Belavin–Drinfeld cohomologies and introduction to classification of quantum groups

Alexander Stolin and Iulia Pop

Department of Mathematical Sciences, University of Göteborg, 41296 Göteborg, Sweden

E-mail: `astolin@chalmers.se; iulia@chalmers.se`

Abstract. In the present article we discuss the classification of quantum groups whose quasi-classical limit is a given simple complex Lie algebra $\mathfrak{g}$. This problem reduces to the classification of all Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$, where $\mathbb{K} = \mathbb{C}((\hbar))$. The associated classical double is of the form $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$, where A is one of the following: $\mathbb{K}[\varepsilon]$, where $\varepsilon^2 = 0$, $\mathbb{K} \oplus \mathbb{K}$ or $\mathbb{K}[j]$, where $j^2 = \hbar$. The first case relates to quasi-Frobenius Lie algebras. In the second and third cases we introduce a theory of Belavin–Drinfeld cohomology associated to any non-skewsymmetric r-matrix from the Belavin–Drinfeld list [1]. We prove a one-to-one correspondence between gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ and cohomology classes (in case II) and twisted cohomology classes (in case III) associated to any non-skewsymmetric r-matrix.

Mathematics Subject Classification (2010): 17B37, 17B62.
Keywords: Quantum groups, Lie bialgebras, classical double, r-matrix.

1. Introduction

Let k be a field of characteristic 0. According to [3], a quantized universal enveloping algebra (or a quantum group) is a topologically free topological Hopf algebra H over the formal power series ring $k[[\hbar]]$ such that $H/\hbar H$ is isomorphic to the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over k.

The quasi-classical limit of a quantum group is a Lie bialgebra. By definition, a Lie bialgebra is a Lie algebra $\mathfrak{g}$ together with a cobracket δ which is compatible with the Lie bracket. Given a quantum group H, with comultiplication Δ, the quasi-classical limit of H is the Lie bialgebra $\mathfrak{g}$ of primitive elements of $H/\hbar H$ and the cobracket is the restriction of the map $(\Delta - \Delta^{21})/\hbar(\mathrm{mod}\hbar)$ to $\mathfrak{g}$.

The operation of taking the semiclassical limit is a functor $SC : QUE \to LBA$ between categories of quantum groups and Lie bialgebras over k. The quantization problem raised by Drinfeld aims at finding a quantization functor, i.e. a functor $Q : LBA \to QUE$ such that $SC \circ Q$ is isomorphic to the identity. Moreover, a quantization functor is required to be universal, in the sense of props.

The existence of universal quantization functors was proved by Etingof and Kazhdan [5, 6]. They used Drinfeld's theory of associators to construct quantization functors for any field k of characteristic zero. Drinfeld introduced the notion of associator in relation to the theory of quasi-triangular quasi-Hopf algebras and showed that associators exist over any field k of characteristic zero. Etingof and Kazhdan proved that for any fixed associator over k one can construct a universal quantization functor. More precisely, let $(\mathfrak{g}, \delta)$ be a Lie bialgebra over k. Then one can associate a Lie bialgebra $\mathfrak{g}_\hbar$ over $k[[\hbar]]$ defined as $(\mathfrak{g} \otimes_k k[[\hbar]], \hbar\delta)$. According to Theorem 2.1 of [6] there exists an equivalence $\widehat{Q}$ between the category $LBA_0(k[[\hbar]])$ of topologically free over $k[[\hbar]]$ Lie bialgebras with $\delta = 0(\mathrm{mod}\hbar)$ and the category $HA_0(k[[\hbar]])$ of topologically free Hopf algebras cocommutative modulo $\hbar$. Moreover, for any $(\mathfrak{g}, \delta)$ over k, one has the following: $\widehat{Q}(\mathfrak{g}_\hbar) = U_\hbar(\mathfrak{g})$.

The aim of the present article is the classification of quantum groups whose quasi-classical limit is a given simple complex Lie algebra $\mathfrak{g}$. Due to the equivalence between $HA_0(\mathbb{C}[[\hbar]])$ and $LBA_0(\mathbb{C}[[\hbar]])$, this problem is equivalent to classification of Lie bialgebra structures on $\mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}[[\hbar]]$. For simplicity, denote $\mathbb{O} := \mathbb{C}[[\hbar]]$, $\mathbb{K} := \mathbb{C}((\hbar))$, $\mathfrak{g}(\mathbb{O}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{O}$ and $\mathfrak{g}(\mathbb{K}) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$.

On the other hand, in order to classify cobrackets on $\mathfrak{g}(\mathbb{O})$ it is enough to classify cobrackets on $\mathfrak{g}(\mathbb{K})$. Indeed, if δ is a Lie bialgebra structure on $\mathfrak{g}(\mathbb{O})$, then it can be naturally extended to $\mathfrak{g}(\mathbb{K})$. Conversely, given a Lie bialgebra structure $\bar{\delta}$ on $\mathfrak{g}(\mathbb{K})$, then by multiplying $\bar{\delta}$ by an appropriate power of $\hbar$, the restriction of $\bar{\delta}$ to $\mathfrak{g}(\mathbb{O})$ is a Lie bialgebra structure on $\mathfrak{g}(\mathbb{O})$.

Now, from the general theory of Lie bialgebras it is known that for each Lie bialgebra structure δ on a fixed Lie algebra L one can construct the corresponding classical double $D(L, \delta)$ which is the vector space $L \oplus L^*$ together with a bracket which is induced by the bracket and cobracket of L, and a non-degenerate invariant bilinear form, see [4]. We consider $L = \mathfrak{g}(\mathbb{K})$ and prove Prop. 2.1 which states that there exists an associative, unital, commutative algebra A, of dimension 2 over $\mathbb{K}$, such that $D(\mathfrak{g}(\mathbb{K}), \delta) \cong \mathfrak{g}(\mathbb{K})\otimes_{\mathbb{K}} A$. In Prop. 2.3 we show that there are three possibilities for A: $A = \mathbb{K}[\varepsilon]$, where $\varepsilon^2 = 0$, $A = \mathbb{K} \oplus \mathbb{K}$ or $A = \mathbb{K}[j]$, where $j^2 = \hbar$.

Due to the correspondence Lie bialgebras–Manin triples, to any Lie bialgebra structure δ on L one can associate a certain Lagrangian subalgebra W of $D(L, \delta)$ which is complementary to L and conversely, any such W produces a Lie cobracket on L. The main problem is to obtain a classification of all such subalgebras W for the three choices of A as above. We investigate each choice of A separately.

For $A = \mathbb{K}[\varepsilon]$, where $\varepsilon^2 = 0$, it turns out that the classification problem is related to that of quasi-Frobenius Lie subalgebras over $\mathbb{K}$.

In case $A = \mathbb{K} \oplus \mathbb{K}$, we introduce Belavin–Drinfeld cohomologies. Namely, for any non-skewsymmetric constant r-matrix r_{BD} from the Belavin–Drinfeld list [1], we associate a cohomology set $H^1_{BD}(r_{BD})$. We prove that there exists a one-to-one correspondence between any Belavin–Drinfeld cohomology and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$. We first consider $\mathfrak{g} = sl(n)$ and show that all cohomologies are trivial. We then discuss the case of orthogonal algebras $\mathfrak{g} = o(n)$, where it turns out that the cohomology associated to the Drinfeld–Jimbo r-matrix is also trivial. We illustrate an example where the cohomology corresponding to another non-skewsymmetric constant r-matrix for $o(2n)$ is non-trivial.

We finally treat the case $A = \mathbb{K}[j]$, where $j^2 = \hbar$. We restrict our analysis to $\mathfrak{g} = sl(n)$ and we show that in this case a cohomology theory can be introduced too. Our result states that there exists a one-to-one correspondence between Belavin–Drinfeld twisted cohomology and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$. We prove that the twisted cohomology corresponding to the Drinfeld–Jimbo r-matrix is trivial.

In the last section of the article we formulate a conjecture stating that the Belavin–Drinfeld cohomology associated to the Drinfeld–Jimbo r-matrix is trivial for any simple complex Lie algebra $\mathfrak{g}$. We also define the quantum Belavin–Drinfeld cohomology and formulate a second conjecture about the existence of a natural correspondence between classical and quantum cohomologies.

2. Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra. Consider the Lie algebras $\mathfrak{g}(\mathbb{O}) = \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{O}$ and $\mathfrak{g}(\mathbb{K}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{K}$.

We have seen that the classification of quantum groups with quasi-classical limit $\mathfrak{g}$ is equivalent to the classification of all Lie bialgebra structures on $\mathfrak{g}(\mathbb{O})$. Moreover, as explained in the introduction, in order to classify Lie bialgebra structures on $\mathfrak{g}(\mathbb{O})$, it is enough to classify them on $\mathfrak{g}(\mathbb{K})$.

Let us assume that $\bar{\delta}$ is a Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$. This cobracket endows the dual of $\mathfrak{g}(\mathbb{K})$ with a Lie bracket. Then one can construct the corresponding classical double $D(\mathfrak{g}(\mathbb{K}),\bar{\delta})$. As a vector space, $D(\mathfrak{g}(\mathbb{K}),\bar{\delta}) = \mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})^*$. As a Lie algebra, it is endowed with a bracket which is induced by the bracket and cobracket of $\mathfrak{g}(\mathbb{K})$. Moreover the canonical symmetric non-degenerate bilinear form on this space is invariant.

