Journal of Physics: Conference Series **570** (2014) 022004 doi:10.1088/1742-6596/570/2/022004

A conformal mapping approach to a root-clustering problem

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Abstract. This paper presents a new approach for matrix root-clustering in sophisticated and multiply-connected regions of the complex plane. The parametric sweeping method and a concept of the closed forbidden region covered by a set of modified three-parametrical Cassini regions are used. A conformal mapping approach was applied to formulate the main results of the paper. An application of the developed method to the problem of matrix root-clustering in a multiply connected region is shown for illustration.

1. Introduction

The properties of control systems depend on the location of roots of a system matrix on the complex plane. In this connection the problem of root-clustering is topical for control systems analysis. Some control techniques like state-space control [1–5] require the roots of a linear part of the system to be localised in a defined regions of the complex plane. The most-used approach this problem is based on the generalized Lyapunov matrix inequality. The necessary and sufficient conditions for matrix root-clustering in Γ-transformable and Ω- transformable algebraic regions obtained in the works of Gutman and colleagues [6] made the fundamental contributions in this direction. The problem of root clustering for robust systems using the generalized Lyapunov matrix inequality approach was investigated in many works, the recent of them are: [7–16].

This paper is devoted to a problem of matrix root-clustering in the sophisticated regions given as an intersection of several algebraic regions of the fourth and smaller order. Such regions can have one or several boundaries. The boundaries may be smooth or piecewise smooth,


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convex or nonconvex; the boundaries may be formed by segments of algebraic curves or by curves enveloping a continual set of curves. The region may be multiply- or simply- connected, nonconvex or convex. We consider more general statement of the problem than in the literature. The main attention is focused on a complement of the root-clustering region to $\mathbb{C}$; these regions are called the *closed forbidden region*. The forbidden region we approximate (sweep) by a set of algebraic subregions; we offer to use a set of simply-connected and multiply-connected three-parametric regions of the fourth order called the *modified Cassini oval regions* as these algebraic subregions.

In the introduced terms the root-clustering region is an intersection of the complements of a set of the forbidden subregions. Necessary and sufficient conditions of matrix spectrum localization in the specified region are obtained as a union of a set of conditions of spectrum location in outside of each forbidden subregion. In this paper we use the conformal mapping approach instead of generalized Lyapunov matrix inequality one to obtain the results.

The paper is laid out as follows. First, general information about Cassini ovals is given and some modifications of Cassini ovals are suggested. Thereafter, a sweeping technique in a root-clustering problem is presented and a conformal mapping of Cassini regions and matrix root-clustering approach are described. The paper is finished with some concluding remarks and acknowledgments.

2. Cassini ovals

In this section we recall some general information about Cassini ovals and introduce some modifications. We present the regions bounded by the modified Cassini ovals as the convenient subregions for covering (sweeping) of forbidden regions.

As it is known, the equation of Cassini ovals includes two parameters (a, c) determining the shape of the ovals.

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = a^4 - c^4 \tag{1}$$

For example, Fig. 1 illustrates the Cassini ovals at $a = [0.6c, 0.8c, c, 1.2c,\ 1.4c, 1.6c, ...]$. In [17] one can find the recent example of application of Cassini ovals in the control theory (for the multi-agent systems).

Let us introduce the modified Cassini ovals, tree-parametric ovals defined by the equation

$$\Gamma(a, c, \mu) = \{s = x + \mathrm{j}y : ((x + \mu)^2 + y^2)^2 + \ + 2c((x + \mu)^2 - y^2) + c^2 - a^2 = 0\},$$

where $a > 0, c \in [-a/2, 0] \cup [a, \infty)$. These ovals are oriented along an imaginary axis at $c > 0$ or along a real axis at $c < 0$. A third parameter μ provides the parallel shift of the ovals along the real axis. Note that we get the parallel shift to the left-hand side at $\mu > 0$ and the parallel shift to the right-hand side at $\mu < 0$.

For numerical calculations (2) can be rewritten as

$$\Gamma = \{x, y : y = (\pm)\sqrt{c - (x + \mu)^2 \pm \sqrt{a^2 - 4c(x + \mu)^2}}\}.$$

Note that it is necessary to consider four different combinations of values of the double marks.

The restriction $c \in [-a/2, 0] \cup [a, \infty)$ means that among different forms of Cassini lines we shall consider only ovals.

Equation (2) in terms of a complex variable s has the form

$$\Gamma = \{s \in \mathbb{C} : ((\bar{s} + \mu)^2 + c)((s + \mu)^2 + c) - a^2 = 0\}, \tag{2}$$

where $a > 0, c \in [-a/2, 0] \cup [a, \infty)$.

