

h-fold Symmetric quasicrystallography from affine Coxeter groups

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Abstract. One can obtain the quasicrystallographic structures from the projection of the higher dimensional lattices into 2D or 3D subspaces. Here we introduce a general technique applicable to any higher dimensional lattice described by the affine Coxeter groups. It is pointed out that the Coxeter number h and the Coxeter exponents play an important role in determining the principal planes onto which the lattice to be projected. The quasicrystal structures obtained by projection display the dihedral symmetry of order $2h$. The projection subspaces are determined by using the eigenvectors and the corresponding eigenvalues of the Cartan matrix. Examples are given for 12-fold symmetric quasicrystal structures obtained by projections of the lattices determined by the affine Coxeter-Weyl groups $W_a(F_4)$, $W_a(B_6)$, and $W_a(E_6)$. The reflection generators R_1 and R_2 of the dihedral group D_{12} can be obtained as the products of the generators of the Coxeter-Weyl groups. It is noted that the quasicrystal structures obtained from the lattices $W_a(F_4)$ and $W_a(B_6)$ are compatible with the experimental data.

1. Introduction

To describe the symmetry of a natural phenomenon the group theory plays a fundamental role. In this paper we will explore the quasicrystal structures obtained by projections of the lattices described by the Coxeter-Weyl groups. Applications of the Lie groups based on the Coxeter-Weyl groups are well known in High Energy Physics. Extension of the standard model $SU(3) \times SU(2) \times U(1)$ [1, 2, 3] of the High Energy Physics to the grand unified theories $SU(5)$ [4], $SO(10)$ [5], and E_6 [6] shed light on the unification of fundamental forces. They are intimately related to their Coxeter-Weyl groups which are the point groups of the higher dimensional lattices described by the affine extensions. One wonders whether the Coxeter-Weyl groups $W(A_4)$, $W(D_5)$, and $W(E_6)$ of these Lie groups play any special role in the study of quasicrystallography. Any Coxeter-Weyl group with Coxeter number h has a maximal dihedral subgroup of order $2h$ which acts faithfully in the Coxeter plane. The cut and project technique applied to the lattices described by the point groups $W(A_4)$ [7, 8, 9] and $W(B_5)$ [10, 11] have been already used for the description of the quasicrystallography with 5-fold symmetries. The lattice described by the affine group $W_a(B_5)$ is the simple cubic lattice in 5D and



may describe 10-fold symmetric quasicrystallography via the cut and project technique. On the other hand, the lattices generated by the affine extension of the Coxeter-Weyl group $W(D_5):C_2 \approx W(C_5) \approx W(B_5)$ (: means semi direct product) are the root and weight lattices of the group $W_a(D_5)$. This proves that the lattice $W_a(D_5)$ can be used to describe both 5-fold and 10-fold symmetric quasicrystallography. The 3D quasicrystallographic structures with icosahedral symmetry can be described by canonical projections of the 6D lattices determined by the affine extension of the Coxeter-Weyl group $W(D_6)$ which embeds the non-crystallographic group $W(H_3)$ as a maximal subgroup [12, 13, 14, 15, 16]. In a recent paper [17] it was proposed that any quasicrystallographic structure with h -fold symmetry can be determined by projections of the higher dimensional lattices onto the relevant Coxeter planes.

The recent developments after the discovery of the icosahedral quasicrystal by D. Shechtman [18] indicate that metallic alloys or other chemical compositions exhibiting 5-fold, 8-fold, 10-fold, 12-fold, and 18-fold symmetries can be composed. A general exposition can be found in the references [19, 20, 21]. A more recent report on the quasicrystallography predicts the existence of a structure of 36-fold symmetry [22]. It seems that here is no limitation on the order of the dihedral group describing the symmetry of the 2D quasicrystal structures. This reminds us the classification of the Coxeter-Weyl groups with different Coxeter numbers.

The Coxeter-Weyl groups are naturally characterized by some integers known as Coxeter exponents [23, 24, 25] and, in particular, each Coxeter-Weyl group has a Coxeter number h which can be used to describe the dihedral subgroup D_h of order $2h$. It is then quite natural to expect that the projection of the higher dimensional lattice determined by the affine Coxeter-Weyl group with Coxeter number h can describe the quasicrystallography with h -fold symmetry. After introducing a general framework, we study the projections of the lattices described by the affine extensions of the Coxeter-Weyl groups $W_a(F_4)$, $W_a(B_6)$, and $W_a(E_6)$ whose Coxeter numbers are given by $h = 12$.

For further study of quasicrystallography a set theoretic approach is initiated by Yves Meyer [26, 27] and later it is developed by Robert V. Moody [28, 29] in the name of *Model Set*.

The paper is organized as follows. Section 2 deals with a general technique for the canonical projections of the lattices determined by the simply-laced Coxeter-Weyl groups. The eigenvalues, eigenvectors and simple roots of the Cartan matrix (Gram matrix) of the Coxeter-Weyl group are used to determine the principal planes and the dihedral subgroup. In Section 3 the canonical projection of the exceptional lattice described by the affine Coxeter-Weyl group $W_a(F_4)$ has been worked out in detail. The Voronoi cell of the group $W(F_4)$ is projected onto the Coxeter plane and the method of canonical projection [30, 11] is used to obtain the 12-fold symmetric quasicrystal structure from the projection of the $W(F_4)$ lattice which is compatible with an experimental observation [31]. Section 4 involves the projections of the hypercubic lattice Z^6 described by the group $W_a(B_6)$ agreeing with the experimental result published recently [32]. In Section 5 the same technique is applied to the root and weight lattices of the group $W_a(E_6)$. Section 6 is devoted for the discussions of the relevant techniques and further extensions of the work.

2. Decomposition of the lattice spaces into principal planes and the canonical projection

Let G be the Coxeter-Dynkin diagram [25] representing the Coxeter-Weyl group of rank n described by the simple roots $\alpha_i, (i = 1, 2, \dots, n)$ with $\text{norm}(\alpha_i, \alpha_i) = 2$. The Cartan matrix (Gram matrix) is defined by the scalar product

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \quad (1)$$

For the simply-laced root systems it is a real and symmetric matrix. The weight vectors ω_i are defined by the relation $(\omega_i, \alpha_j) = \delta_{ij}$ where δ_{ij} is the Kronocker-delta. Weight vectors and the roots are related to each other by the relations

$$\alpha_i = \sum_j C_{ij} \omega_j, \quad \omega_i = \sum_j (C^{-1})_{ij} \alpha_j. \quad (2)$$

The affine extension of the Coxeter-Weyl group $W_a(G)$ can be generated by adjoining a generator r_0 representing the reflection with respect to the hyperplane bisecting the highest root. It can be determined from the extended Coxeter-Dynkin diagram of the Lie group. The root lattice G and the weight lattice G^* are defined respectively by the vectors $p = \sum_{i=1}^n b_i \alpha_i$, $b_i \in \mathbb{Z}$ and

$q = \sum_{i=1}^n a_i \omega_i$, $a_i \in \mathbb{Z}$. A standard notation for the weight vector is $q \equiv (a_1, a_2, \dots, a_n)$, with $a_i \in \mathbb{Z}$. Let

$r_i, (i=1, 2, \dots, n)$ be the reflection generator with respect to the hyperplane orthogonal to the simple root α_i . The highest weight vector $(a_1, a_2, \dots, a_n)$ with $a_i \geq 0$, designates the irreducible representation of a Lie group associated with the Coxeter-Weyl group [33]. Denote by $W(G) = \langle r_1, r_2, \dots, r_n \mid (r_i r_j)^{m_{ij}} = 1 \rangle$ the Coxeter-Weyl group generated by the reflections. The following notation $W(G)\Lambda \equiv \Lambda_G$ represents the orbit generated from the vector Λ by the Coxeter group $W(G)$.

