Bethe vectors for XXX-spin chain

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Abstract. The paper deals with algebraic Bethe ansatz for XXX-spin chain. Generators of Yang-Baxter algebra are expressed in basis of free fermions and used to calculate explicit form of Bethe vectors. Their relation to N-component models is used to prove conjecture about their form in general. Some remarks on inhomogeneous XXX-spin chain are included.

1. Introduction

Algebraic Bethe ansatz has turned out as remarkably sufficient tool in the theory of quantum integrable systems. Its origins come up to the 80's and are connected mainly with Leningrad shool. Since that time, it was used successfuly to solve an amount of models.

In this text we are going to discuss some features of well-known model solved by this method, the so called XXX-spin chain.

In section 2, we repeat some properties of XXX-spin chain in terms of algebraic Bethe ansatz. In sections 3, we introduce free fermions and use them in section 4 to express generators of Yang-Baxter algebra. The aim of this text is to calculate an explicit form of Bethe vectors for XXX-spin chain in fermionic basis which is discussed in sections 5. To prove our conjectures about their form, we are forced to use N-component model in section 6. At the end of this text, in section 7, we mention some results for inhomogeneous case.

2. Algebraic Bethe ansatz for XXX-spin chain

Suppose we have a chain of L nodes. A local Hilbert space $h_j = \mathbb{C}^2$ corresponds to the j-th node. The total Hilbert space of the chain is

$$\mathscr{H} = \prod_{j=1}^{L} \otimes h_j = \prod_{j=1}^{L} \otimes \mathbb{C}^2. \tag{1}$$

Let A be an operator acting on $h = \mathbb{C}^2$. We use the following notation for operator A acting nontrivially only in the space $h_j \subset \mathscr{H}$ and trivially in others throughout the text

$$A_j = \mathbb{I}^{\otimes(j-1)} \otimes A \otimes \mathbb{I}^{\otimes(L-j)}. \tag{2}$$

The basic tool of algebraic Bethe ansatz is Lax operator which is an parameter depending object acting on the tensor product $V_a \otimes h_i$

$$L_{a,i}(\lambda): \quad V_a \otimes h_i \to V_a \otimes h_i \tag{3}$$

explicitly defined as

$$L_{a,i}(\lambda) = (\lambda + \frac{1}{2})\mathbb{I}_{a,i} + \sum_{\alpha=1}^{3} \sigma_a^\alpha S_i^\alpha \tag{4}$$

where σ_a^α are usual Pauli matrices acting in V_a, $S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$ are spin operators on the i-th node and $\mathbb{I}_{a,i}$ is identity matrix in $V_a \otimes h_i$. $L_{a,i}(\lambda)$ can be expressed as a matrix in the auxiliary space V_a

$$L_{a,i}(\lambda) = \begin{pmatrix} \lambda + \frac{1}{2} + S_i^z & S_i^- \\ S_i^+ & \lambda + \frac{1}{2} - S_i^z \end{pmatrix}. \tag{5}$$

Its matrix elements form an associative algebra of local operators acting in the quantum space h_i. Introducing permutation operator P

$$P = \frac{1}{2}\left(\mathbb{I} \otimes \mathbb{I} + \sum_{\alpha=1}^{3} \sigma^\alpha \otimes \sigma^\alpha\right) \tag{6}$$

(here $\mathbb{I}$ denotes 2×2 unit matrix), we can rewrite it as

$$L_{a,i}(\lambda) = \lambda \mathbb{I}_{a,i} + P_{a,i}. \tag{7}$$

Assume two Lax operators $L_{a,i}(\lambda)$ resp. $L_{b,i}(\mu)$ in the same quantum space h_i but in different auxiliary spaces V_a resp. V_b. The product of $L_{a,i}(\lambda)$ and $L_{b,i}(\mu)$ makes sense in the tensor product $V_a \otimes V_b \otimes h_i$. It turns out that there is an operator $R_{ab}(\lambda - \mu)$ acting nontrivially in $V_a \otimes V_b$ which intertwines Lax operators in the following way

$$R_{ab}(\lambda - \mu)L_{a,i}(\lambda)L_{b,i}(\mu) = L_{b,i}(\mu)L_{a,i}(\lambda)R_{ab}(\lambda - \mu). \tag{8}$$

