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Eigenvalue problem for permittivity operator of conductors with the spatial dispersion in a microwave field

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Abstract. Conducting media with spatial dispersion may be described formally by the single operator - operator of dielectric permittivity, which, as it is well known, completely defines the microwave response of conductors with spatial dispersion. So the eigenvalue problem for permittivity operator of conductors and superconductors possessing strong spatial dispersion at low temperatures is of a great importance since the corresponding solutions are the stable waves for constitutive equation in a self-consistent microwave field. Here a wave problem is formulated to search the solutions, which correspond to the eigenvalues of permittivity operator, similar to the problem of wave propagation in hollow waveguides and resonators, but non-self conjugated. Dispersion relations and general solutions are obtained. Significant role of the spatial-type force resonances is considered. Conditions for the spatial resonances are derived. The obtained resonances includes particular solutions corresponding to traditional surface impedance for anomalous skin effect, surface impedance of superconductor, as well as four novel solutions, obviously related to polarization, two of which correspond to the waves with amplitude increasing into the depth of conductor, and two else describe solutions with unusual properties.

1. Introduction

An increase of the mean free path length of the current carriers at sufficiently low temperatures up to values greater than the characteristic penetration depth of electromagnetic field leads to the dominating role of the effects caused by spatial dispersion in the electromagnetic response of cooled conductors [1–6]. The effects that depend on spatial parameters, besides of the obvious interest to such effects especially in structures with small (micro and nano) sizes, open the possibility of investigation of conductivity in novel materials. Microwave effects of the spatial dispersion allow to study the properties of carriers and pairing mechanisms in superconductors, including high$-T_c$ superconductors (HTSC). To deduce this information a description of the surface impedance of superconductor should be in satisfactory agreement with experimental data. The effects of spatial dispersion and anomalous skin effect, which is a characteristic manifestation of the spatial dispersion in the microwave response of cooled conductors, are well described by the textbook kinetic theory [2–6]. Nevertheless, a calculation of the surface resistance R of superconductor for instance, which denotes the real part of complex surface impedance $\widetilde{Z} = R + iX$, in frames of the microscopic model is always complicated since the exact analytical expressions can be found only in few limiting cases. Here and below sign tilde denotes a complex value. The values of the surface resistance in the intermediate regions can be found only by the sophisticated numerical calculations [4]. These complications to a greater extent concern the HTSC since the rigorous microscopic model of microwave response of these materials is not developed yet. An anomalous skin effect is usually considered in details in many textbooks, which review the theoretical and experimental studies of the surface impedance of superconductors (see for example one of the most complete monographs [4]), because of its inherent relation to the problem of surface impedance of superconductors. Actually both the anomalous skin effect in an extremely anomalous limit and the skin effect in superconductors correspond to current carriers with the infinite length of mean free path and differ in other characteristic length parameters. Knowledge of the surface impedance in the normal state under conditions of anomalous skin effect is crucially important for calculations, e. g., of the penetration depth of electromagnetic field in superconducting state from the surface resistance experimental data [4].