At $c = 0$ we have the unit circle. At $c < 0$ we obtain the oval with semi-axis $(\sqrt{a - c})$ and $(\sqrt{a} + c)$, extended along the real axis. At $c = -a/2$ the curvature of the oval in the top and

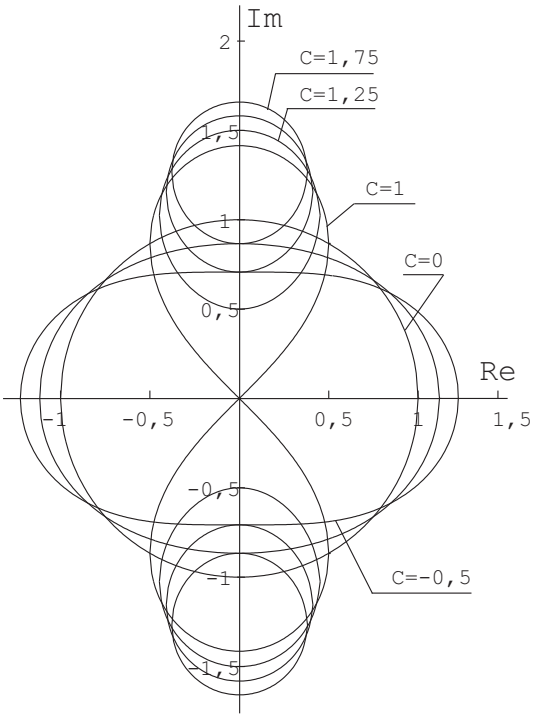


Figure 1. Cassini ovals at $a = [0.6c, 0.8c, c, 1.2c,\ 1.4c, 1.6c, ...]$

in the bottom points are equal to zero. At $c = a$ we have the lemniscate. Finally, at any values $c > a$ we have the pair of ovals symmetrically located at both sides of the real axis having the height of $(\sqrt{c+a} - \sqrt{c-a})$ and having the maximal width of $(a/\sqrt{c})$.

In this subsection we give some definitions.

Suppose the closed region H satisfies the inequality

$$\bar{H}(a, c, \mu) = \{s \in \mathbb{C} : \quad |(s+\mu)^2 + c|^2 - a^2 \leqslant 0\}, \tag{3}$$

where $a > 0, c \in [-a/2, 0] \cup [a, \infty)$. Then the region $\bar{H}(a, c, \mu)$ is called the *modified three-parametric Cassini oval region*. It is clear that curve (2) is the boundary of $\bar{H}$.

Further, the complement of H to $\mathbb{C}$ is called the *external oval region* and denoted by G. The region G satisfies the inequality

$$G(a, c, \mu) = \{s \in \mathbb{C} : f(s) = a^2 - |(s+\mu)^2 + c|^2 < 0\}, \tag{4}$$

where $a > 0, c \in [-a/2, 0] \cup [a, \infty)$.

3. A sweeping technique in a root-clustering problem

In the literature a problem of matrix root - clustering is usually considered as a problem of a membership of all roots to a specified open region D of the complex plane $\mathbb{C}$. Let us consider another treatment of this problem.

3.1. A Modified treatment of a matrix root - clustering problem

Let the closed region $\bar{S}$ be the complement to $\mathbb{C}$ of D, i.e., $\bar{S} = \mathbb{C} \backslash D$; then the region $\bar{S}$ is called *the forbidden region.*

Let us consider 2 cases.

Case 1 Suppose the region $\bar{S}$ is defined as below:

$$\bar{S} = \cup \bar{S}_\alpha \quad \forall \quad \alpha = \overline{1,\nu}, \tag{5}$$

where $\{\bar{S}_\alpha, \quad \forall \alpha = \overline{1,\nu}\}$ is a *final* set of ν intersecting or nonintersecting subregions; the envelope of this set forms the border of $\bar{S}$. Then by definition put

$$\{D_\alpha : \quad D_\alpha = \mathbb{C}\backslash \bar{S}_\alpha \supseteq D\} \quad \forall \quad \alpha = \overline{1,\nu}, \tag{6}$$

where the regions D_α are called *the expanded root-clustering regions*. Note that $D_\alpha \supseteq D$; this follows from $\bar{S}_\alpha \subseteq \bar{S} \Rightarrow D_\alpha = \mathbb{C}\backslash \bar{S}_\alpha \supseteq \mathbb{C}\backslash \bar{S} = D$.

Case 2 Let $\{\bar{\Omega}(\mu), \mu \in [0,\tau]\}$ be a *continual* set of the subregions covering the region $\bar{S}$

$$\bar{S} = \cup \bar{\Omega}(\mu) \quad \forall \quad \mu \in [0,\tau] \tag{7}$$

Then by definition, put

$$\{D(\mu) : \quad D(\mu) = \mathbb{C}\backslash \bar{\Omega}(\mu)\} \quad \forall \quad \mu \in [0,\tau], \tag{8}$$

where $D(\mu) \supseteq D \quad \forall \mu \in [0,\tau]$ is the continual parametric set of the expanded root-clustering regions.

