same approaches dealing with the state variable ratios as well as with the prescribed system constraints [15].

One way to implement above given task is to reformulate the control action manner as the stabilized control which also ensures that the closed-loop system state variables will satisfy the prescribed equality constraints in all time instants except the initial one. Because systems with the state equality constraints generally do not be deemed as descriptor systems [7], the design task has to be interpreted as a singular problem and the associated methods need to be developed to compute the controller parameters. Following the idea of the linear quadratic control, the technique which makes use of the equality constraints formulation for the discrete-time linear systems has been introduced in [8], and was extensively applied in the reconfigurable control design tasks [6], [9]. The reflection of this technique to the stochastic systems constrained in the state variables can be found in [10], and the design conditions based on the system H_∞ norm properties were presented in [5], [11].

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Because the above mentioned control algorithms as a rule use the static decoupling principle in setting the system working points, the prescribed values of equality constraints in the enforced mode are slithered in an additive offset [5]. To become values of the equality constraints linearly dependent on the desired system output values, the subject of the paper is to design the discrete version of PI controller in the structure which ensures the system stability and sets up the required linear dependency of the equality constraints.

Reflecting the fact that the design task is singular (i.e., the direct extension of the Lyapunov principle could give very conservative solutions since resulting LMIs are ill conditioned), the enhanced form of Lyapunov inequality [4] is adapted to obtain the regular set of LMIs. Using the parametrization approach which allows to convert the PI controller synthesis into the set of LMIs, the controller structure takes the optimal performance with respect to the prescribed equality constraints. It should be noted that the control law parameters could be altered while the desired control input values are changed.

The outline of this paper is as follows. Ensuing the introduction given in Sec. I, the system model and the control law structures are briefly described in Sec. II, while Sec. III places the suggested approaches within the context of the existing design requests and the basic preliminaries. In Sec. IV the proposed control design conditions are derived and, subsequently, these results are illustrated in Sec. V by a numerical example. Finally, Sec. VI presents some concluding remarks.

Throughout the paper the notations is narrowly standard in such way that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and the matrix $\boldsymbol{X}$, diag$[\,\cdot\,]$ enters up a block diagonal matrix, rank$(\,\cdot\,)$ remits the rank of a matrix, $\boldsymbol{X}^{\ominus 1}$ is Moore-Penrose inverse of the non-square matrix $\boldsymbol{X}$, for a square matrix $\boldsymbol{X} < 0$ means that $\boldsymbol{X}$ is a symmetric negative definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, Z_+ is the set of all positive integers, $I\!R$ notes the set of real numbers, and $I\!R^n$, $I\!R^{n\times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively.

2. Problem formulation

Through this paper the task is concerned with design of the state feedback constrained in the state variables and controlling the square discrete-time linear dynamic systems given by the set of state equations

$$\boldsymbol{q}(i+1) = \boldsymbol{F}\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i), \tag{1}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i), \tag{2}$$

where $\boldsymbol{q}(i) \in I\!R^n$, $\boldsymbol{u}(i) \in I\!R^r$, $\boldsymbol{y}(i) \in I\!R^m$ are vectors of the state, input and output variables, respectively, $\boldsymbol{F} \in I\!R^{n\times n}$, $\boldsymbol{G} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$ are real matrices and $i \in Z_+$. It is considered that the pair $(\boldsymbol{F}, \boldsymbol{G})$ is controllable.

In practice [2], [3], ratio control can be used to maintain the relationship between two closed-loop state variables, defined as

$$\frac{q_h(i+1)}{q_k(i+1)} = a_h \Rightarrow q_h(i+1) - a_h q_k(i+1) = 0 \tag{3}$$

for all $i \in Z$, or more compactly as

$$\boldsymbol{e}_{hk}^T \boldsymbol{q}(i+1) = 0, \tag{4}$$

$$\boldsymbol{e}_{hk}^T = \left[\begin{array}{cccccccccc} 0_1 & \cdots & 0_{h-1} & 1_h & 0_{h+1} & \cdots & 0_{k-1} & -a_k & 0_{k+1} & \cdots & 0_n \end{array}\right]. \tag{5}$$

Generalization of the above task formulation means the design of the stable closed-loop system with the linear state feedback control of the form

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i), \tag{6}$$

where $\boldsymbol{K} \in \mathbb{R}^{r \times n}$ is the controller feedback gain matrix and the design constraint is given in the matrix equality form

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) = \boldsymbol{E}\boldsymbol{F}_c\boldsymbol{q}(i) = 0 \tag{7}$$

with $\boldsymbol{E} \in \mathbb{R}^{k \times n}$, $\operatorname{rank}\boldsymbol{E} = k \le r$, and

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}. \tag{8}$$

In that sense, $\boldsymbol{E}$ reflects prescribed fixed ratios of two or more state variables.

Considering the square system, i.e., $r = m$, the control law can be formulated as [14]

$$\boldsymbol{u}(i) = -(\boldsymbol{K}_p\boldsymbol{q}(i) + \boldsymbol{K}_s\sum_{j=1}^{i-1}\boldsymbol{e}(j)), \tag{9}$$

$$\boldsymbol{e}(i) = \boldsymbol{w}(i) - \boldsymbol{C}\boldsymbol{q}(i), \tag{10}$$

where $\boldsymbol{K}_p \in \mathbb{R}^{r \times n}$, $\boldsymbol{K}_s \in \mathbb{R}^{m \times m}$, are the control gain matrices and $\boldsymbol{w}(i) \in \mathbb{R}^m$ is the reference output variable vector.