Complications mentioned above in a large extent result from the fact that for an explanation of the microwave spatial effects a microscopic kinetics theory is used to obtain the constitutive equation that relates the current density and the field in a metal, and is deduced on the rigorous base of linearized Boltzmann equation [2–6]. Previously it was noted (see for instance [4]) that rigorous microscopic theory of the effects of spatial dispersion, kinetics theory of anomalous skin effect lead to sophisticated calculations and do not provide an illustrative and clear physical view of processes taking place in metal. Such sophistication of a theoretical description and absence of an illustrative and clear physical view of transport processes enhance a peril of losses of additional or novel wave solutions of the microwave kinetics problem. Really the kinetics, i. e. non-equilibrium approach leads to the electrodynamic problem with dissipation that implies the problem with non-self conjugated differential operator [7]. A theory of non-self conjugated operators is in the stage of development now and is far from that degree of completion, which corresponds to a theory of self conjugated operators [7]. Since the wave problems of the anomalous skin effect in metal and skin effect in superconductor belong to the strongly non-self conjugated electrodynamic problems, not to the so called weak disturbance of self conjugated operators [7], the mentioned peril of losses of solutions is actual. Non-self conjugated problems may possess unusual properties, for instance, may have eigenfunctions without properties of the base functions. Moreover kinetics problems with spatial dispersion are nonlinear on spatial parameters because of the self consistent field coming into the constitutive equation. So the fact that quantitative kinetics theory has given an explanation (original in [6] and replay in [2–5]) of the results obtained in certain experimental situation [8] does not ensure generally an explanation of the results obtained under slightly different experimental conditions [9–13]. To be sure it is important to analyse the possibility of existence of additional solutions of this kinetics problem or of the alternative general approach to resolve the problem.

Microwave effects of the spatial dispersion have been considered recently in [14, 15] and rather general relation has been established there between the eigenvalue of the total permittivity operator and the surface impedance. It elaborates the textbook statement [1] for the situation when the penetration depth δ becomes to be comparable with the length of mean free path l. In this situation the spatial inhomogeneity of the field makes impossible the macroscopic description of the field by use of a permittivity $\widetilde{\varepsilon}$. The $\widetilde{Z}$ surface impedance is the only value, which characterizes the metal properties. However, for its calculation the employment of the kinetics theory is required and a permittivity becomes to be an operator $\widehat{\widetilde{\varepsilon}}$. The found relation between the eigenvalue $\widetilde{\varepsilon}_a$ of the total permittivity operator $\widehat{\widetilde{\varepsilon}}_a$ and the surface impedance corresponds with the usual expression for a transverse wave propagating into a conductor without dispersion $\widetilde{Z} = (\mu_0/\widetilde{\varepsilon}_a)^{1/2}$. A substitution of the eigenvalue of permittivity operator into the

Maxwellian field-stress tensor, which determines the ponderomotive force of Abraham, results to conditions of spatial-type force resonances. The obtained resonances includes a particular solution corresponding to a traditional for anomalous skin effect value of the surface impedance $\widetilde{Z} = R + iX$ with the phase of $\pi/3$ and with the dependence of the module on the microwave frequency $|\widetilde{Z}| \sim \omega^{2/3}$. Taking into account these results and the known conclusion that the operator of dielectric permittivity alone completely defines the microwave response of conductors with spatial dispersion [1–3], the eigenvalue problem for permittivity operator is considered below because of the corresponding solutions are the stable waves for constitutive equation in a self-consistent microwave field due to the problem statement. A wave problem is formulated to search the solutions, which correspond to the eigenvalues of permittivity operator, similar to the problem of wave propagation in hollow waveguides and resonators. Dispersion relations and general solutions are obtained. A significant role of the spatial-type force resonances is considered. Due to the self-consistency of a kinetics problem, which is nonlinear on the spatial parameters, spatial-type force resonances are added to and usually dominate over an influence of boundary conditions. Conditions for the spatial resonances are derived and discussed.

