

Gravity and thermodynamics: fundamental principles and gravothermal instability

G Fragione^{1,2}

¹Department of Physics, University of Rome La Sapienza, Piazzale Aldo Moro 2, 00185, Rome Italy

²Department of Physics, University of Rome Tor Vergata, Via Orazio Raimondo 18, 00173, Rome Italy

E-mail: giacomo.fragione90@gmail.com

Abstract. Globular clusters are stellar systems in which thermodynamics plays a fundamental role. Classically these systems are described by non-relativistic single mass King distribution function, which derives from the Fokker-Planck equation by taking into account stellar collisions and host galaxy tidal confinement. In terms of statistical mechanics the available phase space is restricted and so thermodynamics variables assume different meaning respect to the Boltzmann distribution. Through the introduction of an effective potential, which stands for the galactic tidal forces, a new thermodynamic theory is developed and theoretical energy and specific heat profiles are given for globular clusters.

1. Introduction

Globular clusters (GCs) are stellar systems with masses in the range $10^4 \div 10^6 M_\odot$ and about 10^5 stars. The core radius r_c , i.e the radial coordinate at which the surface brightness becomes one half of the central value, may reach $10 pc$, whereas the tidal radius r_t , i.e. the size of the cluster allowed by the external galactic tidal field, is typically about $50 pc$. Furthermore GCs are nearly spherical systems.

The observations of GCs luminosity profiles [1] show similar curves, depending only on different values of the star concentration, and so they can be fitted by the same empirical law, suggesting a unique distribution function (DF) for all the clusters. This function corresponds approximately to a Boltzmann distribution function (BDF) minus a constant, resulting by the effects of the host galaxy tidal forces, which make GCs be spatially limited and their density profiles vanish in r_t . In this scenario stars going beyond the tidal radius, imposed by the galaxy potential, are lost from the system and evaporate, while, in terms of statistical mechanics, this limit turns to a restriction of the phase space, which is no longer entirely available because stars with velocities larger than a limiting one evaporate from the cluster. On the other hand, stars have the possibility of increasing their velocity due to stellar encounters, important in GCs since the average binary relaxation time is shorter than their absolute age ($10 \div 13 Gyr$). Although the stars evaporation, the velocity distribution remains formally unchanged during the evolution of a GC, varying only the concentration of the system or, equivalently, the dimensionless central gravitational potential W_0 . Therefore the system will evolve through a series of King models [3], every of which is described by the same DF with different values of W_0 , which increases during GCs lifetime. Furthermore the evolution of GCs can be studied as a classical single mass King



model in relation to thermodynamic instability phenomena. In fact stellar encounters strongly contribute to phase space mixing of stellar orbits and so thermodynamics plays a central role in the gravitational equilibrium and stability of these clusters.

Analytically the evolution of GCs is well described by the Fokker-Planck equation, where the nature of collisions in GCs are taken into account, of which the King distribution function (KDF) is a solution [2]. This equation can determine the characteristics of the DF relevant for the equilibrium configurations, whereas the galactic gravitational tidal effects are responsible for the confinement of the cluster. All these effects make the evolution of GCs be described as a series of thermodynamic transformations, where the functional form of the KDF is conserved, while the structural and thermodynamic parameters of the system evolve [6]. Therefore two competitive phenomena determine the evolution of GCs: *i*) stellar encounters, which tend to refresh the high velocities tail in the DF, and *ii*) evaporation of stars, which prevents its formation. This scenario lasts until the so-called gravothermal catastrophe [4], due to the presence of negative specific heat regions caused by the combined effects of gravity and thermodynamics.

In this scenario it is possible to define thermodynamic quantities in analogy with the classical Boltzmann gas, paying attention to the fact that, while a Boltzmann gas is in thermodynamic equilibrium, GCs are in a steady-state equilibrium. Therefore the KDF can be treated as a Boltzmann-like distribution, with an additive term ϕ in the Hamiltonian, called effective potential. This potential is nothing else but the effect of the host galaxy tidal forces and causes the restriction of the phase space available to stars.