Similarly to Lemma 2.1 from [8], one can prove that $D(\mathfrak{g}(\mathbb{K}),\bar{\delta})$ is a direct sum of regular adjoint $\mathfrak{g}$-modules. Combining this result with Prop. 2.2 from [2], one obtains the following

Proposition 2.1. *There exists an associative, unital, commutative algebra A, of dimension 2 over $\mathbb{K}$, such that $D(\mathfrak{g}(\mathbb{K}),\bar{\delta}) \cong \mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$.*

Remark 2.2. The symmetric invariant non-degenerate bilinear form Q on $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ is given in the following way. For arbitrary elements $f_1, f_2 \in \mathfrak{g}(\mathbb{K})$ and $a, b \in A$ we have $Q(f_1 \otimes a, f_2 \otimes b) = K(f_1, f_2) \cdot t(ab)$, where K denotes the Killing form on $\mathfrak{g}(\mathbb{K})$ and $t : A \longrightarrow \mathbb{K}$ is a trace function.

Let us investigate the algebra A. Since A is unital and of dimension 2 over $\mathbb{K}$, one can choose a basis $\{e, 1\}$, where 1 denotes the unit. Moreover, there exist p and q in $\mathbb{K}$ such that $e^2 + pe + q = 0$. Let $\Delta = p^2 - 4q \in \mathbb{K}$. We distinguish the following cases:

(i) Assume $\Delta = 0$. Let $\varepsilon := e + \frac{p}{2}$. Then $\varepsilon^2 = 0$ and $A = \mathbb{K}\varepsilon \oplus \mathbb{K} = \mathbb{K}[\varepsilon]$.

(ii) Assume $\Delta \neq 0$ and has even order as an element of $\mathbb{K}$. This implies that $\Delta = \hbar^{2m}(a_0 + a_1\hbar + a_2\hbar^2 + ...)$, where m is an integer, a_i are complex coefficients and $a_0 \neq 0$.

One can easily check that the equation $x^2 = a_0 + a_1\hbar + a_2\hbar^2 + ...$ has two solutions $\pm x = x_0 + x_1\hbar + x_2\hbar^2 + ...$ in $\mathbb{O}$.

Then $e = -\frac{p}{2} \pm \frac{\hbar^m x}{2}$, which implies that $e \in \mathbb{K}$ and $A = \mathbb{K} \oplus \mathbb{K}$.

(iii) Assume $\Delta \neq 0$ and has odd order as an element of $\mathbb{K}$. We have $\Delta = \hbar^{2m+1}(a_0 + a_1\hbar + a_2\hbar^2 + ...)$, where m is an integer, a_i are complex coefficients and $a_0 \neq 0$.

Again the equation $x^2 = a_0 + a_1\hbar + a_2\hbar^2 + ...$ has two solutions $\pm x = x_0 + x_1\hbar + x_2\hbar^2 + ...$ in $\mathbb{O}$. Since $a_0 \neq 0$, we have $x_0 \neq 0$ and thus x is invertible in $\mathbb{O}$.

Let $j = \hbar^{-m}(2e + p)x^{-1}$. Then $e^2 + pe + q = 0$ is equivalent to $j^2 = \hbar$. Since $A = \mathbb{K}e \oplus \mathbb{K}$ and $2e = \hbar^m xj - p$, $A = \mathbb{K}j \oplus \mathbb{K}$.

We have thus obtained the following result.

Proposition 2.3. *Let $\bar{\delta}$ be an arbitrary Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$. Then $D(\mathfrak{g}(\mathbb{K}),\bar{\delta}) \cong \mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$, where $A = \mathbb{K}[\varepsilon]$ and $\varepsilon^2 = 0$, $A = \mathbb{K} \oplus \mathbb{K}$ or $A = \mathbb{K}[j]$ and $j^2 = \hbar$.*

On the other hand, it is well-known, see for instance [3], that there is a one-to-one correspondence between Lie bialgebra structures on a Lie algebra L and Manin triples $(D(L), L, W)$. For $L = \mathfrak{g}(\mathbb{K})$, this fact implies the following

Proposition 2.4. *There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$ and Lagrangian subalgebras W of $\mathfrak{g}(\mathbb{K}) \otimes_{\mathbb{K}} A$, with respect to the non-degenerate bilinear form Q, and transversal to $\mathfrak{g}(\mathbb{K})$.*

Corollary 2.5. *(i) There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}[\varepsilon])$, $\varepsilon^2 = 0$, and Lagrangian subalgebras W of $\mathfrak{g}(\mathbb{K}[\varepsilon])$, and transversal to $\mathfrak{g}(\mathbb{K})$.*

(ii) There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ and Lagrangian subalgebras W of $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$, and transversal to $\mathfrak{g}(\mathbb{K})$, embedded diagonally into $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$.

(iii) There exists a one-to-one correspondence between Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the classical double is $\mathfrak{g}(\mathbb{K}[j])$, where $j^2 = \hbar$, and Lagrangian subalgebras W of $\mathfrak{g}(\mathbb{K}[j])$, and transversal to $\mathfrak{g}(\mathbb{K})$.

3. Lie bialgebra structures in Case I

Here we study the Lie bialgebra structures δ on $\mathfrak{g}(\mathbb{K})$ for which the corresponding Drinfeld double is isomorphic to $\mathfrak{g}(\mathbb{K}[\varepsilon])$, $\varepsilon^2 = 0$. The problem is to find all subalgebras W of $\mathfrak{g}(\mathbb{K}[\varepsilon])$ satisfying the following conditions:

(i) $W \oplus \mathfrak{g}(\mathbb{K}) = \mathfrak{g}(\mathbb{K}[\varepsilon])$.

(ii) $W = W^{\perp}$, with respect to the following non-degenerate symmetric bilinear form:

$$Q(f_1(\hbar) + \varepsilon f_2(\hbar), g_1(\hbar) + \varepsilon g_2(\hbar)) = K(f_1, g_2) + K(f_2, g_1).$$

Proposition 3.1. *Any subalgebra W of $\mathfrak{g}(\mathbb{K}[\varepsilon])$ satisfying conditions (i) and (ii) from above is uniquely defined by a subalgebra L of $\mathfrak{g}(\mathbb{K})$ together with a non-degenerate 2-cocycle B on L.*

Proof. The proof is similar to that of Th. 3.2 and Cor. 3.3 from [10]. □

Remark 3.2. We recall that a Lie algebra is called quasi-Frobenius if there exists a non-degenerate 2-cocycle on it. It is called Frobenius if the corresponding 2-cocycle is a coboundary. Thus we see that the classification problem for the Lagrangian subalgebras we are interested in contains the classification of Frobenius subalgebras of $\mathfrak{g}(\mathbb{K})$. This question is quite complicated, as it is known from studying Frobenius subalgebras of $\mathfrak{g}$. However, for $\mathfrak{g} = sl(2)$ there is only one Frobenius subalgebra, the standard parabolic one.

4. Lie bialgebra structures in Case II and Belavin-Drinfeld cohomologies

Our task is to classify Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the associated classical double is isomorphic to $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$.

Lemma 4.1. *Any Lie bialgebra structure δ on $\mathfrak{g}(\mathbb{K})$ for which the associated classical double is isomorphic to $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ is a coboundary $\delta = dr$ given by an r-matrix satisfying $r + r^{21} = f\Omega$, where $f \in \mathbb{K}^*$ and $\mathrm{CYB}(r) = 0$.*

We may suppose that $f = 1$. Naturally, we want to classify all such r up to $\mathrm{Ad}(G(\mathbb{K}))$-equivalence. Here $\mathrm{Ad}(G(\mathbb{K}))$ is a group, which acts naturally on $\mathfrak{g}(\mathbb{K})$.

Let $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$. Any Lie bialgebra structure δ over $\mathbb{K}$ can be extended to a Lie bialgebra structure $\overline{\delta}$ over $\overline{\mathbb{K}}$.

According to [1], Lie bialgebra structures on a simple Lie algebra over an algebraically closed field are coboundaries given by non-skewsymmetric r-matrices. These r-matrices have been classified up to $\mathrm{Ad}(G)$-equivalence and they are given in terms of admissible triples.

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the associated root system. We choose a system of generators e_α, $e_{-\alpha}$, h_α such that $K(e_\alpha, e_{-\alpha}) = 1$, for any positive root α. Denote by Ω_0 the Cartan part of Ω. Suppose also that $H \subset \mathrm{Ad}(G)$ is a Cartan subgoup with Lie algebra $\mathfrak{h}$.

Let us recall from [1, 3] that any non-skewsymmetric r-matrix depends on certain discrete and continuous parameters. The discrete one is an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, i.e. an isometry $\tau : \Gamma_1 \longrightarrow \Gamma_2$ where $\Gamma_1, \Gamma_2 \subset \Gamma$ such that for any $\alpha \in \Gamma_1$ there exists $k \in \mathbb{N}$ satisfying $\tau^k(\alpha) \notin \Gamma_1$. The continuous parameter is a tensor $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfying $r_0 + r_0^{21} = \Omega_0$ and $(\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0$ for any $\alpha \in \Gamma_1$. Then the associated r-matrix is given by the following formula

$$r_{BD} = r_0 + \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \sum_{\alpha \in (Span\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_\alpha \wedge e_{-\tau^k(\alpha)}.$$

Now, let us consider an r-matrix corresponding to a Lie bialgebra on $\mathfrak{g}(\mathbb{K})$. Up to $\mathrm{Ad}(G(\overline{\mathbb{K}}))$-equivalence, we have the Belavin–Drinfeld classification. We may assume that our r-matrix is of the form $r_X = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$, where $X \in G(\overline{\mathbb{K}})$ and r_{BD} satisfies the system $r + r^{21} = \Omega$ and $\mathrm{CYB}(r) = 0$. The corresponding bialgebra structure is $\delta(a) = [r_X, a \otimes 1 + 1 \otimes a]$ for any $a \in \mathfrak{g}(\mathbb{K})$.

