27th International Conference on Low Temperature Physics (LT27) IOP Publishing
Journal of Physics: Conference Series **568** (2014) 012014 doi:10.1088/1742-6596/568/1/012014

Study of coupled QED-cavities using the self-consistent Mori projector method

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Abstract. We study the non-equilibrium properties of nonlinear QED-cavities coupled via photon tunneling in the presence of dissipation and coherent pumping. To illustrate the interplay between photon leakage, Rabi oscillations, and coherent photon hopping, we examine the dynamical evolution and the stationary states of finite cavity arrays in the highly nonlinear regime. Finally, we employ a cluster version of the self-consistent Mori projector method and show both its accuracy and efficiency for the determination of local quantities in the highly complex many-body situation at hand.

1. Introduction

The study of light-matter interactions in cavity electrodynamics (QED) has always been a field of high interest [1]. In recent years strong theoretical and experimental progress was made towards the goal of investigating the physics of coupled QED-cavities [2]. An array of such QED-cavities, in which photons can tunnel to neighboring sites, yields a platform to study non-equilibrium many-body problems [3]. The competition between physical processes such as Rabi oscillations driven by an external laser, coherent oscillation between the cavities due to photon tunneling, and dissipation due to the photon loss from the cavities leads to a variety of interesting many-body phenomena, with non-equilibrium phase transitions being one prominent example among them [4, 5].

In this work, we examine the non-equilibrium properties of photons in a one-dimensional array of QED-cavities. The Hamiltonian of the system is described by the Bose Hubbard Hamiltonian together with a term describing a coherent laser drive [6]. The photon loss from the cavities is described by standard Lindblad dissipators [7]. First, we investigate the time evolution of photons in a finite array of cavities to illustrate the dynamical character of this complex system. In this case, we solve the equation of motion for the density matrix numerically. Next, we employ the self-consistent Mori projection method (c-MoP), which was recently developed in Ref. [8], to study the steady state of an infinite cavity array. This method determines the reduced density matrix for a single cavity, or a cluster of a few cavities, in a self-consistent way. We show that the calculated results with two- or three-cavity clusters are in good agreement with a numerically-exact solution obtained by the time-dependent density-matrix renormalization group (t-DMRG)