Let the forbidden region $\bar{S}$ for the matrix spectrum $s(A)$ be a union of the regions (5); then the root-clustering region $D = \mathbb{C}\backslash \bar{S}$ satisfies the equality

$$D = D^*,\ D^* = \cap D_\alpha = \cap(\mathbb{C}\backslash \bar{S}_\alpha),\ \alpha = \overline{1,\nu} \tag{9}$$

i.e., D is the intersection of the set of the expanded root - clustering regions (6).

Let the forbidden region S be a union of the parametric continual set (7); then the root-clustering region D satisfies the equality

$$D = D^*,\ D^* = \cap D(\mu),\ D(\mu) = \mathbb{C}\backslash \bar{\Omega}(\mu). \tag{10}$$

3.2. Conformal mapping of Cassini regions

The important property of the modified Cassini ovals making them suitable for application in root-clustering problems is in the fact that the function of a complex variable

$$z = a[(s+\mu)^2 + c]^{-1}, \quad c > a > 0, \quad \mu \geqslant 0, \tag{11}$$

conformally maps the closed Cassini oval region $\bar{H}(a,c,\mu)$ onto the complement of the closed unit central disk $\{z : |z| \geqslant 1\}$. The reader will easily check that substituting (11) for (3), (4). Note also that function (11) maps the complementary region $G(a,c,\mu) = \mathbb{C}\backslash \bar{H}(a,c,\mu)$ into the open unit disk $\{z : |z| < 1\}$. The bilinear function

$$w = (z+1)(z-1)^{-1}, \tag{12}$$

conformally maps the unit central disk $\{z : |z| < 1\}$ and the exterior disk $\{z : |z| \geqslant 1\}$ respectively into the left half-plane and onto the right half-plane. Substituting (12) for (11), we get

$$w(a,c,\mu) = (a + c + (s+\mu)^2)(a - c - (s+\mu)^2)^{-1}. \tag{13}$$

This function conformally maps the open region $G(a,c,\mu)$ and the closed region $\bar{H}(a,c,\mu)$ respectively into the open left half-pane and onto the right half plane.

3.3. Matrix root-clustering

Suppose the forbidden region $\bar{S}$ is covered by the final set of regions (3), i.e.,

$$\bar{S} = \cup \bar{H}(a_\alpha, c_\alpha, \mu_\alpha), \quad \forall \alpha = \overline{1, \nu}, \tag{14}$$

where

$$\bar{H}(a_\alpha, c_\alpha, \mu_\alpha) = \{s \in \mathbb{C} : |(s + \mu_\alpha)^2 + c_\alpha|^2 - a_\alpha^2 \leqslant 0\} \tag{15}$$

The following functionally transformed matrices correspond to functions (11) and (13):

$$Z_\alpha = a_\alpha((A + \mu_\alpha E)^2 + c_\alpha E)^{-1} \, \forall \, \alpha \in [1, \nu] \tag{16}$$

$$W_\alpha = ((a_\alpha + c_\alpha)E + (A + \mu_\alpha E)^2)((a_\alpha - c_\alpha)E - (A + \mu_\alpha E)^2)^{-1} \tag{17}$$

Using the property of matrix eigenvalues we get the following conditions. All roots of a real matrix A are clustered in $D = \mathbb{C}\backslash\bar{S}$, where $\bar{S}$ is the forbidden region (14), (15), if and only if all eigenvalues of all functionally transformed matrices (16) are clustered in the unit central disk, or if all eigenvalues of all matrices (17) are clustered in the left half-plane.

This leads to the Hurwitz conditions for matrices (17) or Lyapunov inequality for them; or this leads to the conditions of Schur-Cohn for matrices (16).

To illustrate the method we consider an example adopted from a band-filtration problem. Let a band filter has two suppression frequency bands (low and high). In the presented above terms these bands form the multiply connected forbidden region. Now we will obtain the algebraic conditions for matrix root-clustering in outside of this forbidden region. First we approximate the forbidden bands by a set of modified three-parametric Cassini oval regions. We approximate the low frequency band by a set of the simply-connected oval regions and approximate the high band by another set of multiply-connected oval regions As a whole we approximate the forbidden region by two sets of simply-connected and conjugate oval regions with corresponding values of the parameters $\{a, c, \mu\}$. Secondly in the light of the obtained conditions all roots of matrix A are clustered in outside of this forbidden region, if and only if the functionally transformed matrix (17) is Hurwitz for all values of the parameters $a_\alpha, c_\alpha, \mu_\alpha$ corresponding to the defined sets.

4. Conclusion

This paper presents a new conditions for matrix root-clustering in sophisticated and multiply-connected regions of the complex plane using a conformal mapping approach and a parametric sweeping method by a set of modified three-parametrical Cassini regions. An application of the developed method to the problem of matrix root-clustering in a multiply connected region is shown for illustration.

4.1. Acknowledgments

The research was supported by the State Financial Support of the Leading National Universities, no. 074-U01, the Ministry of Education and Science of the Russian Federation, project no. 14.Z50.31.0031, and by Committee of Science and Higher School of St. Petersburg Government.

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