The set of vectors $\Lambda_G = (a_1, a_2, \dots, a_n)_G$ describes the vertices of a polytope and should not be confused with the vector $(a_1, a_2, \dots, a_n)$, with $a_i \geq 0$ used for the designation of the irreducible representation of the associated Lie algebra. The set of simple roots can be decomposed into two sets in such a way that the corresponding reflection generators $r_1, r_2, \dots, r_k$ commute pairwise as well as do the set $r_{k+1}, r_{k+2}, \dots, r_n$ for some $k < n$. Define the generators R_1 and R_2 by $R_1 = r_1 r_2 \dots r_k$ and $R_2 = r_{k+1} r_{k+2} \dots r_n$ [34, 35]. It is easy to show that the generators R_1 and R_2 act as reflections on the simple roots of the Coxeter diagram $I_2(h)$. The Coxeter element $R_1 R_2$ represents a rotation of order h in the plane spanned by the simple roots of the graph $I_2(h)$. The Coxeter exponents m_i , $(i=1, 2, \dots, n)$ [34, 35] are listed in table 1. The eigenvalues of the Cartan matrix can

be written simply as $2[1 - \cos(m_i \frac{\pi}{h})]$. The lattice of even dimension n described by the affine Coxeter group of rank- n can be decomposed into principal planes determined by the pair of orthogonal unit vectors obtained from the eigenvectors of the Cartan matrix corresponding to the eigenvalues $2[1 - \cos(m_i \frac{\pi}{h})]$ and $2[1 - \cos(h - m_i) \frac{\pi}{h}]$.

Since the Cartan matrix of a simply-laced Coxeter-Weyl group is real and symmetric it follows that the eigenvalues are real and the corresponding eigenvectors are orthogonal. Let $\lambda_i, (i=1, 2, \dots, n)$ be the eigenvalue and the $\vec{X}_i$ is the corresponding normalized eigenvector satisfying the eigenvalue equation

$$\sum_j C_{ij} X_{jk} = \lambda_k X_{ik}. \quad (3)$$

The above equation then can be written as

$$(X^T C X)_{ij} = \lambda_i \delta_{ij}. \quad (4)$$

One can define the orthogonal set of unit vectors $\hat{x}_i$ by the relation

$$\hat{x}_i = \frac{1}{\sqrt{\lambda_i}} \sum_j \alpha_j X_{ji} . \quad (5)$$

Table1. Coxeter exponents and the Coxeter number.

	$m_1, m_2, \dots, m_n$	h
A_n	$1, 2, \dots, n$	$n+1$
B_n	$1, 3, 5, \dots, 2n-1$	$2n$
C_n	$1, 3, 5, \dots, 2n-1$	$2n$
D_n	$1, 3, 5, \dots, 2n-3, n-1$	$2(n-1)$
E_6	$1, 4, 5, 7, 8, 11$	12
E_7	$1, 5, 7, 9, 11, 13, 17$	18
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	30
G_2	$1, 5$	6
F_4	$1, 5, 7, 11$	12
H_3	$1, 5, 9$	10
H_4	$1, 11, 19, 29$	30
$I_2(h)$	$1, h-1$	h

The vectors in (5) form an orthonormal set in an n -dimensional Euclidean space. A general proof can be obtained for all Coxeter groups by simple modification of the proof given in the references [36, 34]. A similar relation for the non-simply laced groups can be obtained where

$$\hat{x}_i = \frac{1}{\sqrt{h\lambda_i}} \sum_j X_{ji} \frac{2\alpha_j}{(\alpha_j, \alpha_j)} .$$

Here the eigenvectors are such that the last components are all equal 1.

If we choose $m_1=1$ and $m_n=h-1$ (in the case of the Coxeter-Weyl group D_n this reads $m_1=1$ and $m_{n-1}=h-1$) then the corresponding eigenvalues will be $2[1 \mp \cos(\frac{\pi}{h})]$. Let us denote the corresponding unit vectors by $\hat{x}_1$ and $\hat{x}_n$ ($\hat{x}_{n-1}$ for D_n). Then one can show that the root vectors

$$\beta_1 = \sqrt{2}[\sin(\frac{\pi}{2h})\hat{x}_1 + \cos(\frac{\pi}{2h})\hat{x}_n], \quad \beta_n = \sqrt{2}[\sin(\frac{\pi}{2h})\hat{x}_1 - \cos(\frac{\pi}{2h})\hat{x}_n] \quad (6)$$

determine the Coxeter plane and the simple roots of the Coxeter graph $I_2(h)$ satisfy the Cartan matrix

$$\begin{pmatrix} 2 & -2\cos(\frac{\pi}{h}) \\ -2\cos(\frac{\pi}{h}) & 2 \end{pmatrix}. \quad (7)$$

It is easy to prove that the generators R_1 and R_2 act as reflections on the simple roots β_1 and β_n respectively. Consequently, the Coxeter element $R_1 R_2$ acts as a rotation by $\frac{2\pi}{h}$ in the Coxeter plane determined by the vectors β_1 and β_n . All principal planes can be compactly defined by the pair of simple roots β_i, β_{n+1-i} with

$$\begin{aligned}\beta_i &= \sqrt{2}[\sin(\frac{m_i \pi}{2h})\hat{x}_i + \cos(\frac{m_i \pi}{2h})\hat{x}_{n+1-i}], \\ \beta_{n+1-i} &= \sqrt{2}[\sin(\frac{m_i \pi}{2h})\hat{x}_i - \cos(\frac{m_i \pi}{2h})\hat{x}_{n+1-i}], \quad (i = 1, 2, \dots, \frac{n}{2})\end{aligned}\quad (8)$$

for even n . Therefore each pair of simple roots β_i, β_{n+1-i} determines a Coxeter graph $I_2(\frac{h}{m_i})$ where

the Coxeter element acts as a rotation by the angle $m_i \frac{2\pi}{h}$. The scalar product

$(\beta_i, \beta_{n+1-i}) = -2\cos(\frac{m_i \pi}{h})$ implies that the lattice space is decomposed into orthogonal set of planes

each of which is associated with a pair of Coxeter exponents satisfying $m_i + m_{n+1-i} = h$. The Cartan matrix of an arbitrary Coxeter-Weyl group can be transformed into a block-diagonal matrix where each entry is represented by a 2×2 matrix similar to (7) for even n otherwise it includes a 1×1 matrix with entry 2. The Coxeter element can be represented by a block-diagonal matrix

$$R_1 R_2 = \begin{pmatrix} M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & M_k \end{pmatrix} \quad (9)$$

where each M_k , acting on the set of orthogonal unit vectors $\hat{x}_i$, is either 1, -1 or a 2×2 matrix of the form [37, 11]

$$\begin{pmatrix} \cos(m_i \frac{2\pi}{h}) & -\sin(m_i \frac{2\pi}{h}) \\ \sin(m_i \frac{2\pi}{h}) & \cos(m_i \frac{2\pi}{h}) \end{pmatrix}. \quad (10)$$

The planes determined by (10) are called the principal planes by Engel [37]. The characteristic equation of the Coxeter element is the product of the characteristic equations obtained from (10).