With the definition of the summation state variable

$$\boldsymbol{z}(i) = \sum_{j=0}^{i-1}\boldsymbol{e}(j) \tag{11}$$

it yields

$$\boldsymbol{z}(i+1) = \sum_{j=0}^{i}\boldsymbol{e}(j) = \boldsymbol{e}(i) + \sum_{j=0}^{i-1}\boldsymbol{e}(j), \tag{12}$$

$$\boldsymbol{z}(i+1) = \boldsymbol{w}(i) - \boldsymbol{C}\boldsymbol{q}(i) + \boldsymbol{z}(i), \tag{13}$$

respectively.

To describe the closed-loop system structure with such controller, the extended form of the system description can be constructed as follows

$$\begin{bmatrix} \boldsymbol{q}(i+1) \\ \boldsymbol{z}(i+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{0} \\ -\boldsymbol{C} & \boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{z}(i) \end{bmatrix} + \begin{bmatrix} \boldsymbol{G} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u}(i) + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{bmatrix} \boldsymbol{w}(i), \tag{14}$$

$$\boldsymbol{y}(i) = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(i) \\ \boldsymbol{z}(i) \end{bmatrix} \tag{15}$$

and using the notations

$$\boldsymbol{q}^{\circ T}(i) = \begin{bmatrix} \boldsymbol{q}^T(i) & \boldsymbol{z}^T(i) \end{bmatrix}, \tag{16}$$

$$\boldsymbol{F}^\circ = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{0} \\ -\boldsymbol{C} & \boldsymbol{I}_m \end{bmatrix}, \ \boldsymbol{G}^\circ = \begin{bmatrix} \boldsymbol{G} \\ \boldsymbol{0} \end{bmatrix}, \ \boldsymbol{W}^\circ = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{bmatrix}, \tag{17}$$

$$\boldsymbol{K}^\circ = \begin{bmatrix} \boldsymbol{K}_p & \boldsymbol{K}_s \end{bmatrix}, \quad \boldsymbol{C}^\circ = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{0} \end{bmatrix}, \tag{18}$$

$\boldsymbol{F}^\circ \in \mathbb{R}^{(n+m)\times(n+m)}$, $\boldsymbol{G}^\circ$, $\boldsymbol{W}^\circ \in \mathbb{R}^{(n+m)\times m}$, $\boldsymbol{K}^\circ$, $\boldsymbol{C}^\circ \in \mathbb{R}^{m\times(n+m)}$, then the closed-loop system state-space equations can be written as follows

$$\boldsymbol{q}^\circ(i+1) = \boldsymbol{F}_c^\circ\boldsymbol{q}^\circ(i) + \boldsymbol{W}^\circ\boldsymbol{w}(i), \tag{19}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}^\circ\boldsymbol{q}^\circ(i), \tag{20}$$

where

$$\boldsymbol{F}_c^{\circ} = \boldsymbol{F}^{\circ} - \boldsymbol{G}^{\circ}\boldsymbol{K}^{\circ}. \tag{21}$$

Thus, analogously, the design constraint can be defined as follows

$$\boldsymbol{E}^{\circ}\boldsymbol{q}^{\circ}(i+1) = \boldsymbol{E}^{\circ}\boldsymbol{F}_c^{\circ}\boldsymbol{q}^{\circ}(i) = 0, \tag{22}$$

where $\boldsymbol{E}^{\circ} \in I\!R^{k\times(n+m)}$, $\text{rank}\boldsymbol{E}^{\circ} = k \leq r$

The task is to design the feedback control gain matrices $\boldsymbol{K}$, $\boldsymbol{K}^{\circ}$ in such a way that the closed-loop system is stable and the constraints (7), (22), respectively, are satisfied. Note, such formulated design problem is singular [6].

3. Basic preliminaries

Proposition 1 *[12] If $\boldsymbol{\Pi}$ is a matrix variable and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{\Lambda}$ are known non-square matrices of the appropriate dimensions such that the equality*

$$\boldsymbol{A}\boldsymbol{\Pi}\boldsymbol{B} = \boldsymbol{\Lambda} \tag{23}$$

can be set, then all solution to $\boldsymbol{\Pi}$ means

$$\boldsymbol{\Pi} = \boldsymbol{A}^{\ominus 1}\boldsymbol{\Lambda}\boldsymbol{B}^{\ominus 1} + \boldsymbol{\Pi}^{\bullet} - \boldsymbol{A}^{\ominus 1}\boldsymbol{A}\boldsymbol{\Pi}^{\bullet}\boldsymbol{B}\boldsymbol{B}^{\ominus 1}, \tag{24}$$

where $\boldsymbol{A}^{\ominus 1}$ and $\boldsymbol{B}^{\ominus 1}$ is Moore-Penrose inverse of $\boldsymbol{A}$, $\boldsymbol{B}$, respectively, and $\boldsymbol{\Pi}^{\bullet}$ is an arbitrary matrix of appropriate dimension. □

Proposition 2 *[5] If $\boldsymbol{H} \in I\!R^{n\times n}$ is a real square matrix with non-repeated eigenvalues, satisfying the constraint*

$$\mathbf{e}^T\boldsymbol{H} = \mathbf{0}, \tag{25}$$

then one from its eigenvalues is zero, and (normalized) $\mathbf{e}^T$ is the left raw eigenvector of $\boldsymbol{H}$ associated with the zero eigenvalue. □

Since the formulated design task is singular, the design conditions can be stated using the following lemma which is a derivative to the theorem presented in [4].