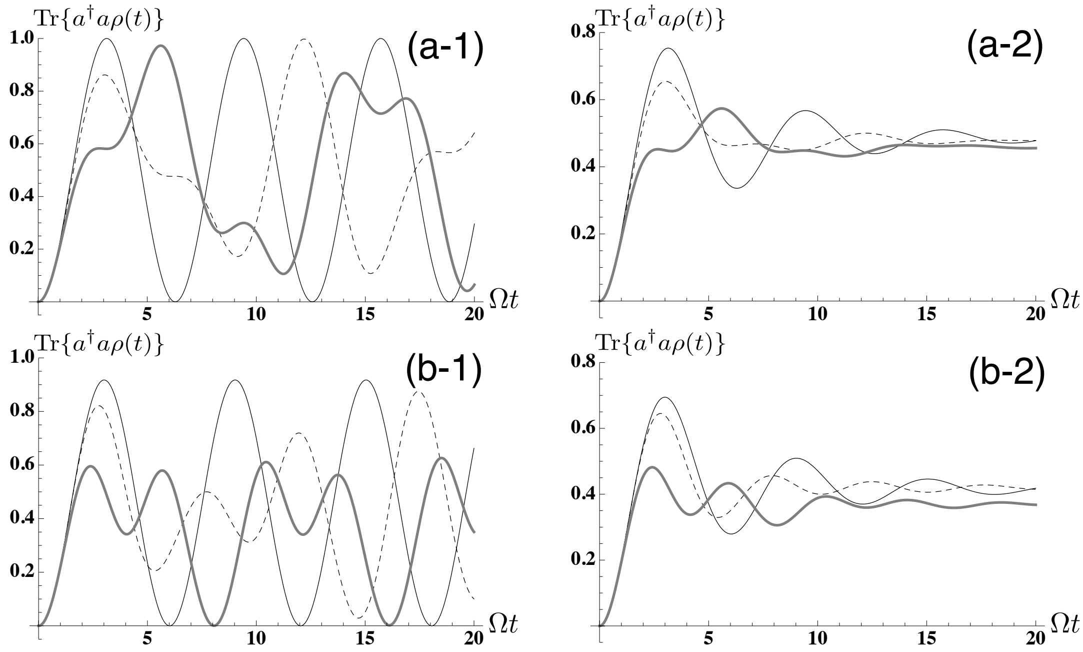

Figure 1. On-site photon numbers in an array of two cavities ($N = 2$), as a function of time. The detuning and dissipation parameters are (a-1) $\Delta = 0$ and $\gamma = 0$, (a-2) $\Delta = 0$ and $\gamma/\Omega = 0.25$, (b-1) $\Delta/\Omega = 0.3$ and $\gamma = 0$, (b-2) $\Delta/\Omega = 0.3$ and $\gamma/\Omega = 0.25$, respectively. The tunneling rate of photons is $J/\Omega = 0$ (solid line), $J/\Omega = 1$ (dashed line), and $J/\Omega = 2$ (grey solid line) in all the panels.

method [8, 9]. In contrast to t-DMRG, however, we emphasize that c-MoP is suitable for any lattice dimensions and arbitrary geometry.

2. Model

We examine an effective model for photons in a one-dimensional array of N QED-cavities, after the qubit degrees of freedom are integrated out [6]. The coherent motion of photons is described by the Bose Hubbard model in a driving laser field of strength $\Omega/2$,

$$H = \sum_{j=1}^{N} \left[\Delta a_j^\dagger a_j + \frac{U}{2} a_j^\dagger a_j^\dagger a_j a_j + \frac{\Omega}{2}(a_j^\dagger + a_j) \right] - J \sum_{j=1}^{N-1} (a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j). \tag{1}$$

Here, the Hamiltonian is written in a frame rotating with the frequency ω of the laser. $\Delta = \omega_{\mathrm{qubit}} - \omega$ is the difference between the qubit excitation frequency and the laser frequency. Photons interact with each other in a cavity with energy U and tunnel to the neighboring cavities with rate J. We restrict ourselves to the case of $U \to \infty$, in which the number of photons is unity or zero in each cavity.

We take the photon loss from the cavities with a constant rate γ into account. The density matrix $\rho(t)$ evolves according to the Liouville equation of motion [10],

$$\dot{\rho}(t) = \mathcal{L}\rho(t) = -i\,[H, \rho(t)] + \frac{\gamma}{2} \sum_{j=1}^{N} \left[2a_j \rho(t) a_j^\dagger - a_j^\dagger a_j \rho(t) - \rho(t) a_j^\dagger a_j \right]. \tag{2}$$

We solve this equation numerically for finite arrays ($N = 2,\ 3,\ 4$) in section 3 and apply the self-consistent Mori projector approach to an infinite array in section 4.

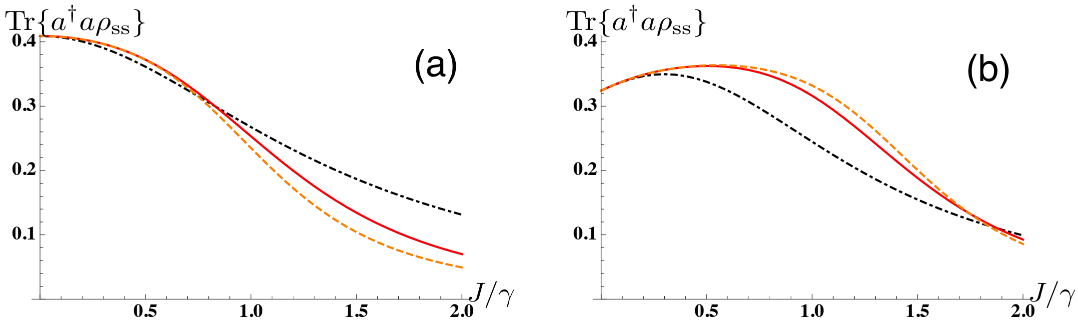

Figure 2. On-site steady-state photon numbers in a finite array of cavities as a function of the tunneling rate J with finite dissipation [$\gamma = (2/3)\Omega$]. The detuning parameter is (a) $\Delta = 0$ and (b) $\Delta/\gamma = 0.6$. The array size is $N = 2$ (dash-dotted line), $N = 3$ (red solid line), and $N = 4$ (orange dashed line) with periodic boundary conditions.

3. Time evolution and steady states of small size arrays

To elucidate the dynamical properties of the system under study, we begin with an array of two QED-cavities ($N = 2$). Figure 1 shows the time evolution of the photon number in a cavity for the initial state, $|\phi(t = 0)\rangle = |0\rangle_1 \otimes |0\rangle_2$. We tune the laser frequency to resonance, i.e. $\Delta = 0$, in panels (a-1) and (a-2) while $\Delta/\Omega = 0.3$ in (b-1) and (b-2). The photon loss from the cavities is absent in (a-1) and (b-1), whereas it is present in (a-2) and (b-2).