Components of the root and weight vectors $p = \sum_{i=1}^n b_i \alpha_i$ and $q = \sum_{i=1}^n a_i \omega_i \equiv (a_1, a_2, \dots, a_n)$ in the orthonormal basis can be easily determined by using (5) as

$$p_i = \sqrt{\lambda_i} \sum_{j=1}^n b_j X_{ji}, \quad q_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^n a_j X_{ji}; \quad a_i, b_i \in \mathbf{Z}. \quad (11)$$

The pair of components (p_i, p_{n+1-i}) and (q_i, q_{n+1-i}) in the plane determined by the root and weight vectors represents respectively the orthogonal projections of the respective lattices. An n -dimensional Euclidean space can be decomposed as the space $E_{\parallel}$ determined by the pair of unit vectors $\hat{x}_1$ and $\hat{x}_n$ and the complement $E_{\perp}$. The canonical projection can be defined as follows: Pick up the set of vectors $V(0)$ [38, 39, 40] representing the Voronoi cell of the root lattice [41, 42] or weight lattice [41] around the origin. Projection of the Voronoi cell into the space $E_{\perp}$ determines a ball whose surface is a sphere S^{n-3} of radius R_0 which is also called window $\mathbf{K}$ [11]. One projects those lattice points which are restricted by the sphere of radius R_0 onto the plane determined by the unit vectors

$\hat{x}_1$ and $\hat{x}_n$. This is a general technique for every higher dimensional lattice. In the following sections we apply this technique for the projections of the lattices determined by the Coxeter-Weyl groups $W_a(F_4)$, $W_a(B_6)$, and $W_a(E_6)$.

The Voronoi regions of the root lattices are discussed in detail in reference [42] and can be easily summarized in our notations as follows. The Voronoi cell $V(0)$ of A_n is a polytope dual to the root polytope $(1,0,0,\dots,0,1)_{A_n}$ and can be written as the union of the orbits $(1,0,0,\dots,0,0)_{A_n} \cup (0,1,0,\dots,0,0)_{A_n} \cup \dots \cup (0,0,0,\dots,0,1)_{A_n}$. For instance, the Voronoi cell of the fcc lattice of A_3 (the root lattice) is a Catalan solid, the rhombic dodecahedron, described by the union of the polytopes $(1,0,0)_{A_3} \cup (0,1,0)_{A_3} \cup (0,0,1)_{A_3}$ dual to the root polytope $(1,0,1)_{A_3}$ [43]. Similarly the Voronoi cell of A_4 is the union of the polytopes $(1,0,0,0)_{A_4} \cup (0,1,0,0)_{A_4} \cup (0,0,1,0)_{A_4} \cup (0,0,0,1)_{A_4}$ dual to the root polytope $(1,0,0,1)_{A_4}$ [44]. One can check that for all simply laced Coxeter-Weyl groups the Voronoi cell of the root lattice is the dual polytope of the root polytope. The Voronoi cells of the weight lattices have been studied in reference [45] that can be expressed in terms of our notation. The Voronoi cell $V(0)$ of the weight lattice A_n^* [44] can be represented by the polytope $\frac{1}{n+1}(1,1,\dots,1)_{A_n}$.

3. Canonical projection of the lattice generated by the exceptional group $W_a(F_4)$

The Coxeter-Dynkin diagram of the exceptional group $W(F_4)$ is shown in figure 1. The order of the group is $|W(F_4)| = 1152$.

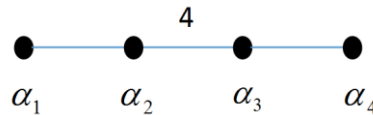


Figure 1. The Coxeter-Dynkin diagram of $W(F_4)$.

It has two long roots (first two from left) and two short roots (last two). The Cartan matrix and its inverse are given as follows

$$C_{F_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad (C_{F_4})^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}. \quad (12)$$

The dual space basis vectors are defined by the relation $(\omega_i, \frac{2\alpha_j}{(\alpha_j, \alpha_j)}) = \delta_{ij}$ where the scalar product in the dual space is defined by the metric tensor $(\omega_i, \omega_j) = G_{ij}$ which is given in the matrix form as

$$G = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 3 & \frac{3}{2} \\ 1 & 2 & \frac{3}{2} & 1 \end{pmatrix}. \quad (13)$$

The F_4 lattice is generated by its short roots and it is self-dual because the inverse matrix in (12) has integer entries only. Therefore the root and the weight lattices coincide up to an action of the Coxeter Weyl group $W_a(F_4)$. We will work with the dual space basis vectors to define a general lattice vector

$q = \sum_{i=1}^4 a_i \omega_i \equiv (a_1, a_2, a_3, a_4)$, $a_i \in \mathbf{Z}$. The F_4 polytopes have been studied in detail in reference [46].

The self-dual regular polytope of F_4 is called the 24-cell and it is either represented by the orbit $(1,0,0,0)_{F_4}$ or $(0,0,0,1)_{F_4}$. In figure 2 we display the orthogonal projection of the Voronoi cell (24-cell) of the group $W(F_4)$.

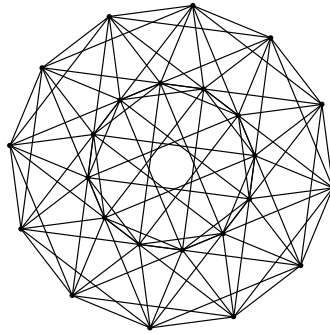


Figure 2. Orthogonal projection of the 24-cell.

The only difference between these two representations of the 24-cell is that norms of the vectors are given by $(\omega_1, \omega_1) = 2$ versus $(\omega_4, \omega_4) = 1$ and one is rotated with respect to the other within the lattice space. A quaternionic representation of the polytope $(1,0,0,0)_{F_4}$ corresponds to the quaternionic elements of the binary tetrahedral group T [47]. The F_4 lattice is generated by the polytope $(0,0,0,1)_{F_4}$ and the scaled copy of the 24-cell $\frac{1}{2}(0,0,0,1)_{F_4}$ [42] represents the Voronoi cell of the lattice. The four orthonormal unit vectors can be constructed by a similar technique studied in the previous section.