Lemma 1 *The static feedback control (6) to the system (1), (2) exists if there exist symmetric positive definite matrices $\boldsymbol{R}$, $\boldsymbol{T} \in I\!R^{n\times n}$ and a matrix $\boldsymbol{Y} \in I\!R^{r\times m}$ such that*

$$\boldsymbol{R} = \boldsymbol{R}^T > 0, \quad \boldsymbol{T} = \boldsymbol{T}^T > 0, \tag{26}$$

$$\begin{bmatrix} -\boldsymbol{T} & \boldsymbol{R}\boldsymbol{F}^T - \boldsymbol{Y}^T\boldsymbol{G}^T \\ \boldsymbol{F}\boldsymbol{R} - \boldsymbol{G}\boldsymbol{Y} & \boldsymbol{T} - 2\boldsymbol{R} \end{bmatrix} < 0. \tag{27}$$

The control law gain matrix is given by the equation

$$\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{R}^{-1} \tag{28}$$

when the above conditions are affirmative. □

The inequality (27) is an enhanced representation of the Lyapunov matrix inequality for the linear discrete-time systems. It is linear with respect to the system variables but does not involve any product of the Lyapunov matrix $\boldsymbol{T}$ and the system matrices $\boldsymbol{F}$, $\boldsymbol{G}$. By introducing a new variable $\boldsymbol{R}$, the matrix products are relaxed to the discrepant products $\boldsymbol{F}\boldsymbol{R}$ and $\boldsymbol{G}\boldsymbol{K}\boldsymbol{R}$ which can be singular. This offers the possibility of this LMI structure to be applied in a singular task analysis. It is obvious that if $\boldsymbol{T} = \boldsymbol{R}$ the standard form of the Lyapunov inequality implies from (27).

4. Constrained control design

4.1. Design task formulation

Prescribed by a matrix $\boldsymbol{E}^\circ \in I\!R^{k\times 2n}$, $\text{rank}\boldsymbol{E}^\circ = k \leq r$, it is considered that the design constraint (22) will be satisfied for all nonzero natural numbers i. Moreover, it is supposed that $\boldsymbol{E}^\circ$ is specified in such a way that the equalities

$$\boldsymbol{E}^\circ(\boldsymbol{F}^\circ - \boldsymbol{G}^\circ\boldsymbol{K}^\circ) = \boldsymbol{0}, \tag{29}$$

$$\boldsymbol{E}^\circ\boldsymbol{F}^\circ = \boldsymbol{E}^\circ\boldsymbol{G}^\circ\boldsymbol{K}^\circ, \tag{30}$$

respectively, can be solved as well as that the closed-loop system matrix $\boldsymbol{F}_c^\circ$ for designed $\boldsymbol{K}^\circ$ is stable (all eigenvalues of the closed-loop system matrix $\boldsymbol{F}_c^\circ$ site in the unit circle in the complex plane $\mathcal{Z}$).

Solving (30) with respect to $\boldsymbol{K}^\circ$ then (24) implies all solutions of $\boldsymbol{K}^\circ$ as follows

$$\boldsymbol{K}^\circ = (\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^{\ominus 1}\boldsymbol{E}^\circ\boldsymbol{F}^\circ + (\boldsymbol{I}_m - (\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^{\ominus 1}\boldsymbol{E}^\circ\boldsymbol{G}^\circ)\boldsymbol{K}^\bullet, \tag{31}$$

where $\boldsymbol{K}^\bullet \in I\!R^{m\times(n+m)}$ is an arbitrary matrix and

$$(\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^{\ominus 1} = (\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^T(\boldsymbol{E}^\circ\boldsymbol{G}^\circ(\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^T)^{-1}, \tag{32}$$

where $(\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^{\ominus 1}$ is Moore-Penrose inverse of $\boldsymbol{E}^\circ\boldsymbol{G}^\circ$.

Thus, it is possible to express (31) as follows

$$\boldsymbol{K}^\circ = \boldsymbol{J}^\circ + \boldsymbol{L}^\circ\boldsymbol{K}^\bullet, \tag{33}$$

where

$$\boldsymbol{J}^\circ = (\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^{\ominus 1}\boldsymbol{E}^\circ\boldsymbol{F}^\circ \tag{34}$$

and

$$\boldsymbol{L}^\circ = \boldsymbol{I}_m - (\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^T(\boldsymbol{E}^\circ\boldsymbol{G}^\circ(\boldsymbol{E}^\circ\boldsymbol{G}^\circ)^T)^{-1}\boldsymbol{E}^\circ\boldsymbol{G}^\circ \tag{35}$$

is the projection matrix (the orthogonal projector onto the null space $\mathcal{N}_{\boldsymbol{E}^\circ\boldsymbol{G}^\circ}$ of $\boldsymbol{E}^\circ\boldsymbol{G}^\circ$ [12]).