Let us take an arbitrary $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$. Then we have $(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r_X), a \otimes 1 + 1 \otimes a]$ and $(\sigma \otimes \sigma)(\delta(a)) = \delta(a)$, which imply that $\sigma(r_X) = r_X + \lambda\Omega$, for some $\lambda \in \overline{\mathbb{K}}$. Let us show that $\lambda = 0$. Indeed, $\Omega = \sigma(\Omega) = \sigma(r_X) + \sigma(r_X^{21}) = r_X + r_X^{21} + 2\lambda\Omega$. Thus $\lambda = 0$ and $\sigma(r_X) = r_X$. Consequently, we get $(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(\sigma(r_{BD})) = r_{BD}$. We recall the following

Definition 4.2. Let r be an r-matrix. The *centralizer* $C(r)$ of r is the set of all $X \in G(\overline{\mathbb{K}})$ satisfying $(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r) = r$.

Theorem 4.3. *Let r_{BD} be an r-matrix from the Belavin–Drinfeld list for $\mathfrak{g}(\overline{\mathbb{K}})$. Suppose that $(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(\sigma(r_{BD})) = r_{BD}$. Then $\sigma(r_{BD}) = r_{BD}$ and $X^{-1}\sigma(X) \in C(r_{BD})$.*

Proof. Consider $r = r_{BD}$ which corresponds to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$ and $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$. Denote $Y := X^{-1}\sigma(X)$ and $s := r - r_0$. Then $(\mathrm{Ad}(Y) \otimes \mathrm{Ad}(Y))(s + \sigma(r_0)) = s + r_0$.

Following [7], p. 43–47, let $F(r) : \mathfrak{g} \longrightarrow \mathfrak{g}$ be the operator defined by $F(r)(x) = r' K(r'', x)$, if $r = \sum r' \otimes r''$ and K is the Killing form on $\mathfrak{g}$. Let

$$\mathfrak{g}_r^\lambda = \bigcup_{n>0} \mathrm{Ker}(F(r) - \lambda)^n.$$

Then

$$\mathfrak{g} = \mathfrak{g}_r^0 \oplus \mathfrak{g}_r' \oplus \mathfrak{g}_r^1,$$

where

$$\mathfrak{g}_r' = \bigoplus_{\lambda \neq 0,1} \mathfrak{g}_r^\lambda.$$

In our case, $\mathfrak{n}_- \subseteq \mathfrak{g}_{s+r_0}^0 \subseteq \mathfrak{b}_-$, $\mathfrak{n}_+ \subseteq \mathfrak{g}_{s+r_0}^1 \subseteq \mathfrak{b}_+$, $\mathfrak{g}_{s+r_0}' \subseteq \mathfrak{h}$, $\mathfrak{g}_{s+r_0}^0 + \mathfrak{g}_{s+r_0}' = \mathfrak{b}_-$, $\mathfrak{g}_{s+r_0}^1 + \mathfrak{g}_{s+r_0}' = \mathfrak{b}_+$, and similarly for $s + \sigma(r_0)$. It can be easily checked that

$$F(\mathrm{Ad}(Y) \otimes \mathrm{Ad}(Y))(r) = \mathrm{Ad}(Y) \circ F(r) \circ \mathrm{Ad}(Y^{-1}).$$

Hence $\mathrm{Ad}(Y)(\mathfrak{g}_{s+\sigma(r_0)}^{0,1}) = \mathfrak{g}_{s+r_0}^{0,1}$ and $\mathrm{Ad}(Y)(\mathfrak{g}_{s+\sigma(r_0)}') = \mathfrak{g}_{s+r_0}'$. Therefore $\mathrm{Ad}(Y)(\mathfrak{b}_\pm) = \mathfrak{b}_\pm$ and $\mathrm{Ad}(Y) \in \mathrm{Ad}(H)(\overline{\mathbb{K}})$. Let us analyse the equality

$$\mathrm{Ad}(Y) \otimes \mathrm{Ad}(Y)(s + \sigma(r_0)) = s + r_0.$$

It follows that

$$\mathrm{Ad}(Y) \otimes \mathrm{Ad}(Y)(s) + \sigma(r_0) = s + r_0.$$

Taking into account that $r_0, \sigma(r_0) \in H^{\otimes 2}$ and

$$(\mathrm{Ad}(Y) \otimes \mathrm{Ad}(Y))(s) = \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \sum_{\beta \in (\mathbb{Z}\Gamma_1)^+} \sum_{n>0} k_{\beta,n} e_\beta \wedge e_{-\tau^n(\beta)},$$

for some integers $k_{\beta,n}$, we deduce that $\sigma(r_0) = r_0$. Thus $\sigma(r) = r$ and $\mathrm{Ad}(Y) \in C(r)$. □

In conclusion, $r_X = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$ induces a Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$ if and only if $X \in G(\overline{\mathbb{K}})$ satisfies the condition $X^{-1}\sigma(X) \in C(r_{BD})$, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$.

Definition 4.4. Let r_{BD} be a non-skewsymmetric r-matrix from the Belavin–Drinfeld list and $C(r_{BD})$ its centralizer. We say that $X \in G(\overline{\mathbb{K}})$ is a *Belavin–Drinfeld cocycle* associated to r_{BD} if $X^{-1}\sigma(X) \in C(r_{BD})$, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$.

We denote the set of Belavin–Drinfeld cocycles associated to r_{BD} by $Z(r_{BD})$. This set is non-empty, always contains the identity.

Definition 4.5. Two cocycles X_1 and X_2 in $Z(r_{BD})$ are called *equivalent* if there exists $Q \in G(\mathbb{K})$ and $C \in C(r_{BD})$ such that $X_1 = QX_2C$.

Definition 4.6. Let $H^1_{BD}(r_{BD})$ denote the set of equivalence classes of cocycles from $Z(r_{BD})$. We call this set the *Belavin–Drinfeld cohomology* associated to the r-matrix r_{BD}. The Belavin–Drinfeld cohomology is said to be *trivial* if all cocycles are equivalent to the identity, and *non-trivial* otherwise.

We make the following remarks:

Remark 4.7. Assume that $X \in Z(r_{BD})$. Then for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$, $\sigma(X) = XC$, for some $C \in C(r_{BD})$. We get $(\mathrm{Ad}_{\sigma(X)} \otimes \mathrm{Ad}_{\sigma(X)})(r_{BD}) = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$. Consequently, $(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$ induces a Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$.

Remark 4.8. Assume that X_1 and X_2 in $Z(r_{BD})$ are equivalent. Then $X_1 = QX_2C$, for some $Q \in G(\mathbb{K})$ and $C \in C(r_{BD})$. This implies that $(\mathrm{Ad}_{X_1} \otimes \mathrm{Ad}_{X_1})(r_{BD}) = (\mathrm{Ad}_{QX_2} \otimes \mathrm{Ad}_{QX_2})(r_{BD})$. In other words, the r-matrices $(\mathrm{Ad}_{X_1} \otimes \mathrm{Ad}_{X_1})(r_{BD})$ and $(\mathrm{Ad}_{X_2} \otimes \mathrm{Ad}_{X_2})(r_{BD})$ are gauge equivalent over $\mathbb{K}$ via an element $Q \in G(\mathbb{K})$.

The above remarks imply the following result.

Proposition 4.9. *Let r_{BD} be a non-skewsymmetric r-matrix over $\overline{\mathbb{K}}$. There exists a one-to-one correspondence between $H^1_{BD}(r_{BD})$ and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ with classical double $\mathfrak{g}(\mathbb{K}) \oplus \mathfrak{g}(\mathbb{K})$ and $\overline{\mathbb{K}}$-isomorphic to $\delta(r_{BD})$.*

5. Belavin-Drinfeld cohomologies for $sl(n)$

Our next goal is to compute $H^1_{BD}(r_{BD})$ for $\mathfrak{g} = sl(n)$. We will first analyse the cohomology associated to the Drinfeld–Jimbo r-matrix r_{DJ}.

Lemma 5.1. *Let $X \in GL(n, \overline{\mathbb{K}})$. Assume that for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$, $X^{-1}\sigma(X) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Then there exist $Q \in GL(n, \mathbb{K})$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$ such that $X = QD$.*

Proof. Let $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$ and $\sigma(X) = XD_\sigma$, where $D_\sigma = \mathrm{diag}(d_1, ...d_n)$. Here d_i depend on σ. Then $\sigma(x_{ij}) = x_{ij}d_j$, for any i, j.

On the other hand, in each column of X there exists a nonzero element. Let us denote these elements by x_{i_11}, ..., x_{i_nn}. For $j = 1$, $\sigma(x_{i1}) = x_{i1}d_1$ and $\sigma(x_{i_11}) = x_{i_11}d_1$. These relations imply that $\sigma(x_{i1}/x_{i_11}) = x_{i1}/x_{i_11}$ for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$ and thus $x_{i1}/x_{i_11} \in \mathbb{K}$, for any i.

Similarly, $x_{i2}/x_{i_22} \in \mathbb{K}$,..., $x_{in}/x_{i_nn} \in \mathbb{K}$, for any i. Let $Q = (k_{ij})$ be the matrix whose elements are $k_{ij} = x_{ij}/x_{i_jj}$, for any i and j.

Thus $X = QD$, where $Q \in GL(n, \mathbb{K})$ and $D = \mathrm{diag}(x_{i_11}, ..., x_{i_nn})$.