The orthogonal components of a general lattice vector in the principle planes $(\hat{x}_1, \hat{x}_4)$ and $(\hat{x}_2, \hat{x}_3)$ can be written respectively as

$$\begin{aligned} q_1 &= \frac{1}{\sqrt{2}\sqrt{2-\sqrt{2+\sqrt{3}}}\sqrt{3+\sqrt{3}}} \left[a_1 + \frac{\sqrt{3}+1}{\sqrt{2}}a_2 + \frac{\sqrt{3}+1}{2}a_3 + \frac{1}{\sqrt{2}}a_4 \right], \\ q_4 &= \frac{1}{\sqrt{2}\sqrt{2+\sqrt{2+\sqrt{3}}}\sqrt{3+\sqrt{3}}} \left[a_1 - \frac{\sqrt{3}+1}{\sqrt{2}}a_2 + \frac{\sqrt{3}+1}{2}a_3 - \frac{1}{\sqrt{2}}a_4 \right], \end{aligned} \quad (14)$$

$$\begin{aligned} q_2 &= \frac{1}{\sqrt{2}\sqrt{2-\sqrt{2-\sqrt{3}}}\sqrt{3-\sqrt{3}}} \left[a_1 + \frac{1-\sqrt{3}}{\sqrt{2}}a_2 + \frac{1-\sqrt{3}}{2}a_3 + \frac{1}{\sqrt{2}}a_4 \right], \\ q_3 &= \frac{1}{\sqrt{2}\sqrt{2+\sqrt{2-\sqrt{3}}}\sqrt{3-\sqrt{3}}} \left[a_1 - \frac{1-\sqrt{3}}{\sqrt{2}}a_2 + \frac{1-\sqrt{3}}{2}a_3 - \frac{1}{\sqrt{2}}a_4 \right]. \end{aligned} \quad (15)$$

When the Voronoi cell $\frac{1}{2}(0,0,0,1)_{F_4}$ is projected onto the plane $(\hat{x}_2, \hat{x}_3)$ it describes a disc of radius $R_0 = \frac{1}{2}(q_2^2 + q_3^2)^{\frac{1}{2}}$. One can select the lattice points which project into this region with their components q_2 and q_3 satisfying the relation $\frac{1}{2}(q_2^2 + q_3^2)^{\frac{1}{2}} \leq R_0$. Now project those points onto the plane $(\hat{x}_1, \hat{x}_4)$. The result of this projection is shown in figure 3. One can repeat the same procedure by interchanging the planes that leads to the same result since the dihedral group acts in both planes faithfully. The quasicrystal structure in figure 3 is strikingly similar observed in the quasicrystals of the Ni-Cr particles [31].

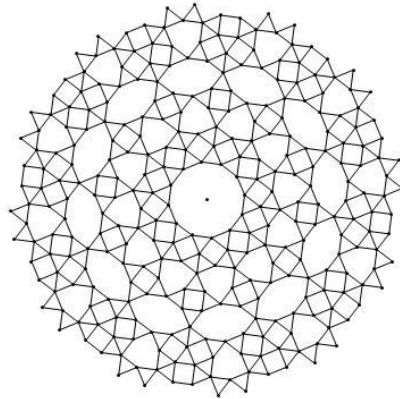


Figure 3. The canonical projection of the F_4 lattice onto the plane $(\hat{x}_1, \hat{x}_4)$.

4. Canonical projection of the lattice generated by the group $W_a(B_6)$

The Coxeter-Dynkin diagram of the group $W(B_6)$ is displayed in figure 4.

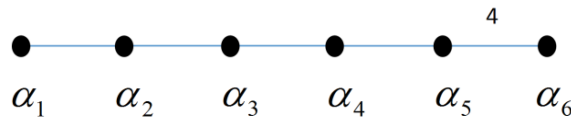


Figure 4. The Coxeter-Dynkin diagram of the Coxeter-Weyl group $W(B_6)$.

Its last root is a short root of norm 1. The Cartan matrix and its inverse are given as follows:

$$C_{B_6} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad (C_{B_6})^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 \end{pmatrix}. \quad (16)$$

The metric tensor in the dual space is given by the matrix $(\omega_i, \omega_j) = G_{ij} = (C^{-1})_{ij} \frac{(\alpha_j, \alpha_j)}{2}$.

The root lattice is a simple cubic lattice in 6D generated by the short roots of B_6 . The lattice can be represented in the weight space with the lattice vectors given by $q = \sum_{i=1}^5 a_i \omega_i + 2a_6 \equiv (a_1, a_2, \dots, 2a_6)$, $a_i \in \mathbf{Z}$. For example, the Voronoi cell is the polytope $(0,0,0,0,0,1)_{B_6}$ which represents a 6D cube with 64 vertices. Its fundamental polytopes are represented by the orbits $(1,0,0,0,0,0)_{B_6}$, $(0,1,0,0,0,0)_{B_6}$, $(0,0,1,0,0,0)_{B_6}$, $(0,0,0,1,0,0)_{B_6}$, $(0,0,0,0,1,0)_{B_6}$, and $(0,0,0,0,0,1)_{B_6}$. For example, the two polytopes $(1,0,0,0,0,0)_{B_6}$ and $(0,0,0,0,0,1)_{B_6}$ are dual of each other. Properties of the fundamental polytopes are described in table 2. The vertices of the 6D cube can also be written as $(0,0,0,0,0,1)_{B_6} = \frac{1}{2}(\pm l_1 \pm l_2 \pm l_3 \pm l_4 \pm l_5 \pm l_6)$ in terms of some suitable orthonormal set of vectors with $(l_i, l_j) = \delta_{ij}$, $(i, j = 1, 2, \dots, 6)$ where the simple roots are defined as $\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = l_3 - l_4, \alpha_4 = l_4 - l_5, \alpha_5 = l_5 - l_6, \alpha_6 = l_6$.

Table 2. Numbers of facets of some of the B_6 polytopes.

Polytope	N_0	N_1	N_2	N_3	N_4	N_5
$(1,0,0,0,0,0)_{B_6}$	12	60	160	240	192	64
$(0,1,0,0,0,0)_{B_6}$	60	480	1,120	1,200	576	76
$(0,0,1,0,0,0)_{B_6}$	160	1,440	2,880	2,160	636	76
$(0,0,0,1,0,0)_{B_6}$	240	1,920	3,200	2,080	636	76
$(0,0,0,0,1,0)_{B_6}$	192	960	1,600	1,200	444	76
$(0,0,0,0,0,1)_{B_6}$	64	192	240	160	60	12

The numbers $N_0 - N_1 + N_2 - N_3 + N_4 - N_5 = 0$ satisfy the Euler equation.

The projections of some of those polytopes onto the Coxeter plane are depicted in figure 5.

