# Hamiltonian structure for a class of parametric coupled systems of the Korteweg-de Vries type

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**Abstract.** We obtain from a lagrangian action describing a class of coupled parametric systems of KdV type its hamiltonian structure. The Poisson algebra arises from second class constraints of the theory and the use of Dirac brackets. The coupled system has relevant applications in physics.

## 1. Introduction

Extensions of the Korteweg-de Vries equation have been object of an extensive study due to its intrinsic mathematical properties and the wide range of applications in which they appear. In particular, the coupled Korteweg-de Vries (KDV) systems are a special class of such extensions, widely used and still deserving attention in view of the relevant open problems related to them [1, 2, 3, 4, 5].

It is known, for example, that coupled KdV systems have, as the KdV equation, many interesting properties, such as associated Bäcklund transformations, multisolitonic solutions, Painlevé property, Lax pairs and hamiltonian formulations, among others.

We remind that the first hamiltonian formulation of KdV [6] was formulated in terms of the Fourier coefficients in the expansion of the defining field. It that sense we have an infinite dimensional hamiltonian equation. It was however necessary, to check the fulfillment of the Jacobi identity, for the Poisson bracket candidate. Later, the method of Dirac was used to obtain the same hamiltonian formulation [7, 8, 9]. The starting point here was a lagrangian formulation in terms of a prepotential of the field which defines the KdV equation and then, via a Legendre transformation and using the Dirac method for singular lagrangians [10], to obtain the second class constraints which induce the Poisson structure. To obtain the second hamiltonian structure it was necessary to consider the associated Miura transformation and again the Dirac method allows to obtain the other hamiltonian structure of the KdV equation. In both cases the Jacobi identity is a direct consequence of the formulation and the hamiltonian structures are the most general ones, because they are defined in terms of the prepotential.

In this work we obtain for a parametric coupled KdV system its hamiltonian structure via a lagrangian formulation and an associated Miura transformation with, of course, the associated coupled Miura system. We follow the method of Dirac for constrained systems. On the way we obtain the natural Casimirs of the theory arising directly from the Dirac construction. We notice that the above results are not a direct consequence of the complexification (or "deformation"

[11]) of the KdV equation. We also notice that this construction is the first step to construct associated pencils of Poisson structures for the given coupled KdV system.

## 2. The parametric coupled KdV system and it hamiltonian structures

The coupled KdV system to be considered is given by

$$u_t + uu_x + u_{xxx} + \lambda vv_x = 0 \tag{1}$$

$$v_t + u_x v + v_x u + v_{xxx} = 0 \tag{2}$$

where $\lambda$ is a real parameter. For the value $\lambda > 0$ the system is equivalent to two decoupled KdV equations. The $\lambda < 0$ case is equivalent to the complex version of KdV equation and finally, for $\lambda = 0$ we have a particular and interesting Hirota-Satsuma type system in the classification given in [12, 13].

In [14] a Bäcklund and a Gardner transformation for the system (1),(2) were obtained , also a permutability theorem was shown, which enables to generate new solutions from old ones. From it an infinite family of multisolitonic and periodic solutions were found.

The Miura transformation together with the Miura system associated to the coupled system (1),(2) are given by

$$u = \mu_x - \tfrac{1}{6}\mu^2 - \tfrac{\lambda}{6}\nu^2$$

$$v = \nu_x - \tfrac{1}{3}\mu\nu \tag{3}$$

and

$$\mu_t + \mu_{xxx} - \tfrac{1}{6}\mu^2\mu_x - \tfrac{\lambda}{6}\nu^2\mu_x - \tfrac{\lambda}{3}\mu\nu\nu_x = 0$$

$$\nu_t + \nu_{xxx} - \tfrac{1}{6}\mu^2\nu_x - \tfrac{\lambda}{6}\nu^2\nu_x - \tfrac{1}{3}\mu\nu\mu_x = 0. \tag{4}$$

The fundamental property of (3) is that it maps solutions of (4) into solutions of (1),(2) for every value of the parameter $\lambda$.