4.2. Control Law Parameter Design

Theorem 1 *The unforced closed-loop system (19), (20) satisfying the constraint (30) is stable if there exist positive definite symmetric matrices $\boldsymbol{R}^\bullet$, $\boldsymbol{T}^\bullet \in I\!R^{(n+m)\times(n+m)}$ and a matrix $\boldsymbol{Y}^\bullet \in I\!R^{m\times(n+m)}$ such that*

$$\boldsymbol{R}^\bullet = \boldsymbol{R}^{\bullet T} > 0, \quad \boldsymbol{T}^\bullet = \boldsymbol{T}^{\bullet T} > 0, \tag{36}$$

$$\begin{bmatrix} -\boldsymbol{T}^\bullet & * \\ \boldsymbol{F}^\bullet\boldsymbol{R}^\bullet - \boldsymbol{G}^\bullet\boldsymbol{Y}^\bullet & \boldsymbol{T}^\bullet - 2\boldsymbol{R}^\bullet \end{bmatrix} < 0, \tag{37}$$

where

$$\boldsymbol{F}^\bullet = \boldsymbol{F}^\circ - \boldsymbol{G}^\circ\boldsymbol{J}^\circ, \quad \boldsymbol{G}^\bullet = \boldsymbol{G}^\circ\boldsymbol{L}^\circ. \tag{38}$$

When the above conditions hold, the control law gain matrices are given as follows

$$\boldsymbol{K}^\bullet = \boldsymbol{Y}^\bullet(\boldsymbol{R}^\bullet)^{-1}, \tag{39}$$

$$\boldsymbol{K}^\circ = \boldsymbol{J}^\circ + \boldsymbol{L}^\circ\boldsymbol{K}^\bullet, \tag{40}$$

respectively, and it yields

$$\boldsymbol{F}_c^\circ = \boldsymbol{F}^\circ - \boldsymbol{G}^\circ\boldsymbol{K}^\circ = \boldsymbol{F}^\bullet - \boldsymbol{G}^\bullet\boldsymbol{K}^\bullet. \tag{41}$$

Here and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof: Substituting (21) into (19) gives

$$\boldsymbol{q}^{\circ}(i+1) = (\boldsymbol{F}^{\circ} - \boldsymbol{G}^{\circ}\boldsymbol{K}^{\circ})\boldsymbol{q}^{\circ}(i) + \boldsymbol{W}^{\circ}\boldsymbol{w}(i) \tag{42}$$

and using (33) then (42) can be rearranged as

$$\boldsymbol{q}^{\circ}(i+1) = (\boldsymbol{F}^{\circ} - \boldsymbol{G}^{\circ}\boldsymbol{J}^{\circ} - \boldsymbol{G}^{\circ}\boldsymbol{L}^{\circ}\boldsymbol{K}^{\bullet})\boldsymbol{q}^{\circ}(i) + \boldsymbol{W}^{\circ}\boldsymbol{w}(i), \tag{43}$$

$$\boldsymbol{q}^{\circ}(i+1) = (\boldsymbol{F}^{\bullet} - \boldsymbol{G}^{\bullet}\boldsymbol{K}^{\bullet})\boldsymbol{q}^{\circ}(i) + \boldsymbol{W}^{\circ}\boldsymbol{w}(i), \tag{44}$$

respectively, where the notations (38) were used. It is evident that (42) and (44) implies (41).

Considering the unforced closed-loop system, (44) can be rewritten as

$$\boldsymbol{F}_c^{\circ}\boldsymbol{q}^{\circ}(t) - \boldsymbol{q}^{\circ}(i+1) = \boldsymbol{0} \tag{45}$$

and for an arbitrary symmetric positive definite matrix $\boldsymbol{Q}^{\circ} \in I\!R^{(n+m)\times(n+m)}$ it yields

$$\boldsymbol{q}^{\circ T}(i+1)\boldsymbol{Q}^{\circ}(\boldsymbol{F}_c^{\circ}\boldsymbol{q}^{\circ}(t) - \boldsymbol{q}^{\circ}(i+1)) = \boldsymbol{0}. \tag{46}$$

Defining the Lyapunov function candidate of the form

$$v(\boldsymbol{q}^{\circ}(i)) = \boldsymbol{q}^{T}(i)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(i) > 0, \tag{47}$$

where $\boldsymbol{P}^{\circ} \in I\!R^{(n+m)\times(n+m)}$ is a symmetric positive definite matrix, then the difference of the Lyapunov function along a trajectory of the unforced closed-loop system is

$$\Delta v(\boldsymbol{q}^{\circ}(i)) = \boldsymbol{q}^{\circ T}(i+1)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(i+1) - \boldsymbol{q}^{\circ T}(i)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(i) < 0 \tag{48}$$

and adding (46) as well as its transposition to (48) it yields

$$\begin{aligned}\Delta v(\boldsymbol{q}^{\circ}(i)) &= \boldsymbol{q}^{\circ T}(i+1)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(i+1) - \boldsymbol{q}^{\circ T}(i)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(i)\\ &+ \boldsymbol{q}^{\circ T}(i+1)\boldsymbol{Q}^{\circ}(\boldsymbol{F}_c^{\circ}\boldsymbol{q}^{\circ}(t) - \boldsymbol{q}^{\circ}(i+1))\\ &+ (\boldsymbol{F}_c^{\circ}\boldsymbol{q}^{\circ}(t) - \boldsymbol{q}^{\circ}(i+1))^T\boldsymbol{Q}^{\circ}\boldsymbol{q}^{\circ}(i+1) < 0.\end{aligned} \tag{49}$$