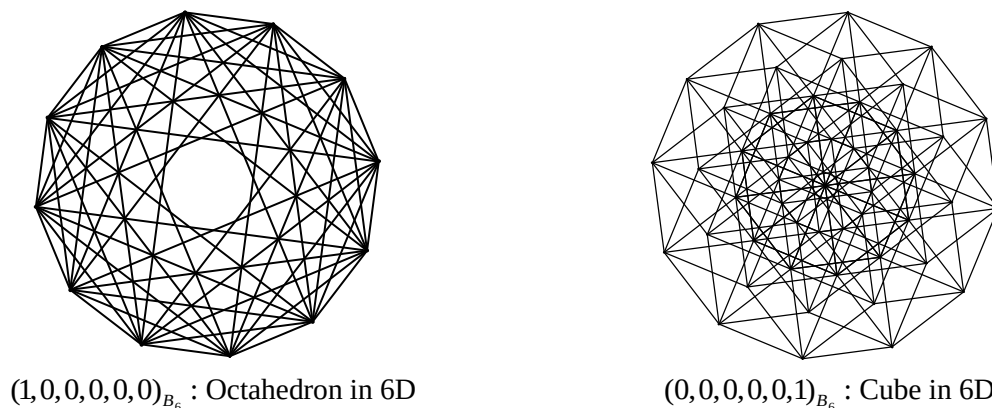


Figure 5. Orthogonal projections of some of the polytopes of $W(B_6)$.

The orthonormal set of basis vectors of the B_6 can be determined as follows

$$\begin{aligned}
\hat{x}_1 &= \frac{1}{\sqrt{12}\sqrt{2-\sqrt{2}+\sqrt{3}}} [\sqrt{2-\sqrt{3}}\alpha_1 + \alpha_2 + \sqrt{2}\alpha_3 + \sqrt{3}\alpha_4 + \sqrt{2+\sqrt{3}}\alpha_5 + 2\alpha_6], \\
\hat{x}_6 &= \frac{1}{\sqrt{12}\sqrt{2+\sqrt{2}+\sqrt{3}}} [-\sqrt{2-\sqrt{3}}\alpha_1 + \alpha_2 - \sqrt{2}\alpha_3 + \sqrt{3}\alpha_4 - \sqrt{2+\sqrt{3}}\alpha_5 + 2\alpha_6], \\
\hat{x}_2 &= \frac{1}{\sqrt{12}\sqrt{2-\sqrt{2}}} [-\sqrt{2}\alpha_1 - 2\alpha_2 - \sqrt{2}\alpha_3 + \sqrt{2}\alpha_5 + 2\alpha_6], \\
\hat{x}_5 &= \frac{1}{\sqrt{12}\sqrt{2+\sqrt{2}}} [\sqrt{2}\alpha_1 - 2\alpha_2 + \sqrt{2}\alpha_3 - \sqrt{2}\alpha_5 + 2\alpha_6], \\
\hat{x}_3 &= \frac{1}{\sqrt{12}\sqrt{2-\sqrt{2}-\sqrt{3}}} [\sqrt{2+\sqrt{3}}\alpha_1 + \alpha_2 - \sqrt{2}\alpha_3 - \sqrt{3}\alpha_4 + \sqrt{2-\sqrt{3}}\alpha_5 + 2\alpha_6] \\
\hat{x}_4 &= \frac{1}{\sqrt{12}\sqrt{2+\sqrt{2}-\sqrt{3}}} [-\sqrt{2+\sqrt{3}}\alpha_1 + \alpha_2 + \sqrt{2}\alpha_3 - \sqrt{3}\alpha_4 - \sqrt{2-\sqrt{3}}\alpha_5 + 2\alpha_6].
\end{aligned} \tag{17}$$

These sets of vectors define three orthogonal planes determined by the pair of vectors $(\hat{x}_1, \hat{x}_6), (\hat{x}_2, \hat{x}_5), (\hat{x}_3, \hat{x}_4)$ in which the Coxeter element acts like rotations with respective angles $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$. Orthogonal projections of a general lattice vector $q = \sum_{i=1}^6 a_i \omega_i$ onto these planes are given

by the following pairs.

The $(\hat{x}_1, \hat{x}_6)$ plane:

$$\begin{aligned}
q_1 &= \frac{1}{\sqrt{12}\sqrt{2-\sqrt{2}+\sqrt{3}}} [\sqrt{2-\sqrt{3}}a_1 + a_2 + \sqrt{2}a_3 + \sqrt{3}a_4 + \sqrt{2+\sqrt{3}}a_5 + 2a_6], \\
q_6 &= \frac{1}{\sqrt{12}\sqrt{2+\sqrt{2}+\sqrt{3}}} [-\sqrt{2-\sqrt{3}}a_1 + a_2 - \sqrt{2}a_3 + \sqrt{3}a_4 - \sqrt{2+\sqrt{3}}a_5 + 2a_6];
\end{aligned} \tag{18}$$

The $(\hat{x}_2, \hat{x}_5)$ plane:

$$\begin{aligned}
q_2 &= \frac{1}{\sqrt{12}\sqrt{2-\sqrt{2}}} [-\sqrt{2}a_1 - 2a_2 - \sqrt{2}a_3 + \sqrt{2}a_5 + 2a_6], \\
q_5 &= \frac{1}{\sqrt{12}\sqrt{2+\sqrt{2}}} [\sqrt{2}a_1 - 2a_2 + \sqrt{2}a_3 - \sqrt{2}a_5 + 2a_6];
\end{aligned} \tag{19}$$

The $(\hat{x}_3, \hat{x}_4)$ plane:

$$\begin{aligned}
q_3 &= \frac{1}{\sqrt{12}\sqrt{2-\sqrt{2}-\sqrt{3}}} [\sqrt{2+\sqrt{3}}a_1 + a_2 - \sqrt{2}a_3 - \sqrt{3}a_4 + \sqrt{2-\sqrt{3}}a_5 + 2a_6], \\
q_4 &= \frac{1}{\sqrt{12}\sqrt{2+\sqrt{2}-\sqrt{3}}} [-\sqrt{2+\sqrt{3}}a_1 + a_2 + \sqrt{2}a_3 - \sqrt{3}a_4 - \sqrt{2-\sqrt{3}}a_5 + 2a_6].
\end{aligned} \tag{20}$$

The Voronoi cell $V(0)$ of the $W_a(B_6)$ lattice, the 6D cube represented by the polytope $(0,0,0,0,0,1)_{B_6}$, is projected into the space $E_\perp$ determined by the 4D sub-space $(\hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5)$ defining a window $\mathbf{K}$. Projection of the strip onto the Coxeter plane $(\hat{x}_1, \hat{x}_6)$ leads to the $W_a(B_6)$ quasicrystal with 12-fold symmetry as shown in figure 6. It is interesting to note that the tiles include rhombi in addition to the usual square and triangle tiles. A recent experimental result [32] confirms this kind of tiling.