□

Proposition 5.2. *For $\mathfrak{g} = sl(n)$, the Belavin–Drinfeld cohomology $H^1_{BD}(r_{DJ})$ associated to r_{DJ} and to the group $GL(n)$ is trivial.*

Proof. It easily follows from the proof of Theorem 4.3 that the center of r_{DJ} is $C(r_{DJ}, GL(n)) = \mathrm{diag}(n, \overline{\mathbb{K}})$. Let us show that any cocycle is equivalent to the identity. Indeed, let $X = (x_{ij})$ be a cocycle from $Z(r_{DJ})$, i.e. $X^{-1}\sigma(X) \in C(r_{DJ})$, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$.

It follows that $X^{-1}\sigma(X) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. According to Lemma 5.1, there exists $Q \in GL(n, \mathbb{K})$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$ such that $X = QD$. This proves that X is equivalent to the identity. □

It turns out that the above result is true not only for r_{DJ}. Given an arbitrary r-matrix r_{BD} from the Belavin–Drinfeld list, the corresponding cohomology is also trivial. First we will take a closer look to the centralizer $C(r_{BD})$ of an r-matrix r_{BD}. Due to Theorem 4.3, the following result holds.

Lemma 5.3. *Let r_{BD} be an arbitrary r-matrix from the Belavin–Drinfeld list. Then*

$$C(r_{BD}) \subseteq \operatorname{diag}(n, \overline{\mathbb{K}}).$$

Remark 5.4. The above result is not true for $o(2n)$ if we consider $O(2n)$ as the gauge group. However, one can easily show that in this case $C(r_{DJ}, O(2n))$ contains all diagonal matrices of $O(2n)$ (we will describe our presentation of $O(n)$ in Section 6).

For $sl(n)$ we are now able to give the exact description of $C(r_{BD})$.

Lemma 5.5. *$C(r_{BD})$ consists of all diagonal matrices $T = \operatorname{diag}(t_1, ..., t_n)$, such that $t_i = s_i s_{i+1}...s_n$, where $s_i \in \overline{\mathbb{K}}$ satisfy the condition: $s_i = s_j$ if $\alpha_i \in \Gamma_1$ and $\tau(\alpha_i) = \alpha_j$.*

Proof. Let us assume that r_{BD} is associated to an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, where $\Gamma_1, \Gamma_2 \subset \{\alpha_1, ..., \alpha_{n-1}\}$. Let $T \in C(r_{BD})$. According to Lemma 5.3, $T \in \operatorname{diag}(n, \overline{\mathbb{K}})$, therefore we put $T = \operatorname{diag}(t_1, ..., t_n)$. Now we note that $T \in C(r_{BD})$ if and only if $(\mathrm{Ad}_T \otimes \mathrm{Ad}_T)(e_{\tau^k(\alpha)} \wedge e_{-\alpha}) = e_{\tau^k(\alpha)} \wedge e_{-\alpha}$ for any $\alpha \in \Gamma_1$ and any positive integer k.

For simplicity, let us take an arbitrary $\alpha_i \in \Gamma_1$ and suppose that $\tau(\alpha_i) = \alpha_j$. We then get $t_i t_{i+1}^{-1} = t_j t_{j+1}^{-1}$. Denote $s_j := t_j t_{j+1}^{-1}$ for each $j \leq n-1$ and $s_n = t_n$. Then $t_j = s_j s_{j+1}...s_n$ and moreover $s_i = s_j$.

□

Theorem 5.6. *For $\mathfrak{g} = sl(n)$, the Belavin–Drinfeld cohomology associated to any r_{BD} is trivial. Any Lie bialgebra structure on $\mathfrak{g}(\mathbb{K})$ is of the form $\delta(a) = [r, a \otimes 1 + 1 \otimes a]$, where r is an r-matrix which is $GL(n, \mathbb{K})$–equivalent to a non-skewsymmetric r-matrix from the Belavin–Drinfeld list.*

Proof. Let X be a cocycle associated to r_{BD} which is a fixed r-matrix from the Belavin–Drinfeld list. Thus $X^{-1}\sigma(X)$ belongs to the centralizer of the r_{BD}. On the other hand, according to Lemma 5.3, $C(r_{BD}) \subseteq \operatorname{diag}(n, \overline{\mathbb{K}})$.

We then obtain that for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$, $X^{-1}\sigma(X)$ is diagonal. By Lemma 5.1, we have a decomposition $X = QD$, where $Q \in GL(n, \mathbb{K})$ and $D \in \operatorname{diag}(\overline{\mathbb{K}})$. Since $\sigma(Q) = Q$, we have $X^{-1}\sigma(X) = (QD)^{-1}\sigma(QD) = D^{-1}Q^{-1}Q\sigma(D) = D^{-1}\sigma(D)$. Recall that $X^{-1}\sigma(X) \in C(r_{BD})$. It follows that $D^{-1}\sigma(D) \in C(r_{BD})$.

Let $D = \operatorname{diag}(d_1, ..., d_n)$. Then $\operatorname{diag}(d_1^{-1}\sigma(d_1), ..., d_n^{-1}\sigma(d_n)) \in C(r_{BD})$. Denote $t_i = d_i^{-1}\sigma(d_i)$ and $T = \operatorname{diag}(t_1, ..., t_n)$. According to Lemma 5.5, $T \in C(r_{BD})$ if and only if $t_i t_{i+1}^{-1} = t_j t_{j+1}^{-1}$. Equivalently, $\sigma(d_i^{-1} d_{i+1} d_j d_{j+1}^{-1}) = d_i^{-1} d_{i+1} d_j d_{j+1}^{-1}$. It follows that $d_i^{-1} d_{i+1} d_j d_{j+1}^{-1} \in \mathbb{K}$. Let $s_i := d_i d_{i+1}^{-1}$ for any i and $s_n = d_n$. Then we get $s_j s_i^{-1} \in \mathbb{K}$.

Let us fix a root $\alpha_{i_0} \in \Gamma_1 \setminus \Gamma_2$ and let $\tau^j(\alpha_{i_0}) = \alpha_j$. Then $s_j s_{i_0}^{-1} \in \mathbb{K}$, for any j. Denote $k_j := s_j s_{i_0}^{-1}$. We have $d_j = s_j s_{j+1}...s_{n-1}s_n = k_j k_{j+1}...k_n s_{i_0}^{n-j+1}$. Let

$$K := \operatorname{diag}(k_1 k_2 ... k_n, k_2 ... k_n, ..., k_n), \quad C := \operatorname{diag}(s_{i_0}^n, s_{i_0}^{n-1}, ..., s_{i_0}).$$

Note that $D = KC$ and $K \in GL(n, \mathbb{K})$. Moreover, according to Lemma 5.5, $C \in C(r_{BD})$.

Summing up, we have obtained that if X is any cocycle associated to r_{BD}, then $X = QD = QKC$, with $QK \in GL(n, \mathbb{K})$, $C \in C(r_{BD})$. This ends the proof.

□

6. Belavin-Drinfeld cohomologies for orthogonal algebras

The next step in our investigation of Belavin–Drinfeld cohomologies is for orthogonal algebras $o(m)$. We begin with the case of Drinfeld–Jimbo r-matrix. In what follows, we will use the following split form of the orthogonal algebra $o(n,\mathbb{C})$ and $o(n,\mathbb{K})$:

$$o(n) = \{A \in gl(n) : A^T S + SA = 0\}.$$

Here S is the matrix with 1 on the second diagonal and zero elsewhere. The group

$$O(n) = \{X \in GL(n) : X^T SX = S\}$$

naturally acts on $o(n)$. However, the center of the Drinfeld–Jimbo r-matrix might be bigger than the Cartan subalgebra of $O(n)$. On the other hand, it follows from Theorem 4.3 that $C(r_{DJ}, SO(n))$ coincides with the Cartan subgroup of $SO(n)$. Our main result about Belavin-Drinfeld cohomologies for orthogonal algebras is the following:

Theorem 6.1. *Let $\mathfrak{g} = o(m)$ and r_{DJ} be the Drinfeld–Jimbo r-matrix. Then $H^1_{BD}(r_{DJ}, SO(m))$ is trivial. Moreover, if m is odd then both $H^1_{BD}(r_{DJ}, O(m))$ and $H^1_{BD}(r_{DJ}, SO(m))$ are trivial.*

Proof. (i) Assume $m = 2n$. On $\overline{\mathbb{K}}^m$ let us fix the bilinear form

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} x_i y_{m+1-i}.$$

Let $X \in O(m, \overline{\mathbb{K}})$ be a cocycle associated to r_{DJ}. Thus $X^{-1}\sigma(X) \in C(r_{DJ})$. Recall that $C(r_{DJ}) = \mathrm{diag}(m, \overline{\mathbb{K}}) \cap SO(m, \overline{\mathbb{K}})$. Therefore $X^{-1}\sigma(X) \in \mathrm{diag}(m, \overline{\mathbb{K}})$. By Lemma 5.1, one has the decomposition $X = QD$, where $Q \in GL(m, \mathbb{K})$ and $D \in \mathrm{diag}(m, \overline{\mathbb{K}})$. Let us write $D = \mathrm{diag}(d_1, ..., d_{2n})$ and denote by q_i the columns of Q. Then $X = QD$ is equivalent to $Q^T SQ = D^{-1}SD^{-1}$, which in turn gives that $B(q_i, q_{i'})d_i d_{i'} = \delta_i^{2n+1-i'}$. We get $B(q_i, q_{i'}) = 0$ if $i + i' \neq 2n + 1$ and $B(q_i, q_{2n+1-i})d_i d_{2n+1-i} = 1$. Let $k_i := B(q_i, q_{2n+1-i})$. Since $Q \in GL(2n, \mathbb{K})$, $k_i \in \mathbb{K}$. Because $k_i^{-1} = d_i d_{2n+1-i}$, it follows that $D = Q_1 D_1$, where

$$Q_1 = \mathrm{diag}(k_1^{-1}, ..., k_n^{-1}, 1..., 1) \text{ and}$$

$$D_1 = \mathrm{diag}(d_1 k_1, ..., d_n k_n, d_{n+1}, ..., d_{2n}).$$

We note that $X = (QQ_1)D_1$, $D_1 \in SO(2n)$ and hence, $D_1 \in C(r_{DJ}, SO(2n))$. Then, clearly $QQ_1 \in SO(2n, \mathbb{K})$. which proves that X is equivalent to the identity.