Thus, introducing the composed vector

$$\boldsymbol{q}_c^{\circ T}(i) = \begin{bmatrix} \boldsymbol{q}^{\circ T}(i) & \boldsymbol{q}^{\circ T}(i+1) \end{bmatrix} \tag{50}$$

and using the notation (41), then (49) can be rewritten as follows

$$\Delta v(\boldsymbol{q}^{\circ}(i)) = \boldsymbol{q}_c^{\circ T}(i)\boldsymbol{P}_c^{\circ}\boldsymbol{q}_c^{\circ}(i) < 0, \tag{51}$$

where

$$\boldsymbol{P}_c^{\circ} = \begin{bmatrix} -\boldsymbol{P}^{\circ} & (\boldsymbol{F}^{\bullet} - \boldsymbol{G}^{\bullet}\boldsymbol{K}^{\bullet})^T\boldsymbol{Q}^{\circ} \\ \boldsymbol{Q}^{\circ}(\boldsymbol{F}^{\bullet} - \boldsymbol{G}^{\bullet}\boldsymbol{K}^{\bullet}) & \boldsymbol{P}^{\circ} - 2\boldsymbol{Q}^{\circ} \end{bmatrix} < 0. \tag{52}$$

Since it is supposed that $\boldsymbol{Q}^{\circ}$ is positive definite, defining the transform matrix $\boldsymbol{W}^{\circ} \in I\!R^{2(n+m)\times 2(n+m)}$ as follows

$$\boldsymbol{W}^{\circ} = \operatorname{diag}\begin{bmatrix} \boldsymbol{R}^{\bullet} & \boldsymbol{R}^{\bullet} \end{bmatrix}, \quad \boldsymbol{R}^{\bullet} = (\boldsymbol{Q}^{\circ})^{-1}, \tag{53}$$

pre-multiplying the both side of (49) by $\boldsymbol{W}^{\circ}$ leads to the inequality

$$\begin{bmatrix} -\boldsymbol{R}^{\bullet}\boldsymbol{P}^{\circ}\boldsymbol{R}^{\bullet} & \boldsymbol{R}^{\bullet}\boldsymbol{F}^{\bullet T} - \boldsymbol{R}^{\bullet}\boldsymbol{K}^{\bullet T}\boldsymbol{G}^{\bullet T} \\ \boldsymbol{F}^{\bullet}\boldsymbol{R}^{\bullet} - \boldsymbol{G}^{\bullet}\boldsymbol{K}^{\bullet}\boldsymbol{R}^{\bullet} & \boldsymbol{R}^{\bullet}\boldsymbol{P}^{\circ}\boldsymbol{R}^{\bullet} - 2\boldsymbol{R}^{\bullet} \end{bmatrix} < 0. \tag{54}$$

Finally, using the notations

$$\boldsymbol{T}^{\bullet} = \boldsymbol{R}^{\bullet}\boldsymbol{P}^{\circ}\boldsymbol{R}^{\bullet}, \quad \boldsymbol{Y}^{\bullet} = \boldsymbol{K}^{\bullet}\boldsymbol{R}^{\bullet}, \tag{55}$$

then (52) implies (37). This concludes the proof. ■

4.3. Static decoupling principle

Because the integral feedback control increases the order of the closed-loop system dynamics, in such cases where this control structure is not appropriate, the principle of static decoupling can be used in the forced mode to set-point the desired values of the closed-loop output variables.

The state control in a forced mode for a square linear system is defined by the control policy

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) + \boldsymbol{W}_w\boldsymbol{w}(i), \tag{56}$$

where $\boldsymbol{w}(i) \in \mathbb{R}^m$ is a desired output vector signal, and $\boldsymbol{W}_w \in \mathbb{R}^{m\times m}$ is the signal gain matrix. Subsequently, the closed-loop system is described as

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{W}_w\boldsymbol{w}(i). \tag{57}$$

If the system (1), (2) satisfy the following condition [17]

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m, \tag{58}$$

$\boldsymbol{W}_w$ can be designed as

$$\boldsymbol{W}_w = (\boldsymbol{C}(\boldsymbol{I}_n - (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}))^{-1}\boldsymbol{G})^{-1}. \tag{59}$$

The constraint (7) is now defined by a matrix $\boldsymbol{E} \in \mathbb{R}^{k\times n}$, rank$\boldsymbol{E} = k \le r$, specified in such a way that the equality

$$\boldsymbol{E}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \boldsymbol{0} \tag{60}$$

can be solved to obtain the stable matrix $\boldsymbol{F}_c$ of the form (8).