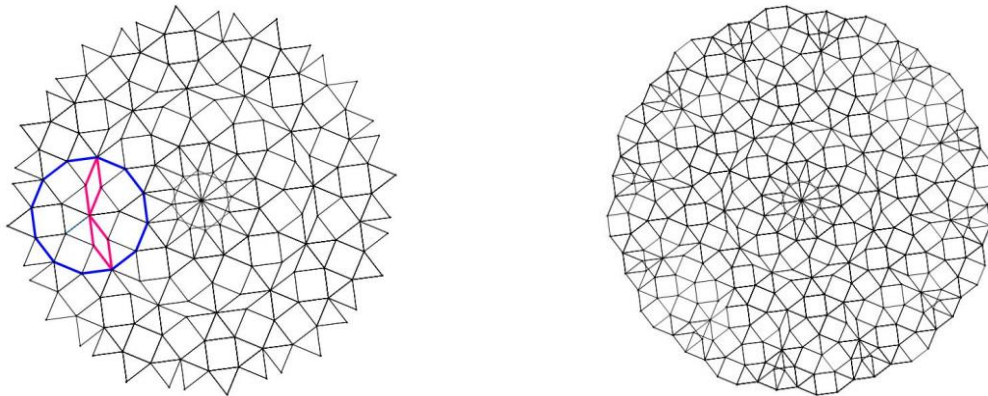


Figure 6. The canonical projection of the lattice $W_a(B_6)$ onto the plane $(\hat{x}_1, \hat{x}_6)$.

5. Canonical projections of the root and weight lattices of $W_a(E_6)$

The Coxeter-Dynkin diagram of the exceptional group $W(E_6)$ is given in figure 7. The Corresponding Cartan matrix and its inverse are given as follows:

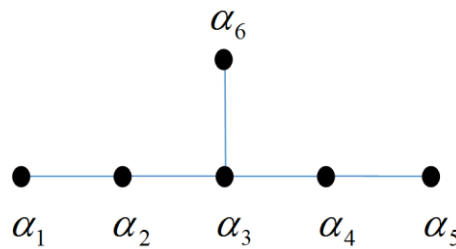


Figure 7. The Coxeter-Dynkin diagram of E_6 .

$$C_{E_6} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \quad (C_{E_6})^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{5}{3} & 2 & \frac{4}{3} & \frac{2}{3} & 1 \\ \frac{5}{3} & \frac{10}{3} & 4 & \frac{8}{3} & \frac{4}{3} & 2 \\ \frac{2}{3} & \frac{4}{3} & 6 & 4 & 2 & 3 \\ \frac{4}{3} & \frac{8}{3} & 4 & \frac{10}{3} & \frac{5}{3} & 2 \\ \frac{2}{3} & \frac{4}{3} & 2 & \frac{5}{3} & \frac{4}{3} & 1 \\ 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix}. \quad (21)$$

The order of the Coxeter-Weyl group generated by the 6, 6×6 matrices is $|W(E_6)| = 2^7 3^4 5 = 51,840$.

An arbitrary vector of the root lattice E_6 is defined by the vector $p = \sum_{i=1}^6 b_i \alpha_i$, $b_i \in \mathbb{Z}$. Similarly, a weight vector belonging to the weight lattice E_6^* is given by the vector $q = \sum_{i=1}^6 a_i \omega_i \equiv (a_1, a_2, \dots, a_6)$, $a_i \in \mathbb{Z}$ and the orbit generated by an arbitrary vector is defined by

$W(E_6)q \equiv q_{E_6} \equiv (a_1, a_2, \dots, a_6)_{E_6}$. If we denote the root and weight lattice vectors by 6-tuples $\vec{b} \equiv (b_1, b_2, \dots, b_6)$ and $\vec{a} \equiv (a_1, a_2, \dots, a_6)$ respectively then we have the relation $\vec{b}C_{E_6} = \vec{a}$. These relations imply that when $\vec{b}$ has integer components it belongs to the root lattice. It also belongs to the weight lattice since $\vec{a}$ would also take integer components. However, for $\vec{a}$ with integer components, $\vec{b}$ may not have integer components. Therefore it is clear that the weight lattice E_6^* admits the root lattice E_6 as a sub-lattice. The orthogonal unit vectors for E_6 obtained from (5) are given explicitly as follows

$$\begin{aligned}\hat{x}_1 &= \frac{1}{\sqrt{2-\sqrt{2+\sqrt{3}}}(2\sqrt{3+\sqrt{3}})}[\alpha_1 + \sqrt{2+\sqrt{3}}\alpha_2 + (1+\sqrt{3})\alpha_3 + \sqrt{2+\sqrt{3}}\alpha_4 + \alpha_5 + \sqrt{2}\alpha_6], \\ \hat{x}_6 &= \frac{1}{\sqrt{2+\sqrt{2+\sqrt{3}}}(2\sqrt{3+\sqrt{3}})}[-\alpha_1 + \sqrt{2+\sqrt{3}}\alpha_2 - (1+\sqrt{3})\alpha_3 + \sqrt{2+\sqrt{3}}\alpha_4 - \alpha_5 + \sqrt{2}\alpha_6], \\ \hat{x}_2 &= \frac{1}{2}[-\alpha_1 - \alpha_2 + \alpha_4 + \alpha_5], \\ \hat{x}_5 &= \frac{1}{2\sqrt{3}}[-\alpha_1 + \alpha_2 - \alpha_4 + \alpha_5], \\ \hat{x}_3 &= \frac{1}{\sqrt{2-\sqrt{2-\sqrt{3}}}(2\sqrt{3-\sqrt{3}})}[-\alpha_1 - \sqrt{2-\sqrt{3}}\alpha_2 - (1-\sqrt{3})\alpha_3 - \sqrt{2-\sqrt{3}}\alpha_4 - \alpha_5 + \sqrt{2}\alpha_6], \\ \hat{x}_4 &= \frac{1}{\sqrt{2+\sqrt{2-\sqrt{3}}}(2\sqrt{3-\sqrt{3}})}[\alpha_1 - \sqrt{2-\sqrt{3}}\alpha_2 + (1-\sqrt{3})\alpha_3 - \sqrt{2-\sqrt{3}}\alpha_4 + \alpha_5 + \sqrt{2}\alpha_6].\end{aligned}\tag{22}$$

We define three orthogonal planes $(\hat{x}_1, \hat{x}_6)$, $(\hat{x}_2, \hat{x}_5)$, and $(\hat{x}_3, \hat{x}_4)$ of the graph $I_2(\frac{h}{m_i})$ with $m_1=1$, $m_2=4$, $m_3=5$ respectively. The generators R_1 and R_2 act like reflection generators on these principal planes and the corresponding Coxeter element acts like rotations by angles $\frac{\pi}{6}$, $\frac{2\pi}{3}$, and $\frac{5\pi}{6}$ respectively.