(ii) Assume $m = 2n + 1$. First, we note that if $X \in O(m)$ is such that $det(X) = -1$, then $det(-X) = 1$. Therefore, it follows from Theorem 4.3 that in this case $C(r_{DJ}, O(2n+1))$ consists of diagonal matrices. By Lemma 5.1, we may write again $X = QD$, where $Q \in GL(n, \mathbb{K})$ and $D \in \mathrm{diag}(m, \overline{\mathbb{K}})$.

Let $k_i := B(q_i, q_{2n+2-i}) \in \mathbb{K}$. Repeating the computations as in (i), one gets $k_i^{-1} = d_i d_{2n+2-i}$. If $i = n + 1$, $d_{n+1}^2 = k_{n+1}^{-1} \in \mathbb{K}$. This implies that either $d_{n+1} \in \mathbb{K}$ or $d_{n+1} \in j\mathbb{K}$, where $j^2 = \hbar$.

Actually we can prove that the second case is impossible.

Let us denote $R = Q^{-1}$ and its rows by r_1,...,r_{2n+1}. Then $X^T SX = S$ is equivalent to $RSR^T = DSD$, which in turn gives the following: $B(r_i, r_{i'}) = 0$, for all $i \neq i'$, $B(r_i, r_i) = d_i d_{2n+2-i}$ for all i.

Let us take an arbitrary orthogonal basis v_1,...,v_{2n+1} in $\mathbb{K}^{2n+1}$ and denote $B(v_i, v_i) = A_i$.

The matrix V with rows v_i satisfies $VSV^T = \mathrm{diag}(A_1, ..., A_{2n+1})$. This relation implies that $A_1...A_{2n+1} = (-1)^n det(V)^2 = (i^n det(V))^2$. Therefore $A_1...A_{2n+1} = l^2$ is a square of some $l \in \mathbb{K}$.

On the other hand, if M is the change of basis matrix from r_i to v_i, then

$$M^T \mathrm{diag}(A_1, ..., A_{2n+1})M = \mathrm{diag}(d_1 d_{2n+1}, ..., d_{n+1}^2, ..., d_{2n+1}d_1).$$

By taking the determinant on both sides, one obtains

$$det(M)^2 A_1...A_{2n+1} = (d_1 d_{2n+1})^2...(d_n d_{n+2})^2 d_{n+1}^2$$

which implies that d_{n+1}^2 is a square in $\mathbb{K}$, and consequently, $d_{n+1} \in \mathbb{K}$.

Let us show that X is equivalent to the trivial cocycle. Consider

$$Q_1 = \mathrm{diag}(k_1^{-1}, ..., k_n^{-1}, d_{n+1}, 1, ..., 1)$$

$$D_1 = \mathrm{diag}(d_1 k_1, ..., d_n k_n, 1, d_{n+2}..., d_{2n+1}).$$

We have $D = Q_1 D_1$ and $D_1 \in O(2n+1, \overline{\mathbb{K}})$. Thus $X = (QQ_1)D_1$, $QQ_1 \in O(2n+1, \mathbb{K})$, $D_1 \in C(r_{DJ})$, i.e. X is equivalent to the trivial cocycle, which completes the proof of triviality of $H_{BD}^1(r_{DJ}, O(m))$.

Finally, the case $H_{BD}^1(r_{DJ}, SO(2n+1))$ can be treated exactly as $H_{BD}^1(r_{DJ}, SO(2n))$.

□

We have just seen that the Belavin–Drinfeld cohomology $H_{BD}^1(r_{DJ})$ is trivial. Regarding Belavin–Drinfeld cohomology $H_{BD}^1(r_{BD}, SO(2n))$ for an arbitrary r_{BD}, we can give an example where this set is non-trivial. Let us denote the simple roots of $o(2n)$ by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, for $i < n$, $\alpha_n = \epsilon_{n-1} + \epsilon_n$, where $\{\epsilon_i\}$ is an orthonormal basis of $\mathfrak{h}^*$. Let $\Gamma_1 = \{\alpha_{n-1}\}$, $\Gamma_2 = \{\alpha_n\}$ and $\tau(\alpha_{n-1}) = \alpha_n$. Denote by r_{BD} the r-matrix corresponding to the triple $(\Gamma_1, \Gamma_2, \tau)$ and s, where $s \in \mathfrak{h} \wedge \mathfrak{h}$ satisfies $((\alpha_{n-1} - \alpha_n)) \otimes 1)(2s) = ((\alpha_{n-1} + \alpha_n)) \otimes 1)\Omega_0$.

Lemma 6.2. *The centralizer $C(r_{BD})$ consists of all diagonal matrices of the form*

$$T = \mathrm{diag}(t_1, ..., t_{n-1}, \pm 1, \pm 1, t_{n-1}^{-1}, ..., t_1^{-1}),$$

for arbitrary nonzero $t_1, t_2 \in \overline{\mathbb{K}}$.

Proof. We already know that $C(r_{BD}, SO(2n)) \subseteq \mathrm{diag}(2n, \overline{\mathbb{K}}) \cap O(2n, \overline{\mathbb{K}})$. Let $T \in C(r_{BD})$, where $T = \mathrm{diag}(t_1, ..., t_n, t_n^{-1}, ..., t_1^{-1})$. Since T commutes with r_0 and r_{DJ}, $T \in C(r_{BD})$ if and only if $(\mathrm{Ad}_T \otimes \mathrm{Ad}_T)(e_{\alpha_n} \wedge e_{\alpha_{n-1}}) = e_{\alpha_n} \wedge e_{\alpha_{n-1}}$. One can check that $(\mathrm{Ad}_T \otimes \mathrm{Ad}_T)(e_{\alpha_n} \wedge e_{\alpha_{n-1}}) = t_n^{-2} e_{\alpha_n} \wedge e_{\alpha_{n-1}}$. Therefore we get $t_n^{-2} = 1$ and the conclusion follows. □

Proposition 6.3. *Let $\mathfrak{g} = o(2n)$ and r_{BD} be the r-matrix corresponding to the triple $(\Gamma_1, \Gamma_2, \tau)$ and $s \in \mathfrak{h} \wedge \mathfrak{h}$ as above. Then $H_{BD}^1(r_{BD}, SO(2n))$ is non-trivial.*

Proof. Assume that $X^{-1}\sigma(X) \in C(r_{BD}, SO(2n))$ for all $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$. By the above lemma, $X^{-1}\sigma(X) = \mathrm{diag}(t_1, ..., t_{n-1}, \pm 1, \pm 1, t_{n-1}^{-1}, ..., t_1^{-1})$.

On the other hand, since $X^{-1}\sigma(X)$ is diagonal, it follows from Theorem 6.1 that there exist $Q \in SO(2n, \mathbb{K})$ and a diagonal matrix $D \in SO(2n, \overline{\mathbb{K}})$ such that $X = QD$. Let $D = \mathrm{diag}(s_1, ..., s_n, s_n^{-1}, ..., s_1^{-1})$. Since $Q \in O(2n, \mathbb{K})$, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$, $\sigma(Q) = Q$. We obtain $X^{-1}\sigma(X) = D^{-1}Q^{-1}Q\sigma(D) = D^{-1}\sigma(D)$, which is equivalent to the following system: $s_i^{-1}\sigma(s_i) = t_i$, for all $i \leq n-1$ and $s_n^{-1}\sigma(s_n) = \pm 1$.

Assume first that there exists σ such that $\sigma(s_n) = -s_n$. Then $s_n \in j\mathbb{K}$. One can check that X is equivalent to $X_0 = \mathrm{diag}(1, ..., 1, j, j^{-1}, 1, ..., 1)$ which is a non-trivial cocycle.

If $\sigma(s_n) = s_n$ for all $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$, then $s_n \in \mathbb{K}$. In this case,

$$D = \mathrm{diag}(s_1, ..., s_{n-1}, 1, 1, s_{n-1}^{-1}, ..., s_1^{-1}) \times \mathrm{diag}(1, ..., 1, s_n, s_n^{-1}, 1, ..., 1),$$

where the first matrix is in $C(r_{BD})$ and the second in $SO(2n, \mathbb{K})$. This proves that X is equivalent to the identity cocycle.

□

7. Lie bialgebra structures in Case III and twisted Belavin-Drinfeld cohomologies

Here we analyse the Lie bialgebra structures on $\mathfrak{g}(\mathbb{K})$ for which the corresponding Drinfeld double is isomorphic to $\mathfrak{g}(\mathbb{K}[j])$, where $j^2 = \hbar$. The question is to find those subalgebras W of $\mathfrak{g}(\mathbb{K}[j])$ satisfying the following conditions:

(i) $W \oplus \mathfrak{g}(\mathbb{K}) = \mathfrak{g}(\mathbb{K}[j])$.

(ii) $W = W^{\perp}$, with respect to the non-degenerate symmetric bilinear form Q given by

$$Q(f_1(\hbar) + jf_2(\hbar), g_1(\hbar) + jg_2(\hbar)) = K(f_1, g_2) + K(f_2, g_1).$$

We will restrict our discussion to $\mathfrak{g} = sl(n)$. We begin with the following remark. The field $\mathbb{K}[j]$ is endowed with a conjugation. For any element $a = f_1 + jf_2$, its conjugate is $\overline{a} := f_1 - jf_2$. If $A = A_1 + jB_1$ and $B = A_2 + jB_2$ are two matrices in $sl(n, \mathbb{K}[j])$, then $Q(A, B) = Tr(A_1B_2 + B_1A_2)$, i.e. the coefficient of j in $Tr(AB)$.