In this paper a generalized form of thermodynamics is obtained due to the introduction of the effective potential in the Hamiltonian. In section 2 the main thermodynamic quantities are obtained through the standard definitions of thermodynamics, stressing the fact there is a splitting of variables due to the presence of the effective potential. In section 3 the thermal, mechanical and chemical equilibria are studied, focusing the attention on the meaning of the two sets of variables. Finally in section 4 the behaviour of the specific heat and of the energy of the system is described, giving the possible theoretical configurations of GCs according to the exposed model, which can lead to the gravothermal catastrophe.

2. Fundamental principles

The KDF [2], describing the energy distribution of an isotropic and spherical system of stars with the same mass m , can be written as

$$f(\varepsilon) = \begin{cases} Ae^{-m\varphi/kT} [e^{-\varepsilon/kT} - e^{-\psi/kT}], & \varepsilon \leq \psi \\ 0, & \varepsilon > \psi \end{cases} \quad (1)$$

where $\psi(r) = m(\varphi_R - \varphi(r))$ is the energy cutoff imposed by the tidal interaction with the host galaxy, while $\varphi(r)$ is the gravitational potential. $A = Be^{(\mu+m\varphi)/kT}$, where B is a constant of normalization and μ the chemical potential. Since its definition, ψ is the energy sufficient to a star to reach the boundary of the system with zero velocity. Merafina & Ruffini studied GCs equilibrium configurations, showing they depend on the value of W_0 and the total mass of the system presents a maximum at $W_0 \simeq 1.35$, evidencing the rise of an instability [5]. Starting from first principles the evolution of GCs can be described as a series of thermodynamic transformations, where the parameters of the system evolve, since binary stellar encounters are important and evaporation takes place [6]. In this scenario thermodynamics is fundamental in the description of GCs and the above KDF can be rewritten as a Boltzmann-like DF, thanks to the introduction of an effective potential $\phi = \phi(\varepsilon, r)$

$$f(\varepsilon) = Ae^{-H/k\theta} \quad (2)$$

where $A = Be^{(\alpha+m\varphi)/k\theta}$, $H = \varepsilon + m\varphi + \phi$ is the total Hamiltonian and θ the thermodynamic temperature. B is a normalization constant, α the thermodynamic chemical potential and ϕ describes the effects of the host galaxy tidal forces. In the case of KDF

$$\phi = -k\theta \ln \left[1 - e^{(\varepsilon-\psi)/k\theta} \right] . \quad (3)$$

The introduction of an additive potential, which goes to infinity at ψ , leads to two different consequences, i.e. the restriction of the phase space and the recover of the BDF form. These effects lead to a natural splitting of thermodynamic variables, which have to be distinguished in thermodynamic quantities, which have the same characteristics of BDF variables and involve the full Hamiltonian, and kinetic quantities, which are the physical variables of the system and involve only the kinetic part of H . Using standard statistical expressions, the kinetic and thermodynamic variables are obtained

$$U_k = \bar{A}V \int_0^\psi f \varepsilon \varepsilon^{1/2} d\varepsilon \quad U = \bar{A}V \int_0^\psi f H \varepsilon^{1/2} d\varepsilon \quad (4)$$

$$kT = \left\langle q_i \frac{\partial \varepsilon}{\partial q_i} \right\rangle \quad k\theta = \left\langle q_i \frac{\partial H}{\partial q_i} \right\rangle \quad (5)$$

$$P = \frac{1}{3} \bar{A} \int_0^\psi f q \frac{d\varepsilon}{dq} d^3q \quad \Pi = \frac{1}{3} \bar{A} \int_0^\psi f q \frac{dH}{dq} d^3q \quad (6)$$

$$\mu = \left. \frac{\partial U_k}{\partial N} \right|_{S,V} \quad \alpha = \left. \frac{\partial U}{\partial N} \right|_{S,V} \quad (7)$$

where q_i are the generalized momenta and \bar{A} is a constant, while the brackets indicate the average over the DF. For number of particles and entropy

$$N = \bar{A}V \int_0^\psi f \varepsilon^{1/2} d\varepsilon \quad (8)$$

$$S = \bar{A}V k \int_0^\psi (1 - \ln f) f \varepsilon^{1/2} d\varepsilon . \quad (9)$$