Resolving (60) then (24) implies all solutions of $\boldsymbol{K}$ as follows

$$\boldsymbol{K} = (\boldsymbol{E}\boldsymbol{G})^{\ominus 1}\boldsymbol{E}\boldsymbol{F} + (\boldsymbol{I}_m - (\boldsymbol{E}\boldsymbol{G})^{\ominus 1}\boldsymbol{E}\boldsymbol{G})\boldsymbol{K}^{\diamond}, \tag{61}$$

where $\boldsymbol{K}^{\diamond} \in \mathbb{R}^{m\times n}$ is an arbitrary matrix and

$$(\boldsymbol{E}\boldsymbol{G})^{\ominus 1} = (\boldsymbol{E}\boldsymbol{G})^T(\boldsymbol{E}\boldsymbol{G}(\boldsymbol{E}\boldsymbol{G})^T)^{-1}, \tag{62}$$

where $(\boldsymbol{E}\boldsymbol{G})^{\ominus 1}$ is Moore-Penrose inverse of $\boldsymbol{E}\boldsymbol{G}$.

Expressing (61) as follows

$$\boldsymbol{K} = \boldsymbol{J} + \boldsymbol{L}\boldsymbol{K}^{\diamond}, \tag{63}$$

where

$$\boldsymbol{J} = (\boldsymbol{E}\boldsymbol{G})^{\ominus 1}\boldsymbol{E}\boldsymbol{F}, \tag{64}$$

$$\boldsymbol{L} = \boldsymbol{I}_m - (\boldsymbol{E}\boldsymbol{G})^T(\boldsymbol{E}\boldsymbol{G}(\boldsymbol{E}\boldsymbol{G})^T)^{-1}\boldsymbol{E}\boldsymbol{G}, \tag{65}$$

then it yields

$$\boldsymbol{F}_c = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{J} - \boldsymbol{G}\boldsymbol{L}\boldsymbol{K}^{\diamond} = \boldsymbol{F}^{\diamond} - \boldsymbol{G}^{\diamond}\boldsymbol{K}^{\diamond}, \tag{66}$$

where

$$\boldsymbol{F}^{\diamond} = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{J}, \quad \boldsymbol{G}^{\diamond} = \boldsymbol{G}\boldsymbol{L}. \tag{67}$$

Supposing that the realization of $(\boldsymbol{F}^{\diamond}, \boldsymbol{G}^{\diamond})$ is controllable, analogously as above this implies the following theorem which links the static decoupling principle to the solutions of feasible LMIs for the discrete equivalent of PI control, both with respect to the in state constrained closed-loop system.

Theorem 2 *The unforced closed-loop system (1), (2) satisfying the constraint (60) is stable if there exist positive definite symmetric matrices* $\boldsymbol{R}^{\diamond}$, $\boldsymbol{T}^{\diamond} \in I\!R^{n\times n}$ *and a matrix* $\boldsymbol{Y}^{\diamond} \in I\!R^{m\times n}$ *such that*

$$\boldsymbol{R}^{\diamond} = \boldsymbol{R}^{\diamond T} > 0, \quad \boldsymbol{T}^{\diamond} = \boldsymbol{T}^{\diamond T} > 0, \tag{68}$$

$$\begin{bmatrix} -\boldsymbol{T}^{\diamond} & * \\ \boldsymbol{F}^{\diamond}\boldsymbol{R}^{\diamond} - \boldsymbol{G}^{\diamond}\boldsymbol{Y}^{\diamond} & \boldsymbol{T}^{\diamond} - 2\boldsymbol{R}^{\diamond} \end{bmatrix} < 0, \tag{69}$$

where $\boldsymbol{F}^{\diamond}$, $\boldsymbol{G}^{\diamond}$ *are defined in (67).*

When the above conditions hold, the control law gain matrices are given as follows

$$\boldsymbol{K}^{\diamond} = \boldsymbol{Y}^{\diamond}(\boldsymbol{R}^{\diamond})^{-1}, \tag{70}$$

$$\boldsymbol{K} = \boldsymbol{J} + \boldsymbol{L}\boldsymbol{K}^{\circ}, \tag{71}$$

where $\boldsymbol{J}$, $\boldsymbol{L}$ *are defined in (64), (65), respectively.*

The proof follows analogously the same steps as the proof of Theorem 1 and therefore is omitted.

4.4. Constrain projection in forced modes

Theorem 3 *If the closed-loop system state variables satisfy the state constraint (29) where* $\boldsymbol{E}^{\circ}$ *takes the form*

$$\boldsymbol{E}^{\circ} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{X} \end{bmatrix}, \tag{72}$$

$\boldsymbol{w}(i) = \boldsymbol{w}_s$ *for all* i *and* $m = r \geq 2$, *then the common state variable vector* $\boldsymbol{q}_d(i) = \boldsymbol{E}^{\circ}\boldsymbol{q}^{\circ}(i)$, $\boldsymbol{q}_d(i) \in I\!R^k$, *of the system (16) in the forced mode attains the zero value vector at all time instants* $i \in Z_+$ *if there exist a nonzero matrix* $\boldsymbol{X} \in I\!R^{k\times m}$ *such that*

$$\boldsymbol{X}\boldsymbol{w}_s = \boldsymbol{0}. \tag{73}$$

Proof: Applying the constraint matrix $\boldsymbol{E}^{\circ}$ on the closed-loop system in the forced mode then (19) implies

$$\boldsymbol{E}^{\circ}\boldsymbol{q}^{\circ}(i+1) = \boldsymbol{E}^{\circ}(\boldsymbol{F}^{\circ} - \boldsymbol{G}^{\circ}\boldsymbol{K}^{\circ})\boldsymbol{q}^{\circ}(i) + \boldsymbol{E}^{\circ}\boldsymbol{W}^{\circ}\boldsymbol{w}(i). \tag{74}$$