Relation (8) describes the so called fundamental commutation relation. The explicit expression for $R_{ab}(\lambda - \mu)$ is

$$R_{ab}(\lambda - \mu) = (\lambda - \mu)\mathbb{I}_{a,b} + P_{a,b} \tag{9}$$

where $\mathbb{I}_{a,b}$ resp. $P_{a,b}$ is identity resp. permutation operator in $V_a \otimes V_b$. In matrix form

$$R_{ab}(\lambda - \mu) = \begin{pmatrix} \lambda - \mu + 1 & 0 & 0 & 0 \\ 0 & \lambda - \mu & 1 & 0 \\ 0 & 1 & \lambda - \mu & 0 \\ 0 & 0 & 0 & \lambda - \mu + 1 \end{pmatrix}. \tag{10}$$

The operator $R_{ab}(\lambda - \mu)$ is called R-matrix and satisfies Yang-Baxter equation

$$R_{ab}(\lambda - \mu)R_{ac}(\lambda)R_{bc}(\mu) = R_{bc}(\mu)R_{ac}(\lambda)R_{ab}(\lambda - \mu) \tag{11}$$

in $V_a \otimes V_b \otimes V_c$. Comparing (7) and (9), we see that Lax operator and R-matrix have exactly the same form.

Yang-Baxter algebra of global operators acting on the Hilbert space $\mathscr{H}$ of the full chain is defined via a monodromy matrix

$$T_a(\lambda) = L_{a,1}(\lambda)L_{a,2}(\lambda)\dots L_{a,L}(\lambda) \tag{12}$$

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which is a product of Lax operators along the chain, i.e. over all quantum spaces h_i. Matrix elements of $T_a(\lambda)$ in the auxiliary space V_a

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{13}$$

provide generators of Yang-Baxter algebra.

Equation (8) provides the following relation for monodromy matrix

$$R_{ab}(\lambda - \mu)T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)R_{ab}(\lambda - \mu) \tag{14}$$

which describes commutation relations for generators of Yang-Baxter algebra $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$. We call it global fundamental commutation relation. Equation (14) implies commutativity of transfer matrices

$$\tau(\lambda)\tau(\mu) = \tau(\mu)\tau(\lambda) \tag{15}$$

where transfer matrix $\tau(\lambda)$ is defined as the trace of monodromy matrix over auxiliary space V_a

$$\tau(\lambda) \equiv \mathrm{Tr}_a T_a(\lambda) = A(\lambda) + D(\lambda). \tag{16}$$

Obviously, transfer matrix $\tau(\lambda)$ is a polynomial of degree L in λ

$$\tau(\lambda) = 2\Big(\lambda + \frac{1}{2}\Big)^L + \sum_{k=0}^{L-2} \lambda^k Q_k. \tag{17}$$

Due to commutativity (15) of transfer matrices, we see that operators Q_k mutually commute

$$[Q_j, Q_k] = 0. \tag{18}$$

Hamiltonian of the chain is expresed in terms of Pauli matrices resp. permutation operators

$$H = \sum_{k=1}^{L} \sum_{\alpha=1}^{3} S_k^\alpha S_{k+1}^\alpha = \frac{1}{2} \sum_{k=1}^{L} P_{k,k+1} - \frac{L}{4} \tag{19}$$

where we impose cyclic condition $S_{L+1} = S_1$ resp. $P_{L,L+1} = P_{L,1}$. It can be expressed as a function of transfer matrix

$$H = \frac{1}{2}\frac{d}{d\lambda} \ln \tau(\lambda)\Big|_{\lambda=-1/2} - \frac{L}{4}. \tag{20}$$

This is the reason why we can say that transfer matrix $\tau(\lambda)$ is a generating function for commuting conserved charges.