Before we study the details of the canonical projection of the E_6 lattices we would like to study the projections of certain polytopes of the group $W(E_6)$. The fundamental polytopes are characterized by the orbits $(1,0,0,0,0,0)_{E_6}$, $(0,1,0,0,0,0)_{E_6}$, $(0,0,1,0,0,0)_{E_6}$, $(0,0,0,1,0,0)_{E_6}$, $(0,0,0,0,1,0)_{E_6}$, and $(0,0,0,0,0,1)_{E_6}$. Their vertices are the vectors of the weight lattice. Two particular polytopes belong to both lattices: the root polytope $(0,0,0,0,0,1)_{E_6}$ and the polytope $(0,0,1,0,0,0)_{E_6}$. These two polytopes have a larger symmetry $Aut(E_6) \approx W(E_6):\gamma$ where γ represents the Dynkin diagram symmetry of order 2. The characteristic properties of these polytopes, such as, numbers of vertices, edges, 2-facets, 3-facets, 4-facets, and 5-facets (denoted respectively by $N_0, N_1, N_2, N_3, N_4, N_5$) are listed in table 3. Their projections onto the plane $(\hat{x}_1, \hat{x}_6)$ are illustrated in figure 9. The dual of the root polytope $(0,0,0,0,0,1)_{E_6}$ is the union of the polytopes $(1,0,0,0,0,0)_{E_6} \cup (0,0,0,0,1,0)_{E_6}$ and it constitutes the Voronoi cell of the root lattice with $N_0=54$, $N_1=702$, $N_2=2,160$, $N_3=2,160$, $N_4=720$, $N_5=72$. The root polytope as well as its dual has a larger symmetry $Aut(E_6) \approx W(E_6):\gamma$. Their projections are shown in figure 8.

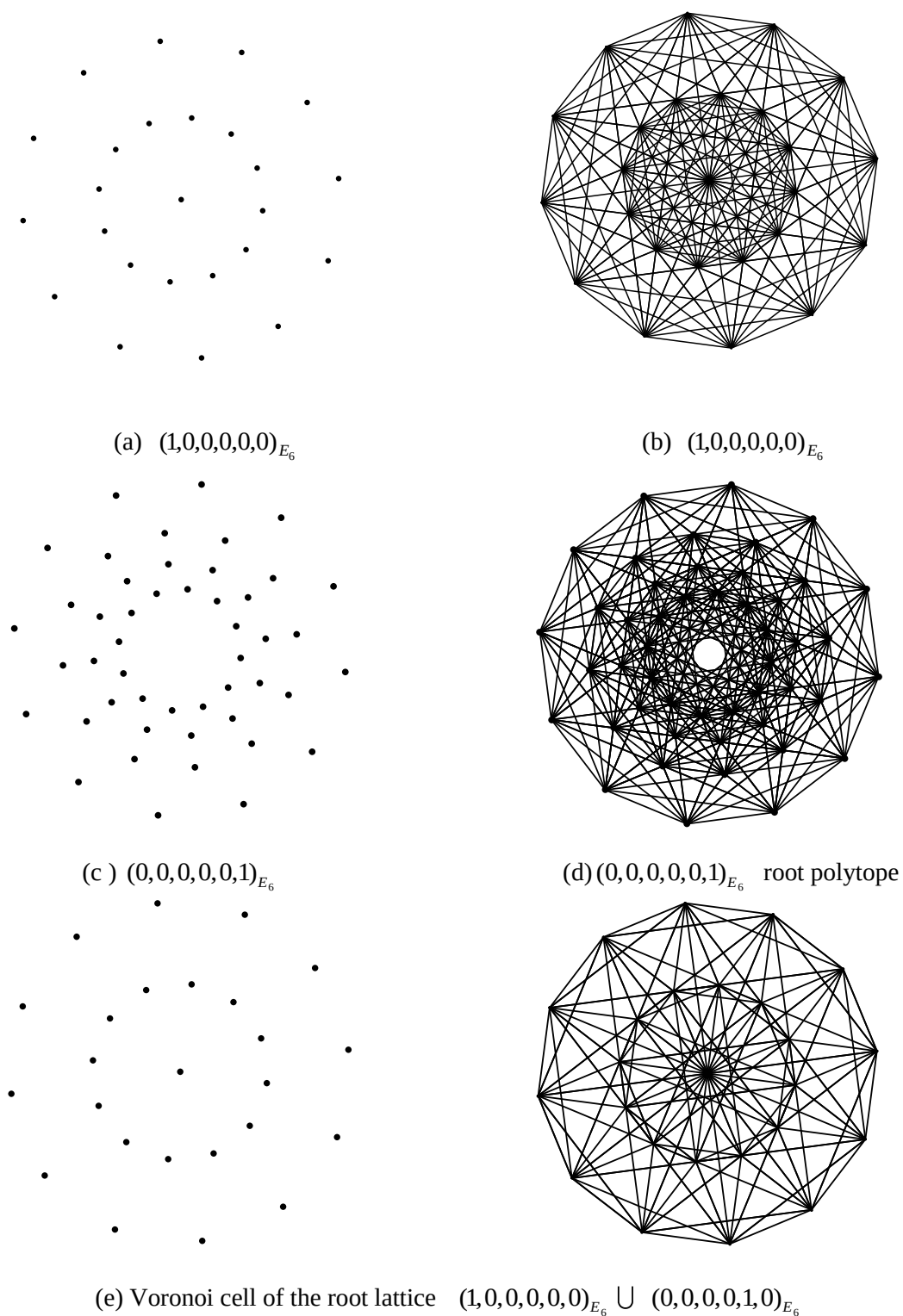


Figure 8. Projections of some of E_6 polytopes onto the Coxeter plane.

Table 3. Numbers of facets of some of the E_6 polytopes.

Polytope	N_0	N_1	N_2	N_3	N_4	N_5
$(1,0,0,0,0,0)_{E_6}$	27	216	720	1,080	648	99
$(0,1,0,0,0,0)_{E_6}$	216	2,160	5,040	4,320	1,350	126
$(0,0,1,0,0,0)_{E_6}$	720	6,480	10,800	6,480	1,566	126
$(0,0,0,1,0,0)_{E_6}$	216	2,160	5,040	4,320	1,350	126
$(0,0,0,0,1,0)_{E_6}$	27	216	720	1,080	648	99
$(0,0,0,0,0,1)_{E_6}$	72	720	2,160	2,160	702	54

Let us choose the plane $(\hat{x}_1, \hat{x}_6)$ as $E_{\parallel}$ and the rest as $E_{\perp}$. As we have explained in Section 2 the canonical projection is carried out by projecting the Voronoi cell $V(0)$ of the root/weight lattice into the space $E_{\perp}$ to obtain a window $\mathbf{K}$. The Voronoi cell of the weight lattice E_6^* [45, 48, 49] is a polytope which can be specified as $\frac{1}{3}(0,0,1,0,0,0)_{E_6}$.