When $\boldsymbol{E}^{\circ}$ takes the form (72) then the equation (74) conditioned by (29) gives

$$\boldsymbol{E}^{\circ}\boldsymbol{q}^{\circ}(i+1) = \boldsymbol{E}^{\circ}\boldsymbol{W}^{\circ}\boldsymbol{w}(i) = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{X} \end{bmatrix}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_m \end{bmatrix}\boldsymbol{w}(i) = \boldsymbol{X}\boldsymbol{w}(i). \tag{75}$$

It is evident that the common state variable vector $q_d(i)$ of the closed-loop system in the forced mode with $\boldsymbol{w}(i) = \boldsymbol{w}_s$ is zero vector if $\boldsymbol{X}\boldsymbol{w}_s = \boldsymbol{0}$ for $\boldsymbol{X} \neq \boldsymbol{0}$. This concludes the proof. ■

Note, considering $m = k = 1$, a solution of this problem exists with the prescribed precision $\varepsilon > 0$ if $\boldsymbol{X} = \varepsilon$, $\varepsilon \in I\!R$.

Theorem 4 *If the closed-loop system state variables satisfy the state constraint (60) and* $\boldsymbol{w}(i) = \boldsymbol{w}_s$ *for all* $i \in Z_+$, *then the common state variable vector* $\boldsymbol{q}_d(i) = \boldsymbol{E}\boldsymbol{q}(i)$, $\boldsymbol{q}_d(i) \in I\!R^k$, *of the system (57) in the forced mode attains the value*

$$\boldsymbol{q}_d = \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}_w\boldsymbol{w}_s. \tag{76}$$

Proof: Using the control policy (56) then

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}_w\boldsymbol{w}(i). \tag{77}$$

Since (77) conditioned by (60) implies

$$\boldsymbol{E}\boldsymbol{q}(i+1) = \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}_w\boldsymbol{w}(i) = \boldsymbol{q}_d(i), \tag{78}$$

it is evident that the common state variable $q_d(i)$ of the closed-loop system in a steady state is linear dependent on the steady value $\boldsymbol{w}_s$. This concludes the proof. ■

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5. Illustrative example

To demonstrate the properties of the proposed approach, the system with two-inputs and two-outputs is used in the example. The matrix parameters of the system are

$$\boldsymbol{F} = \begin{bmatrix} 0.9993 & 0.0987 & 0.0042 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}, \ \boldsymbol{G} = \begin{bmatrix} 0.0010 & 0.0010 \\ 0.0206 & 0.0197 \\ 0.0077 & -0.0078 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 0 \end{bmatrix},$$

respectively, for sampling period $\Delta t = 0.1$ s. The state constraint was specified as

$$\frac{q_1(t) - 0.4\, q_3(t)}{q_2(t)} = 0.1,$$

which implies

$$\boldsymbol{E} = \begin{bmatrix} 1 & -0.1 & -0.4 \end{bmatrix}.$$

Prescribing the desired value of the closed-loop system in the forced mode as follows

$$\boldsymbol{w}_s = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^T,$$

then it can be chosen

$$\boldsymbol{X} = \begin{bmatrix} 5 & 10 \end{bmatrix},$$

$$\boldsymbol{E}^\circ = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 1 & -0.1 & -0.4 & 5 & 10 \end{bmatrix},$$

Solving (36), (37) with respect to the LMI matrix variables $\boldsymbol{R}^\circ$, $\boldsymbol{T}^\circ$, $\boldsymbol{Y}^\circ$ using Self-Dual-Minimization (SeDuMi) package [16] for Matlab, the design conditions imply the control gain matrix parameters

$$\boldsymbol{K}_p = 10^3 \begin{bmatrix} 5.4867 & -0.0335 & -3.8042 \\ 4.1263 & 0.0704 & -2.7825 \end{bmatrix},$$

$$\boldsymbol{K}_s = 10^3 \begin{bmatrix} -1.9209 & -3.8417 \\ -1.3733 & -2.7464 \end{bmatrix},$$

respectively, and the eigenvalues spectrum of the extended closed-loop loop system is

$$\rho(\boldsymbol{A}_c^\circ) = \{\ 0.0000 \quad 0.1389 \quad 0.2104 \quad 0.5153 \quad 0.9483\ \}.$$

Solving (68), (69) the solution of the control design conditions gives the following control gain matrix

$$\boldsymbol{K} = \begin{bmatrix} -181.4457 & -39.5609 & 40.2261 \\ 188.4813 & 58.7340 & -30.9274 \end{bmatrix}$$

and the stable closed-loop eigenvalues spectrum is

$$\rho(\boldsymbol{A}_c) = \{\ 0.0000 \quad 0.8170 \pm 0.0445\,\mathrm{i}\ \}.$$

The previously described control design methods were applied to the simulation benchmark. Since the same input variables in the forced modes were utilized to assess the each controller ability response and to demonstrated performance with respect to asymptotic properties, the results of the proposed design methods can be compared. Moreover, the initial conditions for the state vector $\boldsymbol{q}(i)$ in both cases were simply chosen to be $\boldsymbol{q}(0) = \boldsymbol{0}$ and, moreover, in the system with the summation part was set $\boldsymbol{z}(0) = \boldsymbol{0}$. To complete the forced mode parameters, the signal gain matrix $\boldsymbol{W}_w$ was computed using (59) as follows

$$\boldsymbol{W}_w = \begin{bmatrix} -32.3526 & -124.5375 \\ 30.0938 & 134.0236 \end{bmatrix}.$$