2.1. Global fundamental commutation relations

Let us express global commutation relations (14) for generators of Yang-Baxter algebra. After a simple factorization, R-matrix (10) can be written as

$$R_{ab}(\lambda) = \begin{pmatrix} f(\lambda) & 0 & 0 & 0 \\ 0 & 1 & g(\lambda) & 0 \\ 0 & g(\lambda) & 1 & 0 \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix} \tag{21}$$

where $f(\lambda) = \frac{\lambda+1}{\lambda}$ and $g(\lambda) = \frac{1}{\lambda}$. The matrices $T_a(\lambda)$ resp. $T_b(\mu)$ take the form

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & & B(\lambda) & \\ & A(\lambda) & & B(\lambda) \\ C(\lambda) & & D(\lambda) & \\ & C(\lambda) & & D(\lambda) \end{pmatrix} \quad \text{resp.} \quad T_b(\mu) = \begin{pmatrix} A(\mu) & B(\mu) & & \\ C(\mu) & D(\mu) & & \\ & & A(\mu) & B(\mu) \\ & & C(\mu) & D(\mu) \end{pmatrix}. \tag{22}$$

Comparing matrix elements of (14) on the positions $(1,1)$, $(1,4)$, $(4,1)$, $(4,4)$ we obtain

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0. \tag{23}$$

Comparing $(2,3)$ resp. $(2,2)$

$$[B(\lambda), C(\mu)] = g(\lambda, \mu)\left(D(\mu)A(\lambda) - D(\lambda)A(\mu)\right), \tag{24}$$
$$[A(\lambda), D(\mu)] = g(\lambda, \mu)\left(C(\mu)B(\lambda) - C(\lambda)B(\mu)\right). \tag{25}$$

From comparison of matrix elements $(1,3)$, $(3,4)$, $(2,1)$ resp. $(4,3)$ we obtain

$$A(\mu)B(\lambda) = f(\lambda, \mu)B(\lambda)A(\mu) + g(\mu, \lambda)B(\mu)A(\lambda), \tag{26}$$
$$D(\mu)B(\lambda) = f(\mu, \lambda)B(\lambda)D(\mu) + g(\lambda, \mu)B(\mu)D(\lambda), \tag{27}$$
$$A(\mu)C(\lambda) = f(\mu, \lambda)C(\lambda)A(\mu) + g(\lambda, \mu)C(\mu)A(\lambda), \tag{28}$$
$$D(\mu)C(\lambda) = f(\lambda, \mu)C(\lambda)D(\mu) + g(\mu, \lambda)C(\mu)D(\lambda), \tag{29}$$

where

$$f(\lambda, \mu) = f(\lambda - \mu) = \frac{\lambda - \mu + 1}{\lambda - \mu}, \qquad g(\lambda, \mu) = g(\lambda - \mu) = \frac{1}{\lambda - \mu}. \tag{30}$$

We see that $g(\mu, \lambda) = -g(\lambda, \mu)$.

2.2. *Eigenstates of transfer matrix*

Commutation relations (23)-(30) for Yang-Baxter algebra together with an assumption that the Hilbert space $\mathscr{H}$ has structure of a Fock space are sufficient to encover spectrum of the transfer matrix $\tau(\lambda)$.

There is a pseudovacuum vector $|0\rangle \in \mathscr{H}$ such that $C(\lambda)|0\rangle = 0$ which is an eigenvector of operators $A(\lambda)$ and $D(\lambda)$

$$A(\lambda)|0\rangle = \alpha(\lambda)|0\rangle, \qquad D(\lambda)|0\rangle = \delta(\lambda)|0\rangle. \tag{31}$$

It is a tensor product of local pseudovacua

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_L \tag{32}$$

where $|0\rangle_k = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. It can be easily seen that $\alpha(\lambda) = (\lambda+1)^L$, $d(\lambda) = \lambda^L$.