There is also a splitting of the first principle, Gibbs-Duhem relation, Euler equation and equation of state

$$dU_k = TdS - PdV + \langle \mu_0 \rangle dN + N \langle d\nu \rangle \quad dU = \theta dS - \Pi dV + \alpha dN + N \langle d\phi \rangle \quad (10)$$

$$N \langle d\nu \rangle = SdT - Vdp + Nd\langle \mu_0 \rangle \quad N \langle d\phi \rangle = Sd\theta - Vd\Pi + Nd\alpha \quad (11)$$

$$U_k = TS - PV + N \langle \mu_0 \rangle \quad U = \theta S - \Pi V + N\alpha \quad (12)$$

$$PV = NkT \quad \Pi V = Nk\theta \quad (13)$$

where $\langle d\nu \rangle = d\langle \mu_0 \rangle - \langle d\mu_0 \rangle$. So the introduction of ϕ leads to a natural splitting not only of the thermodynamic variables, but also of the fundamental equations of thermodynamics.

3. Thermal, mechanical and chemical equilibria

Considering the results obtained in the previous section, it is possible to study the thermal, mechanical and chemical equilibria starting from the first principle. First of all two different

types of variations have to be taken into account. The first type, labelled by d , is along a thermodynamic transformation, while the second one, labelled by δ , is along the radial coordinate since quantities depend on r . So it is possible to study two infinitesimal near shells, at distance δr one from each other, among which there is a difference of gravitational potential $\delta\phi$, divided by a separator, which makes the two parts not interact thermally, neither mechanically nor chemically between themselves. Now considering the first principle is true in every shell separately and the total entropy has its maximum value when the equilibrium is established ($dS_{tot} = 0$), for thermal, mechanical and chemical equilibria it will be

$$N, V = \text{const} \rightarrow \delta\theta = 0, \quad k\delta T = R \langle \delta\phi \rangle - (1 - R) m\delta\phi \quad (14)$$

$$S, N = \text{const} \rightarrow \delta\Pi + \frac{N}{V} [\langle \delta\phi \rangle + m\delta\varphi] = 0, \quad \delta P + \rho\delta\varphi = 0 \quad (15)$$

$$V, S = \text{const} \rightarrow \delta\alpha = 0, \quad N\delta\mu = -S\delta T \quad (16)$$

where $R = R(r)$ relates thermodynamic and kinetic variables and is given by

$$R = 1 - \frac{1}{3k\theta} \left\langle q \frac{\partial\phi}{\partial q} \right\rangle \quad (17)$$

As it was revealed before, thermodynamic quantities have the same characteristics of BDF variables. Actually the thermodynamic temperature and chemical potential are constant for all configurations, whereas the kinetic variables are not: the former are only mathematical variables, the latter have physical meaning and must have a decreasing profile which goes to zero at the border of the configuration due to the characteristics of KDF. However the most important thing happens for the kinetic pressure, where the hydrostatic equilibrium comes out naturally from the mechanical equilibrium.

4. Specific heat and energy profiles: the gravothermal catastrophe

Two fundamental quantities for the analysis of GCs stability are the specific heat and the energy, which have a profile in radius alike the case of BDF. For what regards the energy of the system, it is made up of three terms:

$$E_{tot} = E_k + E_g + E_{eff} \quad (18)$$

where the last term takes into account the contribution of ϕ and represents the effect of the tidal forces caused by the host galaxy potential, which imposes the confinement of the system. The specific heat at constant volume is given by $C_V = (dQ/d\theta)_V$ from classical thermodynamics, and from first principle for thermodynamic variables

$$dQ = dU - N \langle dH \rangle \quad (19)$$

where in the second term there is the contribution due to the effective potential. The results from numerical integration are shown in figs. 1-4. From the numerical analysis four different GCs configurations, and their evolution, can be predicted according to different values of W_0 :

- $W_0 < 1.35$ positive specific heat all over the configuration and total positive energy: systems can not evolve towards the gravothermal catastrophe and disrupt due to $E_{tot} > 0$;
- $1.35 < W_0 < 2.3$ positive and negative specific heat regions with total positive energy: systems may evolve towards the gravothermal catastrophe but disrupt due to $E_{tot} > 0$;
- $2.3 < W_0 < 3$ positive and negative specific heat and energy regions with total positive energy and total negative specific heat: systems may evolve towards the gravothermal catastrophe but disrupt due to $E_{tot} > 0$;
- $W_0 > 3$ positive and negative specific heat and energy regions with total negative energy and specific heat: systems can evolve towards the gravothermal catastrophe by increasing W_0 .