In the absence of dissipation due to photon loss, the photon number coherently oscillates. In panel (a-1), the Rabi oscillation changes the photon number between zero and unity for $J = 0$ (solid line). The tunneling rate J complicates the oscillation, reflecting the coexistence of the Rabi oscillation in a cavity and coherent oscillation between the cavities. In panel (b-1) with finite Δ, the Rabi frequency is given by $\sqrt{\Omega^2 + \Delta^2}$ for $J = 0$. The maximum of the photon number is less than unity in this case. In panels (a-2) and (b-2), the oscillation is damped to a steady state by the dissipation. (We also analyze the operator $\mathcal{L}$ in Eq. (1). One of its eigenvalues is zero, the eigenstate of which represents the steady state. The other eigenvalues have a negative real part, which determine the damped oscillation or overdamping.)

Next, we examine the photon numbers in the steady state with finite γ. In Fig. 2, we plot the photon numbers as a function of the tunneling rate J. The size of the array is $N = 2$, 3 or $N = 4$ with periodic boundary conditions. When the resonant condition is satisfied ($\Delta = 0$), the photon number monotonically decreases with J, as shown in Fig. 2(a). The photon number is maximal at $J = 0$ due to the Rabi oscillations in a cavity, which is disturbed by the tunneling between neighboring cavities, resulting in a decrease of the photon number with increasing J. In fact, the tunneling rate J shifts the energy levels of the two cavity system resulting in an effective detuning of the laser drive [6, 11]. Hence, the finite detuning Δ in Fig. 2(b) leads to a maximal photon number at a finite value of J. These features of the photon numbers are commonly observed for $N = 2$, 3, and 4, and also hold true for larger arrays as we show below.

4. Steady state of large size arrays

Now we investigate an infinite one-dimensional array of QED-cavities by applying a cluster version of the self-consistent Mori projection (c-MoP) method [8] to this problem.

In the c-MoP method, the entire many-body system is divided into a subsystem of interest which we refer to as the "system," and the remainder of the lattice which we refer to as the "environment." The closed equations for the reduced density matrix of the "system" are derived by imposing a self-consistent condition to that of the "environment" which holds true because

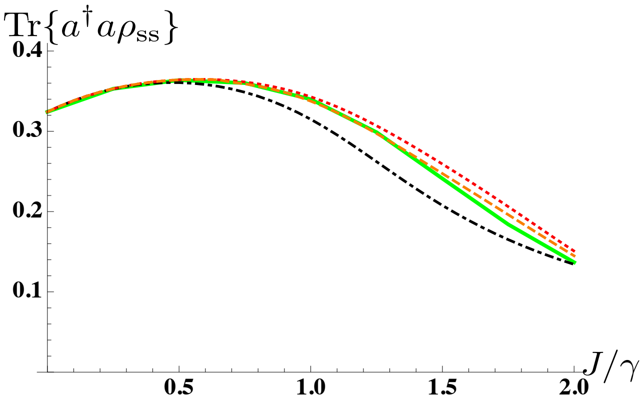

Figure 3. On-site steady-state photon numbers in an infinite array of cavities, as a function of the tunneling rate J with finite dissipation $[\gamma = (2/3)\Omega]$. The detuning parameter is $\Delta/\gamma = 0.6$. The calculation is performed by single-cavity c-MoP (black dash-dotted line), 2-cavity c-MoP cluster (red dotted line), and 3-cavity c-MoP cluster methods (orange dashed line). The data obtained by a t-DMRG integration is shown by the green solid line.

of translation invariance in the full equation of motion, see Eqs. (1) and (2). In our study, the "system" is a single cavity, or a cluster of few cavities, which couples to the "environment" via photon tunneling on the boundaries. For a detailed description, see Ref. [8] and references therein.