Other eigenstates of transfer matrix (16) are of the form

$$|\lambda_1, \ldots, \lambda_M\rangle \equiv B(\lambda_1)B(\lambda_2)\ldots B(\lambda_M)|0\rangle \tag{33}$$

with eigenvalue

$$\Lambda(\lambda, \{\lambda\}) = \alpha(\lambda)\prod_{i=1}^{M} f(\lambda_i, \lambda) + \delta(\lambda)\prod_{i=1}^{M} f(\lambda, \lambda_i). \tag{34}$$

They are called Bethe vectors. For $M \in \mathbb{N}$, we refer to Bethe vector $|\lambda_1, \dots, \lambda_M\rangle$ as M-magnon state. To get eigenstates of the transfer matrix, parameters $\{\lambda\} = \{\lambda_1, \dots, \lambda_M\}$ have to satisfy Bethe equations for all $i \in 1, \dots, M$

$$\alpha(\lambda_i) g(\lambda, \lambda_i) \prod_{\substack{k=1\\k\neq i}}^{M} f(\lambda_k, \lambda_i) + \delta(\lambda_i) g(\lambda_i, \lambda) \prod_{\substack{k=1\\k\neq i}}^{M} f(\lambda_i, \lambda_k) = 0. \tag{35}$$

Explicit form of Bethe equations (35) is

$$\left(\frac{\lambda_i + 1}{\lambda_i}\right)^L = \prod_{\substack{k=1\\k\neq i}}^{M} \frac{\lambda_i - \lambda_k + 1}{\lambda_i - \lambda_k - 1}. \tag{36}$$

3. Free Fermions

Our first aim is to express Bethe vectors in fermionic basis. We start with definition of free fermions. For tensor product of L copies of $\mathbb{C}^2$ we define free fermions as

$$\psi_k \equiv \left(\prod_{j=1}^{k-1} \sigma_j^z\right) \sigma_k^+, \qquad \bar{\psi}_k \equiv \left(\prod_{j=1}^{k-1} \sigma_j^z\right) \sigma_k^-. \tag{37}$$

Commutation relations for the fermions (37) are of the form

$$[\bar{\psi}_i, \psi_j]_+ = \delta_{ij}\mathbb{I}, \quad [\bar{\psi}_i, \bar{\psi}_j]_+ = 0, \quad [\psi_i, \psi_j]_+ = 0. \tag{38}$$

It is a straightforward task to check the following identities

$$\bar{\psi}_{k+1}\psi_k + \bar{\psi}_k\psi_{k+1} + \bar{\psi}_k\bar{\psi}_{k+1} + \psi_{k+1}\psi_k = \sigma_k^x \sigma_{k+1}^x, \tag{39}$$
$$\bar{\psi}_{k+1}\psi_k + \bar{\psi}_k\psi_{k+1} - \bar{\psi}_k\bar{\psi}_{k+1} - \psi_{k+1}\psi_k = \sigma_k^y \sigma_{k+1}^y, \tag{40}$$
$$[\psi_k, \bar{\psi}_k] = \sigma_k^z, \tag{41}$$
$$(1 - 2\bar{\psi}_k\psi_k)(1 - 2\bar{\psi}_{k+1}\psi_{k+1}) = \sigma_k^z \sigma_{k+1}^z. \tag{42}$$

4. Fermionic realization of monodromy matrix

Equation (7) provides us an easy expression for Lax operator. Identity operator $\mathbb{I}$ is a member of algebra of fermions. Therefore, it remains to know a fermionic realization only for permutation operator $P_{a,i}$.

Let us start with permutation operator $P_{k,k+1}$ permuting just the neighboring vector spaces h_k and h_{k+1}. Due to identities (39)-(42) and definition of permutation operator (6), it is straightforward to check that

$$P_{k,k+1} = \mathbb{I} + \bar{\psi}_{k+1}\psi_k + \bar{\psi}_k\psi_{k+1} - \bar{\psi}_k\psi_k - \bar{\psi}_{k+1}\psi_{k+1} + 2\bar{\psi}_k\psi_k\bar{\psi}_{k+1}\psi_{k+1}. \tag{43}$$