The 720 vertices of the polytope are determined by the action of the group elements on the vector $\frac{1}{3}(0,0,1,0,0,0)$ and using the formula

$$R_0 = \frac{1}{3}(q_2^2 + q_3^2 + q_4^2 + q_5^2)^{\frac{1}{2}} \quad (23)$$

one obtains the window $\mathbf{K}$ in the form of a 3-sphere. We illustrate the projection of the root lattice in figure 9. They are the same point distributions in which (b) represents the tiling with minimal distance and (c) represents a partial tiling which is exactly the same tiling obtained from the projection of F_4 lattice that is a sublattice in the root lattice E_6 .

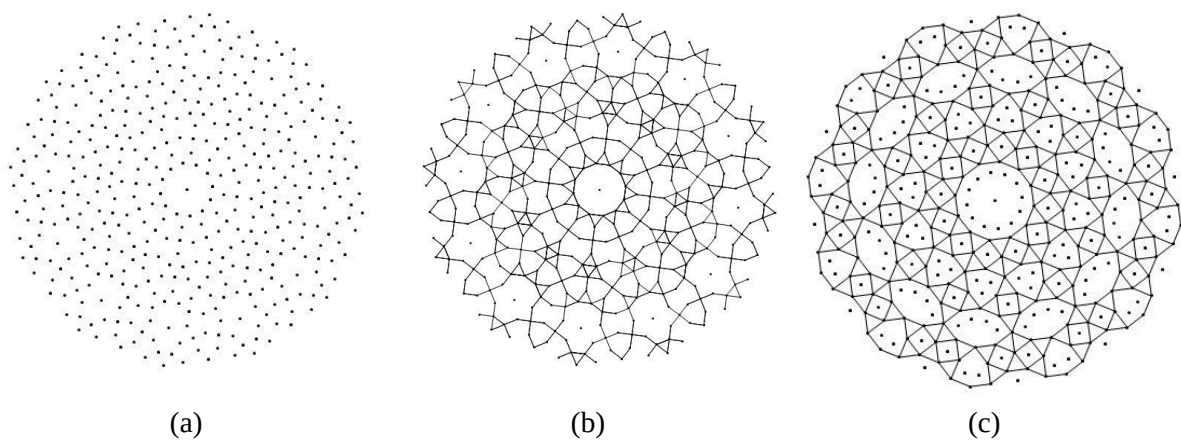


Figure 9. Projection of root lattice of E_6 .

The projection of the weight lattice is depicted in figure 10.

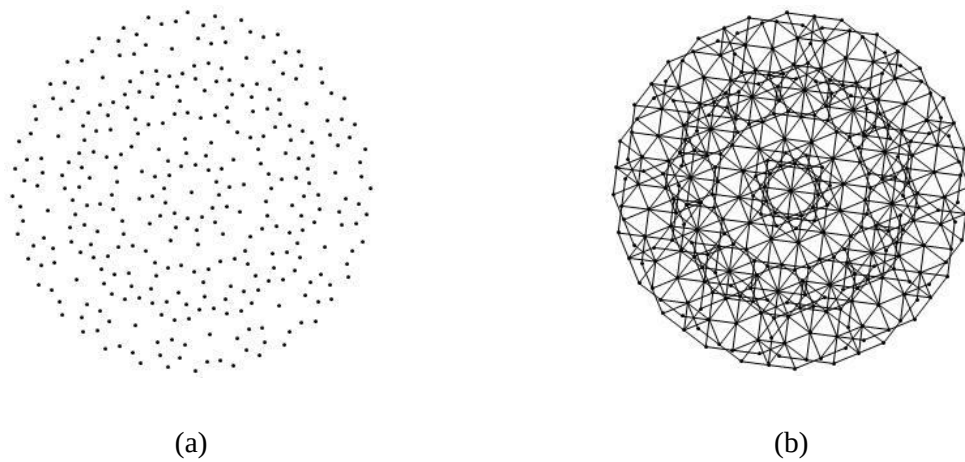


Figure 10. Canonical projection of the weight lattice of E_6 onto the plane $(\hat{x}_1, \hat{x}_6)$.

Now let us define the $E_{\parallel}$ plane by the pair of vectors $(\hat{x}_2, \hat{x}_5)$. The representation of the Coxeter group in the plane $(\hat{x}_2, \hat{x}_5)$ is such that $(R_1 R_2)^3 = 1$. Therefore the dihedral group is represented by the group $D_3 \approx S_3 \approx W(A_2)$ which is not maximal. Consequently, the 4-dimensional Euclidean sub-space $E_{\perp}$ is determined by the orthogonal planes $(\hat{x}_1, \hat{x}_6)$ and $(\hat{x}_3, \hat{x}_4)$. This time we consider only the weight lattice projection in which the window K is determined by the projection of the Voronoi cell $\frac{1}{3}(0, 0, 1, 0, 0, 0)_{E_6}$ into the space $E_{\perp}$. After repeating the previous procedure we project the lattice points into the plane determined by the vectors $(\hat{x}_2, \hat{x}_5)$ and obtain the planar lattice as shown in figure 11. It is a honeycomb lattice as expected because the projection has been made onto a plane where the crystallographic subgroup $W(A_2) \subset W(E_6)$ acts. It is interesting to observe that the honeycomb lattice exists in nature as the graphene sheet.

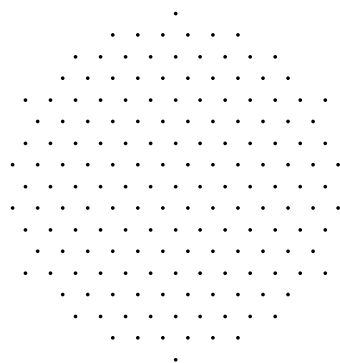


Figure 11. The honeycomb lattice obtained by projection of E_6 lattice.

6. Conclusion

This paper has introduced a general technique applicable to the projections of all higher dimensional lattices generated by the affine Coxeter-Weyl groups. It was shown that the eigenvalues, the corresponding eigenvectors, and the simple roots of the Cartan matrix (Gram matrix) play an important role in the projection technique. The maximal dihedral subgroup of the Coxeter-Weyl group can be generated by two generators whose product defines the Coxeter element of the Coxeter-

Weyl group. It was noted that the Coxeter group acts, in the lattice space spanned by the orthogonal unit vectors, as a reducible representation of the dihedral group in the block diagonal form of 2×2 irreducible representations and/or as 1×1 matrix depending on the rank of the Coxeter-Weyl group. If the dihedral subgroup acting on the principal plane (Coxeter plane) is non-crystallographic the point distribution represents a quasicrystal with h -fold symmetry where h is the Coxeter number; otherwise it represents a crystallographic point distribution such as square and/or hexagonal lattices. We have studied examples of 12-fold symmetry by projection of the lattices $W_a(F_4)$, $W_a(B_6)$, and $W_a(E_6)$. It turned out that the structures obtained by the projections of the lattices $W_a(F_4)$ and $W_a(B_6)$ are compatible with the 12-fold symmetric quasicrystal structure observed in Ni-Cr particles [31] and in dodecagonal quasicrystal formation BaTiO₃ [32] respectively. The projection technique we introduced can be applied to any lattice described by the affine Coxeter group.

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