2. Microwave response of conductors with spatial dispersion

When the spatial effects in a microwave response of continuous media are described by the methods of macroscopic electrodynamics, which operates the strength of electric field $\widetilde{\mathbf{E}}$ and the electric induction $\widetilde{\mathbf{D}}$, it is assumed that a characteristic scale of the correlation length between the values of $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{D}}$ at different points of the space is much longer than the mean interparticle distance ($\sim n^{-1/3}$) [1-3]. This condition allows to apply a macroscopic description of the spatial dispersion in terms of a permittivity even in media with strong dispersion. Microscopic theories, which use, as a rule, the linearized Boltzmann kinetics equation, are applied for a deduction of a constitutive equation, which relates $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{D}}$ in metals [2-5]. Generally this is an operator relation: an absolute permittivity operator $\widehat{\widetilde{\varepsilon}}_a$ in a constitutive equation connects the vector functions $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{D}}$. This operator $\widehat{\widetilde{\varepsilon}}_a$, which depends on spatial variables, completely defines alone a reaction of a medium to an external microwave electromagnetic field. An inclusion of a concept of the relative magnetic permeability $\widetilde{\mu}$ in the Maxwell equations is not advantageous when the spatial dependencies are taken into account [1-3]. In media without dispersion in addition to the dielectric polarization $\widetilde{\mathbf{P}}$, which is incorporated in the definitions of $\widetilde{\mathbf{D}}$ and $\widetilde{\varepsilon}_a$, the magnetic polarization (or magnetization strength) $\widetilde{\mathbf{M}}$ is introduced as well, which comes in definitions of the magnetic field strength $\widetilde{\mathbf{H}}$ and of the absolute magnetic permeability $\widetilde{\mu}_a = \widetilde{\mu}\,\mu_0$, where μ_0 is the magnetic permeability of vacuum. It is assumed that an average microscopic current may be represented by a sum of two terms $\partial\widetilde{\mathbf{P}}/\partial t$ and rot $\widetilde{\mathbf{M}}$; in a plane wave, a combination of which is a Fourier expansion, these terms are reduced to $i\omega\widetilde{\mathbf{P}}$ and $i[\widetilde{\mathbf{k}} \times \widetilde{\mathbf{M}}]$. However, when the spatial dispersion being present, all parameters of the field depend already on the spatial parameter $\widetilde{\mathbf{k}}$ – the wave vector, the representation by two terms is excessive and the proportionality $\widetilde{\mathbf{B}} = \mu_0\widetilde{\mathbf{H}}$ must be used as a simple constitutive equation to relate the induction and the strength of magnetic field. For this choice the current and the charge density occur to be introduced in the definitions of the polarization $\widetilde{\mathbf{P}}$, the induction $\widetilde{\mathbf{D}}$ and the operator $\widehat{\widetilde{\varepsilon}}_a$, which in general case depend on both the electric $\widetilde{\mathbf{E}}$ and magnetic $\widetilde{\mathbf{B}}$ fields. However, the field $\widetilde{\mathbf{B}}$ may be expressed as a function of $\widetilde{\mathbf{E}}$ by use of the first pair of Maxwell equations [1], which include only these two values related by operations of differentiation. So the polarization $\widetilde{\mathbf{P}}$, the induction $\widetilde{\mathbf{D}}$ and the operator $\widehat{\widetilde{\varepsilon}}_a$ will be expressed as depending only on $\widetilde{\mathbf{E}}$.

The operator of absolute permittiviy $\widehat{\widetilde{\varepsilon}}_a$, which relates the vector functions $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{D}}$, and completely defines the microwave response of the medium to the incident field, is determined in microscopic theories by an infinite system of equations of motion for individual particles together with the field equations. For description of such systems the probabilistic methods are used and kinetic equation for the distribution function with the self-consistent field should be solved [2–6]. Taking account of the dominant role of an operator of dielectric permittivity $\widehat{\widetilde{\varepsilon}}_a$, a problem of searching the wave solutions in a microwave response of conductors with spatial dispersion, which correspond to the eigenvalues of this operator is of a great importance.

3. Eigenvalue problem for permittivity operator

We shall find the wave eigensolutions compliant with the eigenvalue of permittivity operator $\widehat{\widetilde{\varepsilon}}_a$, i. e. with the numerical complex value $\widetilde{\varepsilon}_a$ in the first of constitutive equations

$$\widetilde{\mathbf{D}} = \widehat{\widetilde{\varepsilon}}_a \widetilde{\mathbf{E}} = \widetilde{\varepsilon}_a \widetilde{\mathbf{E}}\ ; \qquad \widetilde{\mathbf{B}} = \mu_0 \widetilde{\mathbf{H}}, \tag{1}$$