Permutation operator $P_{j,k}$ in non-neighboring vector spaces h_j, h_k where $j < k - 1$, becomes a non-local in terms of fermions. Using properties of Pauli matrices, it can be rewritten as

$$P_{j,k} = \frac{1}{2}(\mathbb{I} + \sigma_j^z \sigma_k^z) + (\sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+). \tag{44}$$

The first part is local even in the terms of fermions

$$\frac{1}{2}(\mathbb{I} + \sigma_j^z \sigma_k^z) = \mathbb{I} - \bar{\psi}_k\psi_k - \bar{\psi}_j\psi_j + 2\bar{\psi}_k\psi_k\bar{\psi}_j\psi_j, \tag{45}$$

but the second part is nonlocal

$$\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+ = (\psi_j\bar{\psi}_k + \bar{\psi}_j\psi_k)\prod_{l=j}^{k-1}\sigma_l^z = (\psi_j\bar{\psi}_k + \bar{\psi}_j\psi_k)\prod_{l=j}^{k-1}(\mathbb{I} - 2\bar{\psi}_l\psi_l). \tag{46}$$

The nonlocality of $P_{j,k}$ resp. $R_{j,k}(\lambda)$ is a serious problem. There appear difficulties when we attempt to express monodromy matrix (12) in terms of such nonlocal operators. We need to avoid the nonlocality.

Let us remind that $L_{a,i}(\lambda) = R_{a,i}(\lambda)$. For R-matrix $R_{ab}(\lambda)$ satisfying Yang-Baxter equation (11) we can define the matrix $\hat{R}_{ab}(\lambda) = R_{ab}(\lambda)P_{ab}$ which satisfies

$$\hat{R}_{ab}(\lambda)\hat{R}_{bc}(\lambda+\mu)\hat{R}_{ab}(\mu) = \hat{R}_{ab}(\mu)\hat{R}_{bc}(\lambda+\mu)\hat{R}_{ab}(\lambda). \tag{47}$$

Substituting $L_{a,i}(\lambda) = \hat{R}_{a,i}(\lambda)P_{a,i}$ in monodromy matrix (12), we obtain very convenient expression

$$\begin{aligned} T_a(\lambda) = L_{a,1}(\lambda)L_{a,2}(\lambda)\dots L_{a,L}(\lambda) = \hat{R}_{a,1}(\lambda)P_{a,1}\hat{R}_{a,2}(\lambda)P_{a,2}\dots\hat{R}_{a,L}(\lambda)P_{a,L} = \\ = \hat{R}_{a,1}(\lambda)\hat{R}_{1,2}(\lambda)\dots\hat{R}_{L-1,L}(\lambda)P_{L-1,L}\dots P_{1,2}P_{a,1}. \end{aligned} \tag{48}$$

It contains operators $\hat{R}_{k,k+1}$ resp. $P_{k,k+1}$ acting only in the neighboring spaces $h_k \otimes h_{k+1}$. Fermionic realization of R-matrix $\hat{R}_{k,k+1}(\lambda)$ is

$$\hat{R}_{k,k+1}(\lambda) = \lambda P_{k,k+1} + \mathbb{I} = (\lambda+1)\mathbb{I} + \lambda(\bar{\psi}_{k+1}\psi_k + \bar{\psi}_k\psi_{k+1} - \bar{\psi}_k\psi_k - \bar{\psi}_{k+1}\psi_{k+1} + 2\bar{\psi}_k\psi_k\bar{\psi}_{k+1}\psi_{k+1}). \tag{49}$$

To get fermionic represenation of Yang-Baxter algebra means to express monodromy matrix (48) as 2×2 matrix in the auxiliary space $V_a = \mathbb{C}^2$. For this purpose, we rewrite (48) as

$$T_a(\lambda) = \hat{R}_{a,1}(\lambda)X(\lambda)P_{a,1} \tag{50}$$

where the operator $X(\lambda)$

$$X(\lambda) = \hat{R}_{1,2}(\lambda)\dots\hat{R}_{L-1,L}(\lambda)P_{L-1,L}\dots P_{1,2} \tag{51}$$

acts nontrivially only in the quantum spaces $\mathscr{H} = h_1 \otimes \cdots \otimes h_L$ and is a scalar in the auxiliary space V_a. Moreover, we know due to equations (43) and (49) how to express $X(\lambda)$ in the terms of fermions.