Lemma 7.1. *Let L be the subalgebra of $sl(n, \mathbb{K}[j])$ which consists of all matrices $Z = (z_{ik})$ satisfying $z_{ik} = \overline{z}_{n+1-i,n+1-k}$. Then L and $sl(n, \mathbb{K})$ are isomorphic via a conjugation of $sl(n, \mathbb{K}[j])$.*

Proof. Assume that $Z = (z_{ik})$ satisfies $z_{ik} = \overline{z}_{n+1-i,n+1-k}$. Then $Z = S\overline{Z}S$, where S is the matrix with 1 on the second diagonal and zero elsewhere.

Choose a matrix $X \in GL(n, \mathbb{K}[j])$ such that $\overline{X} = XS$. Then $\overline{XZX^{-1}} = XS\overline{Z}SX^{-1} = XZX^{-1}$ which implies that $XZX^{-1} \in sl(n, \mathbb{K})$. Conversely, if $A \in sl(n, \mathbb{K})$, then $Z = X^{-1}AX$ satisfies the condition $Z = S\overline{Z}S$. □

From now on we will identify $sl(n, \mathbb{K})$ with L. Let us find a complementary subalgebra to L in $sl(n, \mathbb{K}[j])$. Let us denote by H the Cartan subalgebra of L. If we identify the Cartan subalgebra of $sl(n, \mathbb{K}[j])$ with $\mathbb{K}^{2(n-1)}$, then H is a Lagrangian subspace of $\mathbb{K}^{2(n-1)}$. Choose a Lagrangian subspace H_0 of $\mathbb{K}^{2(n-1)}$ such that H_0 has trivial intersection with H. Let N^+ be the algebra of upper triangular matrices of $sl(n, \mathbb{K}[j])$ with zero diagonal. Consider $W_0 = H_0 \oplus N^+$. We immediately obtain the following

Lemma 7.2. *The subalgebra W_0 as above satisfies conditions (i) and (ii), where $sl(n, \mathbb{K})$ is identified with L as in Lemma 7.1.*

Proposition 7.3. *Any Lie bialgebra structure on $sl(n, \mathbb{K})$ for which the classical double is isomorphic to $sl(n, \mathbb{K}[j])$ is given by an r-matrix which satisfies $CYB(r) = 0$ and $r + r^{21} = j\Omega$.*

Proof. Let W_0 be as in the above lemma. By choosing two dual bases in W_0 and $sl(n, \mathbb{K})$ respectively, one can construct the corresponding r-matrix r_0 over $\overline{\mathbb{K}}$. It is easily seen that r_0 satisfies the system $CYB(r_0) = 0$ and $r_0 + r_0^{21} = j\Omega$.

Let us suppose that W is another subalgebra of $sl(n, \mathbb{K}[j])$, satisfying conditions (i) and (ii). Then the corresponding r-matrix over $\overline{\mathbb{K}}$ is obtained by choosing dual bases in W and $sl(n, \mathbb{K})$ respectively. We have $r + r^{21} = a\Omega$ for some $a \in \mathbb{K}[j]$. On the other hand, the classical double of the Lie bialgebras corresponding to r and r_0 is the same. This implies that r and r_0 are twists of each other and therefore $a = j$. □

Now, we recall that, over $\overline{\mathbb{K}}$, all r-matrices are gauge equivalent to the ones from Belavin–Drinfeld list. It follows that there exists a non-skewsymmetric r-matrix r_{BD} and $X \in GL(n, \overline{\mathbb{K}})$ such that $r = j(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$.

Consider an arbitrary $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$. Since δ is a cobracket on $sl(n, \mathbb{K})$, $(\sigma \otimes \sigma)(\delta(a)) = \delta(a)$ and $(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r), a \otimes 1 + 1 \otimes a]$.

At this point it is worth recalling that $Gal(\overline{\mathbb{K}}/\mathbb{K}) \cong \hat{\mathbb{Z}} = \varprojlim(\mathbb{Z}/n\mathbb{Z})$ (see [9]). Clearly, the subgroup $2\hat{\mathbb{Z}}$ acts trivially on $\mathbb{K}[j]$. Assume that $\sigma \in 2\hat{\mathbb{Z}}$. Exactly as in section 4, it follows that $\sigma(r) = r$ and if $r = (\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(jr_{BD})$ with $X \in GL(n, \overline{\mathbb{K}})$, then $\sigma(X) = XD(\sigma)$.

Since $Gal(\overline{\mathbb{K}}/\mathbb{K}[j]) \cong 2\hat{\mathbb{Z}} \cong \hat{\mathbb{Z}}$, we can use the same arguments as in the proof of Lemma 5.1 to obtain the following result

Lemma 7.4. *Let $X \in GL(n, \overline{\mathbb{K}})$. Assume that for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K}[j])$, $X^{-1}\sigma(X) \in \operatorname{diag}(n, \overline{\mathbb{K}})$. Then there exists $P \in GL(n, \mathbb{K}[j])$ and $D \in \operatorname{diag}(n, \overline{\mathbb{K}})$ such that $X = PD$.*

Now let us consider the action of $\sigma_2 \in Gal(\mathbb{K}[j]/\mathbb{K})$, $\sigma_2(a + bj) = a - bj := \overline{a + bj}$. Our identities imply that $\sigma_2(r) = r + \alpha\Omega$, for some $\alpha \in \overline{\mathbb{K}}$. Let us show that $\alpha = -j$. Indeed, since $r + r^{21} = j\Omega$, we also have $\sigma_2(r) + \sigma_2(r^{21}) = -j\Omega$. Combining these relations with $\sigma_2(r) = r + \alpha\Omega$, we get $\alpha = -j$ and therefore $\sigma_2(r) = r - j\Omega = -r_{21}$.

Recall now that $r = j(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$. It follows that $X \in GL(n, \overline{\mathbb{K}})$ must satisfy the identity $(\mathrm{Ad}_{X^{-1}\sigma_2(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_2(X)})(\sigma(r_{BD})) = r^{21}_{BD}$. Using the same arguments as in the proof of Theorem 4.3, we obtain

Proposition 7.5. *Any Lie bialgebra structure on $sl(n, \mathbb{K})$ for which the classical double is $sl(n, \mathbb{K}[j])$ is given by an r-matrix $r = j(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)(r_{BD})$, where r_{BD} is a non-skewsymmetric r-matrix from the Belavin–Drinfeld list and $X \in GL(n, \overline{\mathbb{K}})$ satisfies*

$$(\mathrm{Ad}_{X^{-1}\sigma_2(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_2(X)})(r_{BD}) = r^{21}_{BD}$$

$$(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD},$$

for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K}[j])$,

Definition 7.6. Let r_{BD} be a non-skewsymmetric r-matrix from the Belavin–Drinfeld list. We call $X \in G(\overline{\mathbb{K}})$ a *Belavin–Drinfeld twisted cocycle* associated to r_{BD} if $(\mathrm{Ad}_{X^{-1}\sigma_2(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma_2(X)})(r_{BD}) = r^{21}_{BD}$ and $(\mathrm{Ad}_{X^{-1}\sigma(X)} \otimes \mathrm{Ad}_{X^{-1}\sigma(X)})(r_{BD}) = r_{BD}$, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K}[j])$.

The set of Belavin–Drinfeld twisted cocycle associated to r_{BD} will be denoted by $\overline{Z}(r_{BD})$.

Now, let us restrict ourselves to the case $r_{BD} = r_{DJ}$. In order to continue our investigation, let us prove the following

Lemma 7.7. *Let S be the matrix with 1 on the second diagonal and zero elsewhere. Then*

$$r^{21}_{DJ} = (\mathrm{Ad}_S \otimes \mathrm{Ad}_S) r_{DJ}.$$

Proof. We recall that r_{DJ} is given by the following formula:

$$r_{DJ} = \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2}\Omega_0.$$

First note that $(\mathrm{Ad}_S \otimes \mathrm{Ad}_S)(e_{ij} \otimes e_{ji}) = e_{n+1-i,n+1-j} \otimes e_{n+1-j,n+1-i}$, which is a term in r^{21}_{DJ}, if $i > j$. On the other hand, since Ω_0 is the Cartan part of the invariant element Ω, we get $(\mathrm{Ad}_S \otimes \mathrm{Ad}_S)\Omega_0 = \Omega_0$. This could also be proved by using the identity $\Omega_0 = n\sum_{i=1}^n e_{ii} \otimes e_{ii} - I \otimes I$, where I denotes the identity matrix of $GL(n, \mathbb{K})$. Then $r^{21}_{DJ} = (\mathrm{Ad}_S \otimes \mathrm{Ad}_S) r_{DJ}$ holds. □

Remark 7.8. $\overline{Z}(r_{DJ})$ is non-empty. Indeed, choose $X \in GL(n, \mathbb{K}[j])$ such that $\sigma_2(X) = XS$. Then $X \in \overline{Z}(r_{DJ})$.

Corollary 7.9. *Let X be a Belavin–Drinfeld twisted cocycle associated to r_{DJ}. Then $X = PD$, where $P \in GL(n, \mathbb{K}[j])$ and $D \in \operatorname{diag}(n, \overline{\mathbb{K}})$. Moreover, $\sigma_2(P) = PSD_1$, where $D_1 \in \operatorname{diag}(n, \mathbb{K}[j])$.*

Proof. Since X is a twisted cocycle, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K}[j])$, $X^{-1}\sigma(X) \in C(r_{DJ})$. Recall that $C(r_{DJ}) = \mathrm{diag}(n, \overline{\mathbb{K}})$. By Lemma 7.4, we have $X = PD$, where $P \in GL(n, \mathbb{K}[j])$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Lemma 7.7 implies that $D_2 := S^{-1}X^{-1}\sigma_2(X) \in \mathrm{diag}(n, \overline{\mathbb{K}})$.