Now we construct the hamiltonian structure of the given coupled KdV system. We consider the two lagrangians $L_1 = \int_0^T dt \int_{-\infty}^{+\infty} dx \, \mathcal{L}_1$, $L_2 = \int_0^T dt \int_{-\infty}^{+\infty} dx \, \mathcal{L}_2$ where

$$\mathcal{L}_1 = -\frac{1}{2}w_x w_t - \frac{1}{6}w_x{}^3 + \frac{1}{2}w_{xx}{}^2 - \frac{\lambda}{2}w_x y_x{}^2 - \frac{\lambda}{2}y_x y_t + \frac{\lambda}{2}y_{xx}{}^2, \tag{5}$$

$$\mathcal{L}_2 = -\frac{1}{2}w_x y_t - \frac{1}{2}w_t y_x - \frac{1}{2}w_x^2 y_x - y_x w_{xxx} - \frac{\lambda}{6}y_x^3 \tag{6}$$

and

$$u(x,t) = w_x(x,t)$$
$$v(x,t) = y_x(x,t)$$

where $w$ and $y$ are the prepotentials for $u$ and $v$ respectively. For the lagrangian defined by (5) it is necessary to impose $\lambda \neq 0$.

By taking independent variations of $L_1$ and $L_2$ with respect to $w$ and to $y$ we obtain the field equations, they are given by (1) and (2).

We start our construction with lagrangian $L_1$. We introduce the conjugate momenta associated to $w$ and $y$, we denote them $p$ and $q$ respectively, we have

$$p = \frac{\delta\mathcal{L}_1}{\delta w_t} = -\frac{1}{2}w_x \quad , \quad q = \frac{\delta\mathcal{L}_1}{\delta y_t} = -\frac{\lambda}{2}y_x.$$

We define

$$\phi_1 \equiv p + \frac{1}{2}w_x \quad , \quad \phi_2 = q + \frac{\lambda}{2}y_x.$$

We notice that $\phi_1$ and $\phi_2$ do not have any $w_t$ nor any $y_t$ dependence, hence $\phi_1 = \phi_2 = 0$ are constraints on the phase space. It turns out that these are the only constraints on the phase space. They are second class contraints. The hamiltonian may be obtain directly from $\mathcal{L}_1$ by performing a Legendre transformation,

$$\mathcal{H}_1 = pw_t + qy_t - \mathcal{L}_1.$$

We obtain

$$\mathcal{H}_1 = \frac{1}{6}w_x^3 - \frac{1}{2}w_{xx}^2 + \frac{\lambda}{2}w_x y_x^2 - \frac{\lambda}{2}y_{xx}^2$$

and the corresponding hamiltonian $H_1 = \int_{-\infty}^{+\infty} dx\, \mathcal{H}_1$.

We introduce a Poisson structure on the phase space defined by

$$\begin{aligned}
\{w(x), p(\hat{x})\}_{PB} &= \delta(x - \hat{x}) \\
\{y(x), q(\hat{x})\}_{PB} &= \delta(x - \hat{x})
\end{aligned}$$

with all other brackets between these variables being zero.

The resulting phase space is a constrained phase space and for that reason it is necessary to introduce Dirac brackets. The Dirac brackets between two functionals $F$ and $G$ on phase space is defined as

$$\{F, G\}_{DB} = \{F, G\}_{PB} - \left\langle \left\langle \{F, \phi_i(x')\}_{PB}\, \mathbb{C}_{ij}(x', x'')\, \{\phi_j(x''), G\}_{PB} \right\rangle_{x'} \right\rangle_{x''} \tag{7}$$

where $<>_{x'}$ denotes integration on $x'$ from $-\infty$ to $+\infty$. It turns out, after some calculations, that

$$\{u(x), u(\hat{x})\}_{DB} = -\partial_x \delta(x - \hat{x}) \ , \ \{v(x), v(\hat{x})\}_{DB} = -\frac{1}{\lambda}\partial_x \delta(x - \hat{x})$$

$$\{u(x), v(\hat{x})\}_{DB} = 0.$$

We notice that this Poisson bracket is not well defined for $\lambda = 0$. We have already assume $\lambda \neq 0$.