Permutation matrix (6) can be rewritten as

$$\begin{aligned} P_{a,1} = \frac{1}{2}\Big(\mathbb{I}\otimes\mathbb{I} + \sigma^x\otimes\sigma^x + \sigma^y\otimes\sigma^y + \sigma^z\otimes\sigma^z\Big) = \\ = \frac{1}{2}\left[\begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I}\end{pmatrix} + \begin{pmatrix} 0 & \sigma^x \\ \sigma^x & 0\end{pmatrix} + \begin{pmatrix} 0 & -i\sigma^y \\ i\sigma^y & 0\end{pmatrix} + \begin{pmatrix} \sigma^z & 0 \\ 0 & -\sigma^z\end{pmatrix}\right] = \\ = \begin{pmatrix} \frac{1}{2}(\mathbb{I}+\sigma^z) & \frac{1}{2}(\sigma^x - i\sigma^y) \\ \frac{1}{2}(\sigma^x + i\sigma^y) & \frac{1}{2}(\mathbb{I}-\sigma^z)\end{pmatrix} = \begin{pmatrix} \psi_1\bar{\psi}_1 & \bar{\psi}_1 \\ \psi_1 & \bar{\psi}_1\psi_1\end{pmatrix} = \begin{pmatrix} \mathbb{I} - N_1 & \bar{\psi}_1 \\ \psi_1 & N_1\end{pmatrix} \end{aligned} \tag{52}$$

where we have used (37), (41), and $N_1 = \bar{\psi}_1\psi_1$. Hence, we get

$$\hat{R}_{a,1}(\lambda) = \mathbb{I}_{a,i} + \lambda P_{a,1} = \begin{pmatrix} (\lambda+1)\mathbb{I} - \lambda N_1 & \lambda\bar{\psi}_1 \\ \lambda\psi_1 & \lambda N_1 + \mathbb{I}\end{pmatrix}. \tag{53}$$

Using (52) and (53), monodromy matrix (50) can be written in the following form

$$T_a(\lambda) = \begin{pmatrix} (\lambda+1)\mathbb{I} - \lambda N_1 & \lambda\bar{\psi}_1 \\ \lambda\psi_1 & \lambda N_1 + \mathbb{I} \end{pmatrix} X(\lambda) \begin{pmatrix} \mathbb{I} - N_1 & \bar{\psi}_1 \\ \psi_1 & N_1 \end{pmatrix} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (54)$$

where

$$A(\lambda) = (\lambda + 1 - \lambda N_1)X(\lambda)(1 - N_1) + \lambda\bar{\psi}_1 X(\lambda)\psi_1, \quad (55)$$
$$B(\lambda) = (\lambda + 1 - \lambda N_1)X(\lambda)\bar{\psi}_1 + \lambda\bar{\psi}_1 X(\lambda)N_1, \quad (56)$$
$$C(\lambda) = \lambda\psi_1 X(\lambda)(1 - N_1) + (\lambda N_1 + 1)X(\lambda)\psi_1, \quad (57)$$
$$D(\lambda) = \lambda\psi_1 X(\lambda)\bar{\psi}_1 + (\lambda N_1 + 1)X(\lambda)N_1. \quad (58)$$

5. Fermionic realization of Bethe vectors

The goal of our text is to find expression for Bethe vectors (33). For this purpose, fermionic realization (56) of the creation operator $B(\lambda)$ is convenient. The operator $X(\lambda) = \hat{R}_{12}(\lambda)\ldots\hat{R}_{L-1,L}(\lambda)P_{L-1,L}\ldots P_{12}$ can be written in terms of fermions due to equations (43) and (49). It can be easily seen that

$$\psi_k \left|0\right\rangle = 0 \quad (59)$$

for all $k = 1, \ldots, L$.