in the similar manner as in the problem of wave propagation in hollow waveguides and resonators [1, 4]. For monochromatic fields the Maxwell equations for vectors $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$ in homogeneous medium with spatial dispersion [1–3] may be written, taking into account (1), as

$$\begin{aligned} \text{rot}\ \widetilde{\mathbf{B}} = \mu_0\ \text{rot}\ \widetilde{\mathbf{H}} = \mu_0\ (\partial\widetilde{\mathbf{D}}/\partial t)\ ; \qquad \text{div}\ \widetilde{\mathbf{D}} = \text{div}\ \widetilde{\mathbf{E}} = 0\ ; \\ \text{rot}\ \widetilde{\mathbf{E}} = -(\partial\widetilde{\mathbf{B}}/\partial t)\ ; \qquad \text{div}\ \widetilde{\mathbf{B}} = 0\ . \end{aligned} \tag{2}$$

The temporal factor $e^{i\omega t}$ is canceled out of equations below. After a differentiation with respect to time and a substitution of eqs (1) in (2), one will receive two equations

$$\text{rot}\ \widetilde{\mathbf{B}} = i\omega\mu_0\ \widetilde{\mathbf{D}} = i\omega\mu_0\widetilde{\varepsilon}_a\ \widetilde{\mathbf{E}}\ ; \qquad \text{rot}\ \widetilde{\mathbf{E}} = -i\omega\ \widetilde{\mathbf{B}} = -i\omega\mu_0\ \widetilde{\mathbf{H}}. \tag{3}$$

If we express $\widetilde{\mathbf{B}}$ from the second of eqs (3) and substitute it into the first of eqs (3), we get rot rot $\widetilde{\mathbf{E}} = \omega^2\mu_0\widetilde{\varepsilon}_a\ \widetilde{\mathbf{E}}$ or, expanding the operation of double rotation, grad div $\widetilde{\mathbf{E}}-$div grad $\widetilde{\mathbf{E}} = \omega^2\mu_0\widetilde{\varepsilon}_a\ \widetilde{\mathbf{E}}$. One can substitute in this expression the equality div $\widetilde{\mathbf{E}} = 0$ from (2) and introduce a new parameter $\widetilde{k}_d^2$ as $\omega^2\mu_0\widetilde{\varepsilon}_a = \widetilde{k}_d^2$ that result in the uniform, nonlinear in spatial parameter, vector wave equation $\nabla^2\widetilde{\mathbf{E}} + \widetilde{k}_d^2\,\widetilde{\mathbf{E}} = \nabla^2\widetilde{\mathbf{E}} + \omega^2\mu_0\widetilde{\varepsilon}_a\ \widetilde{\mathbf{E}} = 0$, where ∇^2- is the Laplace operator. If we take into account the definition of the gradient of a vector field with respect to a vector [16–18] ($\mathbf{a}$ grad) $\mathbf{V} = (\mathbf{a}\cdot\nabla)\,\mathbf{V}$, where $\mathbf{a}$ – the arbitrary constant vector and $\mathbf{V}$ – the vector field, the Laplace operator in the previous equation can be written in the form of an euclidean scalar product of two Hamilton operators div grad $\widetilde{\mathbf{E}} = (\nabla\cdot\nabla)\,\widetilde{\mathbf{E}} = \nabla^2\widetilde{\mathbf{E}}$. So we can seek a solution of the Laplace operator, which corresponds to the eigenvalue in form of the euclidean square of some vector $\widetilde{\mathbf{k}}$ in the real three-dimensional euclidean space, but generally with complex coordinates. This vector – the wave vector may be considered as a vector of the complex euclidean space [18]. Derivatives are taken on real variables here.

The equation for $\widetilde{\mathbf{B}}$ is obtained analogously. These vector equations in all cases fall into six scalar wave equations of the same type for components of vectors $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$. Each of scalar equations can be solved by the method of variables separation. Let us assume, for example, in the rectangle coordinate system only one component of the electric field $\widetilde{E}_x$ parallel to the metal surface is presented.