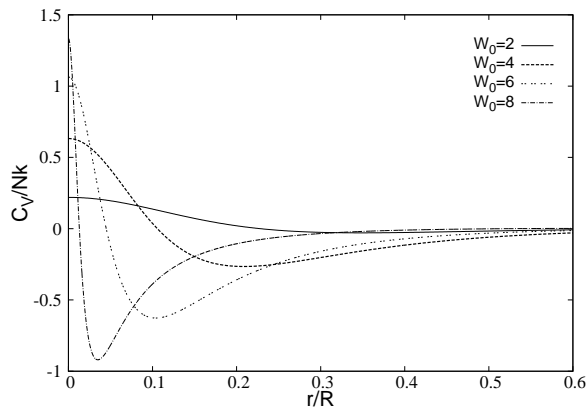


Figure 1. Radial profiles of the specific heat for different values of W_0 .

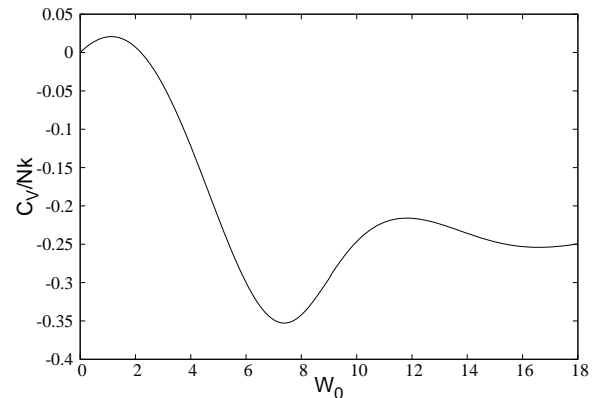


Figure 2. Total specific heat in function of W_0 . Note it is negative for $W_0 > 2.3$.

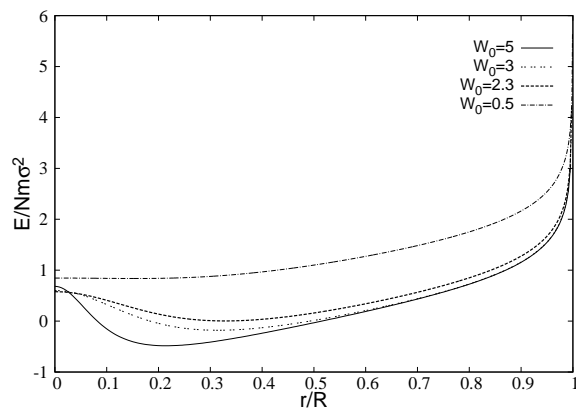


Figure 3. Radial profiles of the energy for different values of W_0 .

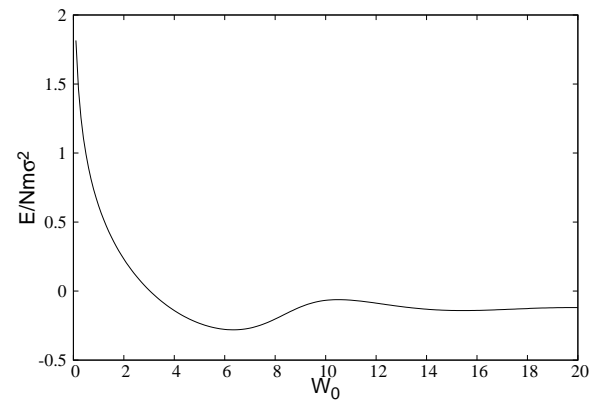


Figure 4. Total energy in function of W_0 . Note it is negative for $W_0 > 3$.

5. Conclusions

The theoretical model exposed in this paper predicts negative and positive specific heat regions, so being self-consistent, for GCs, which can evolve towards the gravothermal catastrophe without an external bath of heat, alike the Lynden-Bell & Wood model [4]. The positive specific heat core could justify the presence of post core-collapsed objects. The system gives the value of $W_0 \simeq 6.9$ as the limit for the gravothermal instability and finds also preliminary observational evidences, by performing statistical analysis using Harris data [7][8].

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