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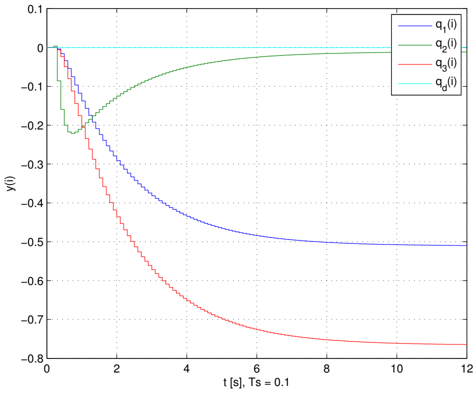

Figure 1. State responses for the control designed pursuant to Theorem 1.

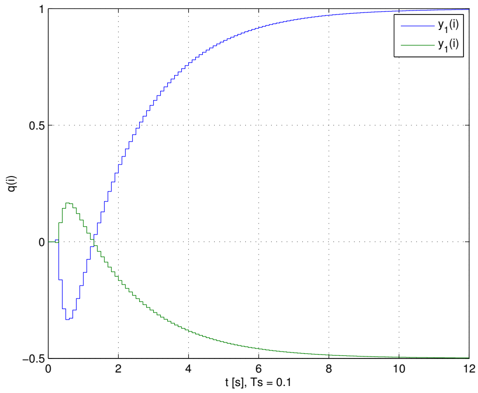

Figure 2. Output responses for the control designed pursuant to Theorem 1.

It was verified, that the condition (62) is satisfied at all time instant except initial time in such a way that

$$\boldsymbol{q}_d = \boldsymbol{E}\boldsymbol{W}^{\circ}\boldsymbol{w}_s = 0, \quad \boldsymbol{q}_d = \boldsymbol{E}\boldsymbol{G}\boldsymbol{W}_w\boldsymbol{w}_s = -0.2032.$$

(see $q_d(i)$ in Fig. 1, Fig. 3, respectively).

The closed-loop responses for the equally prescribed desired system outputs are shown in Fig. 1 to Fig. 4. From these figures it is obvious that the both control laws, which parameters were obtained by the solutions of the LMI problems specified by Theorem 1 and Theorem 2, respectively, can successfully provide for the closed-loop system steady-state properties and asymptotic dynamics. Due to the relatively large eigenvalues that causes the summation part of the control law (9), the response of the system under control designed using Theorem 1 is considerably slower then is the response of the system with the control law computed pursuant to Theorem 2. On the other hand, this disadvantage is compensated by zero value of the variable $\boldsymbol{q}_d(i)$. It should be noted that the $\boldsymbol{K}^{\circ} = [\boldsymbol{K}_p,\ \boldsymbol{K}_s]$ can be altered while the desired control input $\boldsymbol{w}_s$ is changed.

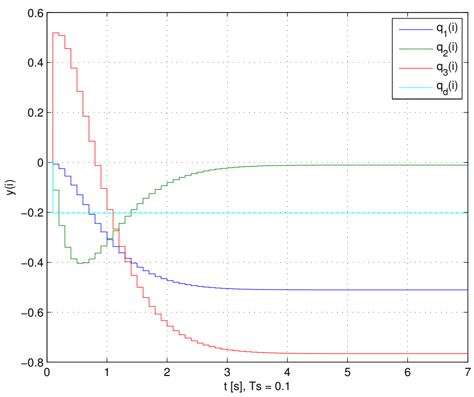

Figure 3. State responses for the control designed pursuant to Theorem 2.

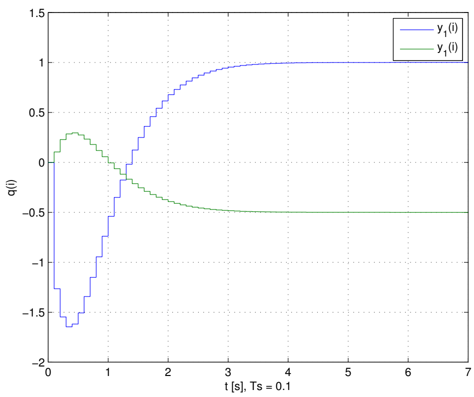

Figure 4. Output responses for the control designed pursuant to Theorem 2.

6. Concluding remarks

The paper presents the novel approach for synthesis the discrete-time equivalent PI controller which is destined to control the discrete-time linear MIMO systems in such a way that the closed-loop state variables satisfy the prescribed equality constraints. To circumvent the singular design task, the resulting LMIs do not involve any product of the Lyapunov matrix and the system matrices, which provides a suitable way for the control law determination, conditioned by the singular closed-loop system matrix. Design conditions are established in the terms of LMI as a feasible problem, accomplishing the manipulation in the manner giving guaranty the asymptotic stability of the closed-loop system and satisfying the prescribed equality constraints in the forced regime. Comparing the proposed PI state-space design with the principle based on the static decoupling, the presented illustrative example also confirms the effectiveness of the proposed design method.

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