Since $S^{-1}D^{-1}P^{-1}\sigma_2(P)\sigma_2(D) = D_2$, we obtain $P^{-1}\sigma_2(P) = DSD_2\sigma_2(D^{-1})$.

Let $D_1 := S^{-1}DSD_2\sigma_2(D^{-1}) \in \mathrm{diag}(n, \overline{\mathbb{K}})$. Then $\sigma_2(P) = PSD_1$ and $D_1 \in \mathrm{diag}(n, \mathbb{K}[j])$. □

Definition 7.10. Let X_1 and X_2 be two Belavin–Drinfeld twisted cocycles associated to r_{DJ}. We say that they are *equivalent* if there exists $Q \in GL(n, \mathbb{K})$ and $D \in \mathrm{diag}(n, \overline{\mathbb{K}})$ such that $X_1 = QX_2D$.

Remark 7.11. Assume that X is a twisted cocycle associated to r_{DJ}. By Corollary 7.9, $X = PD$ and is equivalent to the twisted cocycle $P \in GL(n, \mathbb{K}[j])$.

Definition 7.12. Let $\overline{H}^1_{BD}(r_{DJ})$ denote the set of equivalence classes of twisted cocycles associated to r_{DJ}. We call this set the *Belavin–Drinfeld twisted cohomology* associated to the r-matrix r_{DJ}.

Remark 7.13. If X_1 and X_2 are equivalent twisted cocycles, then the corresponding r-matrices $r_1 = j(\mathrm{Ad}_{X_1} \otimes \mathrm{Ad}_{X_1})(r_{DJ})$ and $r_2 = j(\mathrm{Ad}_{X_2} \otimes \mathrm{Ad}_{X_2})(r_{DJ})$ are gauge equivalent via $Q \in GL(n, \mathbb{K})$.

Proposition 7.14. *There is a one-to-one correspondence between $\overline{H}^1_{BD}(r_{DJ})$ and gauge equivalence classes of Lie bialgebra structures on $sl(n, \mathbb{K})$ with classical double $sl(n, \mathbb{K}[j])$ and $\overline{\mathbb{K}}$-isomorphic to $\delta(r_{DJ})$.*

Let $m = [\frac{n+1}{2}]$. Denote by J the matrix with elements $a_{kk} = 1$, for $k \leq m$, $a_{kk} = -j$ for $k \geq m+1$, $a_{k,n-k+1} = 1$, for $k \leq m$ and $a_{k,n-k+1} = j$ for $k \geq m+1$.

Theorem 7.15. *For $\mathfrak{g} = sl(n)$, the Belavin–Drinfeld twisted cohomology $\overline{H}^1_{BD}(r_{DJ})$ is non-empty and consists of one element, the class of J.*

Proof. Let X be a twisted cocycle associated to r_{DJ}. By Remark 7.11, X is equivalent to a twisted cocycle $P \in GL(n, \mathbb{K}[j])$, associated to r_{DJ}. We may therefore assume from the beginning that $X \in GL(n, \mathbb{K}[j])$. We will prove that X and J are equivalent, i.e. $X = QJD'$, for some $Q \in GL(n, \mathbb{K})$ and $D' \in \mathrm{diag}(n, \mathbb{K}[j])$. The proof will be done by induction.

For $n = 2$, consider $J = \begin{pmatrix} 1 & 1 \\ j & -j \end{pmatrix}$. Suppose $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{K}[j])$ satisfies $\overline{X} = XSD$ with $D = \mathrm{diag}(d_1, d_2) \in GL(2, \mathbb{K}[j])$. The identity is equivalent to the following system: $\overline{a} = bd_1$, $\overline{b} = ad_2$, $\overline{c} = dd_1$, $\overline{d} = cd_2$. Assume that $cd \neq 0$. Let $a/c = a' + b'j$. Then $b/d = a' - b'j$. One can immediately check that $X = QJD'$, where $Q = \begin{pmatrix} a' & b' \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{K})$, $D' = \mathrm{diag}(c, d) \in \mathrm{diag}(2, \mathbb{K}[j])$.

For $n = 3$, consider $J = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ j & 0 & -j \end{pmatrix}$ and let $X = (a_{ij}) \in GL(3, \mathbb{K}[j])$ satisfy $\overline{X} = XSD$, with $D = \mathrm{diag}(d_1, d_2, d_3) \in GL(3, \mathbb{K}[j])$. The identity is equivalent to the following system: $\overline{a_{11}} = d_1a_{13}$, $\overline{a_{21}} = d_1a_{23}$, $\overline{a_{31}} = d_1a_{33}$, $\overline{a_{12}} = d_2a_{12}$, $\overline{a_{22}} = d_2a_{22}$, $\overline{a_{32}} = d_2a_{32}$, $\overline{a_{13}} = d_3a_{11}$, $\overline{a_{23}} = d_3a_{21}$, $\overline{a_{33}} = d_3a_{31}$. Assume that $a_{21}a_{22}a_{23} \neq 0$.

Let $a_{11}/a_{21} = b_{11} + b_{13}j$ and $a_{31}/a_{21} = b_{31} + b_{33}j$. Then $a_{13}/a_{23} = b_{11} - b_{13}j$ and $a_{33}/a_{23} = b_{31} - b_{33}j$. On the other hand, let $b_{12} := a_{12}/a_{22}$ and $b_{32} := a_{32}/a_{22}$. Note that $b_{12} \in \mathbb{K}$, $b_{32} \in \mathbb{K}$. One can immediately check that $X = QJD'$, where

$$Q = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 1 & 1 & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in GL(3, \mathbb{K}),\ D' = \mathrm{diag}(a_{21}, a_{22}, a_{23}) \in \mathrm{diag}(3, \mathbb{K}[j]).$$

For $n > 3$, we proceed by induction. Let us denote our matrix $J \in GL(n, \mathbb{K}[j])$ by J_n. We are going to prove that if $X \in GL(n, \mathbb{K}[j])$ satisfies $\overline{X} = XSD$, then using elementary row operations with entries from $\mathbb{F}$ and multiplying columns by proper elements from $\mathbb{K}[j]$ we can bring X to J_n.

We will need the following operations on a matrix

$$M = \{m_{pq}\} \in \mathrm{Mat}(n):$$

1. $u_n(M) = \{m_{pq},\ p, q = 2, 3, ..., n-1\} \in \mathrm{Mat}(n-2)$;
2. $g_n(M) = \{m_{pq}\} \in \mathrm{Mat}(n+2)$, where m_{pq} are already defined for $p, q = 1, 2, ...n$, $m_{00} = m_{n+1,n+1} = 1$ and the rest $m_{0,a} = m_{a,0} = m_{n+1,a} = m_{a,n+1} = 0$.

It is clear that $u_n(X)$ satisfies the twisted cocycle condition. However, its determinant might vanish. To avoid this complication, we note that columns $2, 3, ..., n-1$ of X are linearly independent. Applying elementary row operations (in fact, they are permutations) we obtain a new cocycle X_1, which is equivalent to X and such that $u_n(X_1)$ is a cocycle in $GL(n-2, \mathbb{K}[j])$. Then, by induction, there exist $Q_{n-2} \in GL(n-2, \mathbb{K})$ and a diagonal matrix D_{n-2} such that

$$Q_{n-2} \cdot u_n(X_1) \cdot D_{n-2} = J_{n-2}.$$

Let us consider $X_n = g_{n-2}(Q_{n-2}) \cdot X_1 \cdot g_{n-2}(D_{n-2})$. Clearly, X_n is a twisted cocycle equivalent to X and $u_n(X_n) = J_{n-2}$.

Applying elementary row operations with entries from $\mathbb{K}$ and multiplying by a proper diagonal matrix, we can obtain a new cocycle $Y_n = (y_{pg})$ equivalent to X with the following properties:
1. $u_n(Y_n) = J_{n-2}$;
2. $y_{12} = y_{13} = ... = y_{1,n-1} = 0$ and $y_{n2} = y_{n3} = ... = y_{n,n-1} = 0$;
3. $y_{11} = y_{1n} = 1$, here we use the fact that if $y_{pq} = 0$, then $y_{p,n+1-q} = 0$.

It follows from the cocycle condition $\overline{Y_n} = Y_n \cdot S \cdot \mathrm{diag}(h_1, ..., h_n)$ that $h_1 = h_n = 1$ and hence, $y_{n1} = \overline{y_{nn}}$.

Now, we can use the first row to achieve $y_{n1} = -y_{nn} = j$ and after that, we use the first and the last rows to annihilate $\{y_{k1},\ k = 2, ..., n-1\}$. Then the set $\{y_{kn},\ k = 2, ..., n-1\}$ will automatically vanish. We have obtained J_n from X and thus, have proved that X is equivalent to J_n.