From them we obtain the Hamilton equations, which are of course the same as (1),(2):

$$\begin{aligned}
u_t &= \{u, H_1\}_{DB} = -uu_x - u_{xxx} - \lambda v v_x \\
v_t &= \{v, H_1\}_{DB} = -u_x v - v_x u - v_{xxx}.
\end{aligned} \tag{8}$$

Moreover, we may obtain directly the Dirac bracket of any two functionals $F(u, v)$ and $G(u, v)$ from (7) using the above bracket relations for $u$ and $v$. We notice that the observables $F$ and $G$ in (7) may be functionals of $w, y, p$ and $q$, not only of $u$ and $v$. In this sense the phase space approach for singular lagrangians provides the most general space of observables. The same comment will be valid for the phase space construction using lagrangians $L_2$ and $L_1^M, L_2^M$ in the following.

We notice that that by construction $\phi_1$ and $\phi_2$ as well as any functional of them, in all the cases we have considered, are Casimirs of the Poisson structure defined in terms of the Dirac brackets. In fact,

$$\begin{aligned}
\{F, \phi_1)\}_{DB} &= 0 \\
\{F, \phi_2)\}_{DB} &= 0
\end{aligned}$$

for any functional $F$ on phase space. This is a general property of the Dirac bracket.

We now consider the lagrangian $L_2$ and its associated hamiltonian structure. In this case we denote the conjugate momenta to $w$ and $y$ by $\hat{p}$ and $\hat{q}$ respectively. We have

$$\hat{p} = -\frac{1}{2}y_x \quad , \quad \hat{q} = -\frac{1}{2}w_x.$$

The constraints become in this case

$$\widehat{\phi_1} = \hat{p} + \frac{1}{2}y_x = 0 \quad , \quad \widehat{\phi_2} = \hat{q} + \frac{1}{2}w_x = 0.$$

The corresponding Poisson brackets between $\hat{\phi}_i$ and $\hat{\phi}_j, i, j = 1, 2$, are given by

$$\left\{\widehat{\phi_1}(x), \widehat{\phi_1}(x')\right\}_{PB} = 0 \quad , \quad \left\{\widehat{\phi_2}(x), \widehat{\phi_2}(x')\right\}_{PB} = 0,$$
$$\left\{\widehat{\phi_1}(x), \widehat{\phi_2}(x')\right\}_{PB} = \partial_x \delta(x - x').$$

The corresponding construction of the Dirac brackets yields

$$\{u(x), u(\hat{x})\}_{DB} = 0 \quad , \quad \{v(x), v(\hat{x})\}_{DB} = 0,$$
$$\{u(x), v(\hat{x})\}_{DB} = -\partial_x \delta(x - \hat{x}).$$

The hamiltonian $H_2 = \int_{-\infty}^{+\infty} dx\, \mathcal{H}_2$ is given by the hamiltonian density

$$\mathcal{H}_2 = \frac{1}{2}w_x^2 y_x + y_x w_{xxx} + \frac{\lambda}{6}y_x^3.$$

The Hamilton equations

$$u_t(x) = \{u(x), H_2\}_{DB} \quad , \quad v_t(x) = \{v(x), H_2\}_{DB}$$

now using the corresponding Dirac brackets yield the same fields equations (1),(2) for any $\lambda$. We have thus constructed two hamiltonian functionals and associated Poisson bracket structures. These two hamiltonian structures arise directly from the basic lagrangians $L_1$ and $L_2$. We will now construct two additional hamiltonian structures by considering the Miura transformation.

The hamiltonians $H_1, H_2$ and $H_1^M, H_2^M$ in the following, were presented in [11].