For the "system" of a single cavity, which we denote by the index n_0, we need to solve the following algebraic equation of motion for the steady state of the reduced density matrix $\rho^{ss}_{n_0}$,

$$\begin{aligned} 0 = &\mathcal{L}_{n_0}\rho^{ss}_{n_0} + iZJ\left[a_{n_0}\mathrm{Tr}_n\{a^\dagger_n\rho^{ss}_n\} + a^\dagger_{n_0}\mathrm{Tr}_n\{a_n\rho^{ss}_n\}, \rho^{ss}_{n_0}\right] \\ &-\left(ZJ^2\sum_{j=\{+,-\}}\left[a^\dagger_{n_0}, \int_0^\infty d\tau\, e^{\tau\mathcal{L}_{n_0}}\left(d_j(\rho^{ss}_n)\,\delta a^j_{n_0}\rho^{ss}_{n_0} - s_j(\rho^{ss}_n)\,\rho^{ss}_{n_0}\delta a^j_{n_0}\right)\right] + \mathrm{H.c}\right), \end{aligned} \tag{3}$$

where Z is the number of nearest neighbors, $a^+_n = a^\dagger_n$, $a^-_n = a_n$, $\delta a^j_n = a^j_n - \mathrm{Tr}_n\{a^j_n\rho^{ss}_n\}$, and

$$d_j(\rho^{ss}_n) = \mathrm{Tr}_n\{a_n e^{\tau\mathcal{L}_n}(\delta a^j_n)^\dagger\rho^{ss}_n\}, \tag{4}$$

$$s_j(\rho^{ss}_n) = \mathrm{Tr}_n\{a_n e^{\tau\mathcal{L}_n}\rho^{ss}_n(\delta a^j_n)^\dagger\}. \tag{5}$$

In Eq. (3), ρ_n is the reduced density matrix for site n. The Liouville operator $\mathcal{L}_{n_0}(\cdot) = -i[H_{n_0}, (\cdot)] + \gamma/2[2a_{n_0}(\cdot)a^\dagger_{n_0} - a^\dagger_{n_0}a_{n_0}(\cdot) - (\cdot)a^\dagger_{n_0}a_{n_0}]$ describes the dynamics of the "system" without the coupling to the "environment." As a self-consistent condition, we replace ρ_n by ρ_{n_0} in Eqs. (3)–(5), which yields closed algebraic equations for ρ_{n_0}.

We make three comments. (i) The c-MoP method up to first order in the "system"-"environment" coupling J, see the first two terms on the right hand side of Eq. (3), is equivalent to the mean-field approximation [12, 13]. The term proportional to J^2 explicitly takes system-environment correlations into account, and thus, we gain a much more accurate result than the results obtained with a mean-field Ansatz [8]. (ii) When the "system" is a cluster consisting of two or three neighboring cavities, we obtain very similar self-consistent equations for the reduced density matrix. The larger size of the "system" yields the better approximation. (iii) The numerical task to solve for $\rho^{ss}_{n_0}$ in Eq. (3) is comparable to the numerical task to solve

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a mean-field equation which can be done very efficiently as it only involves the dimension of the local Hilbert space. Even the integral over the dynamical map $e^{\tau\mathcal{L}_{n_0}}$ only demands the diagonalization of $\mathcal{L}_{n_0}$ which is again a local operator. In contrast to DMRG methods which work only efficient in 1D, the c-MoP equations do not change their structure by going from one-dimensional arrays to higher dimensional lattices and are therefore well suited for the treatment of problems with an arbitrary lattice geometry.

Figure 3 shows the calculated results for the photon numbers in the steady state by the c-MoP method for different values of the tunneling rate J. The "system" consists of a single cavity (black dash-dotted line), a two-cavity cluster (red solid line), and a three-cavity cluster (orange dashed line). Further we show the result obtained by a t-DMRG integration of Eq. (2) with $N = 21$ lattice sites and open boundary conditions. From this t-DMRG numerics we extract the reduced density matrix for the central site $n_0 = 11$ and the corresponding photon number (green solid line). Although we find a deviation from this numerically-exact result in the case of single-site c-MoP, the calculation with two- or three-site c-MoP clusters shows a good agreement with the t-DMRG result. This clearly indicates the suitability of the c-MoP method to examine this non-equilibrium many-body problem.

5. Conclusion

We have studied the non-equilibrium many-body problem of a one-dimensional array of QED-cavities. For a finite array, we have elucidated the complex and rich interplay between Rabi oscillations in a cavity driven by the laser, coherent oscillation induced by tunneling of photons between neighboring cavities, and the dissipative effects of photon losses from the cavities, leading to the relaxation of the array to a non-equilibrium steady state. For an infinite array, we have adopted the c-MoP method to calculate the reduced density matrix of subsystem in a self-consistent way. We have found that the two- or three-cavity c-MoP cluster method is a powerful tool to study the complex many-body behavior of coupled QED-cavities on a lattice in any dimension and arbitrary lattice geometry.

Acknowledgments

This work is supported by a Grant-in-Aid for Scientific Research (S) (No. 26220711) from Japan Society for the Promotion of Science and by the collaborative research center 631 of the German research foundation (DFG).

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