Designating the three member in the Pythagoras theorem for a square of wave-vector in orthogonal coordinates by the constants $-\widetilde{k}^2$ (and $\widetilde{\varepsilon}_{ax}$ for a more illustrative physical view): $-\widetilde{k}_x^2 = -\omega^2\mu_0\widetilde{\varepsilon}_{axx}$, $-\widetilde{k}_y^2 = -\omega^2\mu_0\widetilde{\varepsilon}_{axy}$, $-\widetilde{k}_z^2 = -\omega^2\mu_0\widetilde{\varepsilon}_{axz}$, one will have $\widetilde{k}_d^2 - \widetilde{k}_x^2 - \widetilde{k}_y^2 - \widetilde{k}_z^2 = 0$ or $\widetilde{\varepsilon}_a = \widetilde{\varepsilon}_{axx} + \widetilde{\varepsilon}_{axy} + \widetilde{\varepsilon}_{axz}$. The general solution of this equations can be written in the form similar to the following $\widetilde{E}_x = \widetilde{C}_5\exp(-i\widetilde{k}_z z) + \widetilde{C}_6\exp(i\widetilde{k}_z z)$.

4. Spatial force resonance and additional waves

The strongly inhomogeneous in space field can be considered as static in a comparison with the reaction of the conducting medium, characterized by the Fermi velocity $v_F \sim 0.01\,c$,

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where c is the speed of light. A reaction to an external field is spatially non-local, and the state of the system is determined by the correlation of the applied forces. In the quasistatic consideration, the maximum perturbation of the conducting medium will be caused by that of possible spatial configurations of the medium, which corresponds to the influence of dominating acting force caused by the incident field. The force (per unit volume) can be calculated by the Maxwellian field-stress tensor, each component of which is the spatial density of the corresponding momentum flux $\widetilde{\mathbf{\Pi}} = [\widetilde{\mathbf{D}} \times \widetilde{\mathbf{B}}]$. Derivative $\partial\widetilde{\mathbf{\Pi}}/\partial t$ determines the Abraham force (except for the force acting on the field inside the conductor and the effects of dispersion) [1, 3].

Substituting the impedance value into the momentum flux equation one can deduce on the boundary surface

$$\widetilde{\mathbf{\Pi}} = [\widetilde{\mathbf{D}} \times \widetilde{\mathbf{B}}] = \widetilde{D}_x(0)\widetilde{B}_y(0)\mathbf{k} = \widetilde{\varepsilon}\widetilde{E}_x(0)\sqrt{\mu_0\widetilde{\varepsilon}}\widetilde{E}_x(0)\mathbf{k} = (\mu_0^{1/2}\widetilde{\varepsilon}^{3/2})\widetilde{E}_x^2(0)\mathbf{k}. \tag{4}$$

The spatial-type force resonance corresponds to a real value of the coefficient in parentheses. This condition can be expressed as the equations $\widetilde{\varepsilon} = |\widetilde{\varepsilon}|e^{i\alpha}$, $\widetilde{\varepsilon}^{3/2} = |\widetilde{\varepsilon}|^{3/2}e^{3i\alpha/2}$, $3i\alpha/2 = i\pi n$, $n = 0, 1, 2$, $\alpha_{0,1,2} = 0, 2\pi/3, 4\pi/3$. The resonance in metals corresponds to the phase $\alpha_2 = 4\pi/3$, at which the spatial force resonance takes place. Phases $\alpha_0 = 0$ and $\alpha_1 = 2\pi/3$ refer to continuous wave and wave with amplitude growing deep into conductor, correspondingly. When the phase of $\widetilde{\varepsilon}$ is $4\pi/3$, and the phase of $\widetilde{Z} = (\mu_0/\widetilde{\varepsilon})^{1/2}$ is $\pi/3$ the ratio of the absolute values of real and imaginary parts of both becomes equal to $\sqrt{3}$.