For the Miura system given by (4) we have the lagrangians $L_1^M$, $L_2^M$ defined in terms of the lagrangians densities

$$\mathcal{L}_1^M = -\frac{1}{2}\sigma_t\sigma_x - \frac{\lambda}{2}\rho_t\rho_x - \frac{1}{2}\sigma_x\sigma_{xxx} - \frac{\lambda}{2}\rho_x\rho_{xxx} + \frac{1}{72}\sigma_x{}^4 - \frac{\lambda^2}{72}\rho_x{}^4 + \frac{\lambda}{12}\rho_x^2\sigma_x^2 \qquad (9)$$

and

$$\mathcal{L}_2^M = -\frac{1}{2}\sigma_t\rho_x - \frac{1}{2}\sigma_x\rho_t - \sigma_{xxx}\rho_x + \frac{1}{18}\sigma_x{}^3\rho_x + \frac{\lambda}{18}\rho_x{}^3\sigma_x \qquad (10)$$

where $\mu = \sigma_x, \nu = \rho_x$. As is previous case it is necessary to impose $\lambda \neq 0$ for the lagrangian defined by (9). Using the method of Dirac for singular lagrangians we can deduce the two additional hamiltonian structures corresponding to (9) and (10) for the coupled KdV system given by (1) and (2). They are given by:

$$\{u(x), u(\hat{x})\}_{DB} = \partial_{xxx}\delta(x,\hat{x}) + \frac{1}{3}u_x\delta(x,\hat{x}) + \frac{2}{3}u\partial_x\delta(x,\hat{x})$$

$$\{v(x), v(\hat{x})\}_{DB} = \frac{1}{\lambda}\partial_{xxx}\delta(x,\hat{x}) + \frac{1}{3\lambda}u_x\delta(x,\hat{x}) + \frac{2}{3\lambda}u\partial_x\delta(x,\hat{x})$$

$$\{u(x), v(\hat{x})\}_{DB} = \frac{1}{3}v_x\delta(x,\hat{x}) + \frac{2}{3}v\partial_x\delta(x,\hat{x})$$

and

$$\{u(x), u(\hat{x})\}_{DB} = \frac{\lambda}{3}v_x\delta(x,\hat{x}) + \frac{2\lambda}{3}v\partial_x\delta(x,\hat{x})$$

$$\{v(x), v(\hat{x})\}_{DB} = \frac{1}{3}v_x\delta(x,\hat{x}) + \frac{2}{3}v\partial_x\delta(x,\hat{x})$$

$$\{u(x), v(\hat{x})\}_{DB} = \partial_{xxx}\delta(x,\hat{x}) + \frac{1}{3}u_x\delta(x,\hat{x}) + \frac{2}{3}u\partial_x\delta(x,\hat{x})$$

and the corresponding hamiltonians are given by

$$H_1^M = \int_{-\infty}^{+\infty} \left(v^2 - u^2\right)\,dx$$

$$H_2^M = \int_{-\infty}^{+\infty} (-uv).$$

We recall that it is necessary to assume $\lambda \neq 0$ to consider the hamiltonian structure related with $L_1^M$.

We have then obtained four self-adjoint operators which define the hamiltonian basic structures of the coupled KdV system given by (1),(2). We mention that this is the starting construction in order to obtain a more general formulation in terms of pencils of Poisson structures, that is, families of Poisson brackets depending on a given parameter. Another interesting related problem is to consider supersymmetric extensions of the given coupled KdV system.

The functions on phase space $\phi_1$ and $\phi_2$ defining the second class for each hamiltonian structure are Casimirs of the Poisson algebra since they commute with all observables, this is a direct consequence of the Dirac bracket construction.

## 3. Conclusions

We obtained four basic self-adjoint hamiltonians for the given coupled KdV system and its corresponding hamiltonian stuctures, using the method of Dirac for singular lagrangians. We also commented that this construction is the first step in order to obtain a full hamiltonian structure for the given system in terms of pencils of Poisson structures.

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