□

Example 7.16. *For $\mathfrak{g} = sl(2)$, the Belavin–Drinfeld list of non-skewsymmetric constant r-matrices consists of only one class, $r_{DJ} = e \otimes f + \frac{1}{4} h \otimes h$, where $e = e_{12}$, $f = e_{21}$ and $h = e_{11} - e_{22}$. One can easily determine the corresponding class of gauge equivalent Lie bialgebra structures on $sl(2, \mathbb{K})$ with classical double $sl(2, \mathbb{K}[j])$ and $\overline{\mathbb{K}}$-isomorphic to $\delta(r_{DJ})$. Indeed, since any twisted cocycle is equivalent to J, it follows that a class representative is $\delta_0 = dr_0$, where*

$$r_0 = j(\mathrm{Ad}_J \otimes \mathrm{Ad}_J) r_{DJ}.$$

A straightforward computation gives

$$r_0 = \frac{j\Omega}{2} + \frac{1}{4} h \wedge e + \frac{\hbar}{4} f \wedge h.$$

We conclude that any Lie bialgebra structure on $sl(2, \mathbb{K})$ with classical double $sl(2, \mathbb{K}[j])$ is gauge equivalent to that given by dr_0, multiplied by a constant from $\mathbb{K}$.

Remark 7.17. In case $sl(2)$, it follows that r_{DJ} and r_0, multiplied by some constants of $\mathbb{K}$, provide all gauge non-equivalent Lie bialgebra structures on $sl(2, \mathbb{K})$ of types II and III and, consequently, two families of non-isomorphic Hopf algebra structures on $U(sl(2, \mathbb{C}))[[\hbar]]$. Moreover, in some sense, these two structures exhaust all Hopf algebra structures on $U(sl(2, \mathbb{C}))[[\hbar]]$ with a non-trivial Drinfeld associator (see also conjectures below).

Journal of Physics: Conference Series **563** (2014) 012030 doi:10.1088/1742-6596/563/1/012030

Remark 7.18. The next step would be to compute the Belavin–Drinfeld twisted cohomology corresponding to an arbitrary r-matrix r_{BD}. Unlike untwisted cohomology, it might happen that even $\overline{Z}(r_{BD})$ is empty as we will see in next publications.

8. Conjectures

8.1. Belavin–Drinfeld cohomology conjecture

Let $\mathfrak{g}$ be a simple Lie algebra and $G = \mathrm{Ad}(\mathfrak{g})$ be the corresponding adjoint group, which is the connected component of the group unit element modulo its center. Let $C(r_{BD})$ be the subgroup of elements of $G(\overline{\mathbb{K}})$ which act trivially on r_{BD}.

Definition 8.1. We say that $X \in G(\overline{\mathbb{K}})$ is a *Belavin–Drinfeld cocycle* associated to r_{BD} if $X^{-1}\sigma(X) \in C(\overline{\mathbb{K}}, r_{BD})$, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$.

Definition 8.2. Two Belavin–Drinfeld cocycles X_1 and X_2 are *equivalent* if $X_1 = QX_2C$, where $Q \in G(\mathbb{K})$ and $C \in C(r_{BD})$.

Let us denote the set of equivalence classes by $H^1_{BD}(r_{BD}, G)$.

Conjecture 8.3. *Let $\mathfrak{g}$ be a simple Lie algebra and r_{DJ} the Drinfeld–Jimbo r-matrix. Then $H^1_{BD}(r_{DJ}, G)$ is trivial.*

Remark 8.4. We have already proved the conjecture for $sl(n)$ and $o(n)$.

8.2. Quantization conjecture

Let L be a finite dimensional Lie algebra over $\mathbb{C}$ and δ a Lie bialgebra structure on $L(\mathbb{K})$ such that $\delta = 0(\mathrm{mod}\hbar)$. Let $(U_\hbar(L), \Delta_\hbar)$ be the corresponding quantum group. Let $G(\mathbb{K}) = \mathrm{Ad}(L(\mathbb{K}))$ and $G(\overline{\mathbb{K}}) = \mathrm{Ad}(L(\overline{\mathbb{K}}))$.

Let us define the centralizer $C(\overline{\mathbb{K}}, \delta)$. Consider the classical double $D(L(\mathbb{K}), \delta)$. Clearly, δ can be extended to a Lie bialgebra structure $\overline{\delta}$ on $L(\overline{\mathbb{K}})$ and $D(L(\overline{\mathbb{K}}), \overline{\delta})$ contains $D(L(\mathbb{K}), \delta)$, more precisely $D(L(\overline{\mathbb{K}}), \overline{\delta}) = D(L(\mathbb{K}), \delta) \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. The universal classical r-matrix $r_\delta = \sum e_i \otimes e^i$ is the same for $D(L(\mathbb{K}), \delta)$ and $D(L(\overline{\mathbb{K}}), \overline{\delta})$.

Definition 8.5. The centralizer $C(\overline{\mathbb{K}}, \delta)$ consists of all $X \in G(\overline{\mathbb{K}})$ such that

$$(\mathrm{Ad}_X \otimes \mathrm{Ad}^*_X)(r_\delta) = r_\delta + \alpha\Omega,$$

where Ω is an invariant element of $D(L(\overline{\mathbb{K}}), \overline{\delta})^{\otimes 2}$ and Ad^* is the coadjoint representation on $(L(\overline{\mathbb{K}}))^*$. Equivalently, $(\mathrm{Ad}_X \otimes \mathrm{Ad}_X)\delta(\mathrm{Ad}_X^{-1}(l)) = \delta(l)$, for any $l \in L$.

Definition 8.6. We say that $X \in G(\overline{\mathbb{K}})$ is a *Belavin–Drinfeld cocycle* associated to δ if $\sigma(X) = XC$, where $C \in C(\overline{\mathbb{K}}, \delta)$.

Two cocycles, associated to δ, X_1 and X_2 are *equivalent* if $X_1 = QX_2C$, where $Q \in G(\mathbb{K})$ and $C \in C(\overline{\mathbb{K}}, \delta)$.

The set of equivalence classes will be denoted by $H^1_{BD}(G, \delta)$.

Now let us define quantum Belavin–Drinfeld cohomology. The quantum group $(U_\hbar(L), \Delta_\hbar)$ is defined over $\mathbb{O} = \mathbb{C}[[\hbar]]$. We extend the Hopf structures of $U_\hbar(L)$ to $U_\hbar(L, \mathbb{K}) = U_\hbar(L) \otimes_{\mathbb{O}} \mathbb{K}$ and $U_\hbar(L, \overline{\mathbb{K}}) = U_\hbar(L) \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. By abuse of notation, $\Delta_\hbar$ denotes all three comultiplications.

Definition 8.7. Let P be an invertible element of $U_\hbar(L, \overline{\mathbb{K}})$. We say that it belongs to $C(U_\hbar(L), \Delta_\hbar)$ if, for all $a \in U_\hbar(L)$,

$$(P \otimes P)\Delta_\hbar(P^{-1}aP)(P^{-1} \otimes P^{-1}) = \Delta_\hbar(a).$$

Denote $F_P := (P \otimes P)\Delta_\hbar(P^{-1}) \in U_\hbar(L, \overline{\mathbb{K}})^{\otimes 2}$.

Definition 8.8. P is called a *quantum Belavin–Drinfeld cocycle* if $\sigma(P) = PC$, for any $\sigma \in Gal(\overline{\mathbb{K}}/\mathbb{K})$ and some $C \in C(U_\hbar(L), \Delta_\hbar)$.

Two quantum cocycles P_1 and P_2 are *equivalent* if $P_2 = QP_1C$ where Q is an invertible element of $U_\hbar(L, \mathbb{K})$ and $C \in C(U_\hbar(L), \Delta_\hbar)$.

Remark 8.9. On $U_\hbar(L)$ consider the comultiplications

$$\Delta_{\hbar,P_1}(a) = F_{P_1}\Delta_\hbar(a)F_{P_1}^{-1},\ \Delta_{\hbar,P_2}(a) = F_{P_2}\Delta_\hbar(a)F_{P_2}^{-1}.$$

Clearly,

$$\Delta_{\hbar,P_2}(a) = (Q \otimes Q)\Delta_{\hbar,P_1}(Q^{-1}aQ) \cdot (Q^{-1} \otimes Q^{-1}).$$

Since $Q \in U_\hbar(L(\mathbb{K}))$, it is natural to call $\Delta_{\hbar,P_1}$ and $\Delta_{\hbar,P_2}$ $\mathbb{K}$–equivalent comultiplications.

The set of equivalence classes of quantum Belavin–Drinfeld cocycles will be denoted by $H^1_{q-BD}(\Delta_\hbar)$.

Conjecture 8.10. *There is a natural correspondence between* $H^1_{BD}(G, \delta)$ *and* $H^1_{q-BD}(\Delta_\hbar)$.

Acknowledgment. The authors are grateful to V. Kac, P. Etingof and V. Hinich for valuable suggestions.

References

[1] Belavin A and Drinfeld V 1984 Triangle equations and simple Lie algebras *Math. Phys. Rev.* **4** 93
[2] Benkart G and Zelmanov E 1996 Lie algebras graded by finite root systems and intersection matrix algebras *Invent. math.* **126** 1
[3] Drinfeld V G 1997 Quantum groups *Proceedings ICM (AMS, Berkeley 1996)* **1** 798
[4] Drinfeld V G 1983 Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations. (Russian) *Dokl. Akad. Nauk SSSR* **268** 285
[5] Etingof P and Kazhdan D 1996 Quantization of Lie bialgebras I *Sel. Math. (NS)* **2** 1
[6] Etingof P and Kazhdan D 1998 Quantization of Lie bialgebras II *Sel. Math. (NS)* **4** 213
[7] Etingof P, Schiffmann O 1988 Lectures on Quantum Groups. International Press, Cambridge
[8] Montaner F, Stolin A, Zelmanov E 2010 Classification of Lie bialgebras over current algebras *Sel. Math. (NS)* **16** 935
[9] Serre, J P 1979 Local fields. Springer-Verlag, New York
[10] Stolin A 1999 Some remarks on Lie bialgebra structures on simple complex Lie algebras *Comm. Alg.* **27** (9) 4289