The spatial-type force resonance may also correspond to an imaginary value of a coefficient in the expression for momentum flux $\widetilde{\mathbf{\Pi}}$. This condition agrees with equalities $\widetilde{\varepsilon} = |\widetilde{\varepsilon}|e^{i\beta}$, $\widetilde{\varepsilon}^{3/2} = |\widetilde{\varepsilon}|^{3/2}e^{3i\beta/2}$, $3i\beta/2 = i(\pi n + \pi/2)$, $n = 0, 1, 2$, $\beta_{0,1,2} = \pi/3, \pi, 5\pi/3$. Phases of $\widetilde{\varepsilon}$, equal to π, and $\widetilde{Z} = (\mu_0/\widetilde{\varepsilon})^{1/2}$, equal to $\pi/2$, correspond to the surface impedance of a nearly completely superconducting component with a close to an ideal reflection. The phase $\pi/3$ corresponds to the wave with an amplitude, growing deep into a conductor, and the phase $5\pi/3$ for $\widetilde{\varepsilon}$, which would correspond to an unusual phase $\pi/6$ for the surface impedance $\widetilde{Z} = (\mu_0/\widetilde{\varepsilon})^{1/2}$, principally is possible in normal metals and its registration is of interest.

References

[1] Landau L D and Lifshiz E M 1992 *L D Landau and E M Lifshiz Theoretical Physics vol 8, Electrodynamics of Continuous Media* (Moscow: Nauka) [in Russian].
[2] Lifshiz E M and Pitaevskii L P 1979 *L D Landau and EM Lifshiz Theoretical Physics vol 10, Physical Kinetics* (Moscow: Nauka) [in Russian]
[3] Silin V P and Rukhadze A A 1961 *Electromagnetic Properties of Plasma and Plasma-Like Media* (Moscow: Gosatomizdat) [in Russian].
[4] Mende F F and Spitsyn A I 1985 *Surface Impedance of Superconductors* (Kiev: Naukova Dumka) [in Russian]
[5] Abrikosov A A 1987 *Osnovy Teorii Metallov (Fundamentals of the Theory of Metals)* (Moscow: Nauka) [in Russian. Translated into English (Amsterdam: North-Holland, 1988)]
[6] Reuter G E H and Sondheimer E H 1948 *Proc. Roy. Soc. A* **195**, 336
[7] Ilyinskii A S and Slepyan G Y 1983 *Oscillations and Waves in Dissipative Electrodynamic Systems* (Moscow: Moscow University) [in Russian]
[8] Pippard A B 1947 *Proc. Roy. Soc. A* **191** 385
[9] Shcherbakov A S *at al* 1987 *Fiz.Met.Met.* **64** 735
[10] Shcherbakov A S *at al* 1987 *Fiz.Met.Met.* **64** 742
[11] Apakina V N *at al* 1997 *Physica C* **282-287** 1585
[12] Chernyaev A P *at al* 2008 *Journal of Physics and Chemistry of Solids* **69** 3313
[13] Dresvyannikov M A *at al* 2013 *Proc. of SPIE* **8700** 87000I
[14] Volchkov N A, Karuzskii A L and Perestoronin A V 2010 *JETP* **111** 292
[15] Karuzskii A L *at al* 2012 *J. Phys.: Conf. Ser.* **400** 022048
[16] Bronshtein I N and Semendyaev K A 1995 *Teubner-Taschenbuch der Mathematik* (Leipzig-Stuttgart: Teubner)
[17] Kochin N E 1965 *Vector Analysis and Fundamentals of the Tensor Analysis* (Moscow: Nauka) [in Russian]
[18] Rashevskii P K 1967 *Riemannian Geometry and Tensor Analysis* (Moscow: Nauka) [in Russian]