If we are able to write $B(\lambda)$ in normal form our work would be simple. Unfortunately, it seems as a rather difficult task. Instead, we have to use the "weak approach," i.e. to apply $B(\lambda)$ on the pseudovacuum $\left|0\right\rangle$ and try to commute the fermions ψ_k to the right and see what remains.

For our purposes, we need the following set of useful identities, which follow from equations (52) and (53)

$$P_{k,k+1}\,\bar{\psi}_k = \bar{\psi}_{k+1} - \bar{\psi}_{k+1}N_k + \bar{\psi}_k N_{k+1}, \quad (60)$$
$$\hat{R}_{k,k+1}(\lambda)\bar{\psi}_{k+1} = \bar{\psi}_{k+1} + \lambda\bar{\psi}_k + \lambda\bar{\psi}_{k+1}N_k - \bar{\psi}_k N_{k+1} \quad (61)$$

where $N_k = \bar{\psi}_k\psi_k$, again. We can see that

$$P_{k,k+1}\bar{\psi}_k \left|0\right\rangle = \bar{\psi}_{k+1}\left|0\right\rangle, \quad (62)$$
$$\hat{R}_{k,k+1}(\mu)\bar{\psi}_{k+1}\left|0\right\rangle = \bar{\psi}_{k+1}\left|0\right\rangle + \mu\bar{\psi}_k\left|0\right\rangle, \quad (63)$$

and

$$\hat{R}_{k,k+1}(\mu)\left|0\right\rangle = (\mu+1)\left|0\right\rangle, \quad (64)$$
$$P_{k,k+1}\left|0\right\rangle = \left|0\right\rangle. \quad (65)$$

For higher magnons, we need also

$$\hat{R}_{k,k+1}(\lambda)\bar{\psi}_k\bar{\psi}_{k+1} = (\lambda+1)\bar{\psi}_k\bar{\psi}_{k+1} \quad (66)$$

and

$$\hat{R}_{l,l+1}(\lambda)\ldots\hat{R}_{k-1,k}(\lambda)\bar{\psi}_k\left|0\right\rangle = \lambda^{k-l}\bar{\psi}_l\left|0\right\rangle + \frac{\lambda^{k-l}}{\lambda+1}\sum_{j=1}^{k-l}\left(\frac{\lambda+1}{\lambda}\right)^j \bar{\psi}_{l+j}\left|0\right\rangle, \quad (67)$$

$$\hat{R}_{l,l+1}(\lambda)\ldots\hat{R}_{k,k+1}(\lambda)\bar{\psi}_k\left|0\right\rangle = \hat{R}_{l,l+1}(\lambda)\ldots\hat{R}_{k-1,k}(\lambda)\bar{\psi}_k\left|0\right\rangle + \lambda(\lambda+1)^{k-l}\bar{\psi}_{k+1}\left|0\right\rangle. \quad (68)$$

5.1. 1-magnon
It is obvious that the second term in (56) annihilates the pseudovacuum state. Then, using (67), we get for 1-magnon state

$$\begin{aligned}|\mu\rangle \equiv B(\mu)\,|0\rangle &= (\mu+1-\mu N_1)\hat{R}_{12}(\mu)\dots\hat{R}_{L-1,L}(\mu)P_{L-1,L}\dots P_{12}\bar{\psi}_1\,|0\rangle = \\ &= (\mu+1-\mu N_1)\hat{R}_{12}(\mu)\dots\hat{R}_{L-1,L}(\mu)\bar{\psi}_L\,|0\rangle = \\ &= (\mu+1-\mu N_1)\Big[\mu^{L-1}\bar{\psi}_1\,|0\rangle + \frac{\mu^{L-1}}{\mu+1}\sum_{j=1}^{L-1}\Big(\frac{\mu+1}{\mu}\Big)^j\bar{\psi}_{j+1}\Big]\,|0\rangle = \\ &= \frac{\mu^L}{\mu+1}\sum_{k=1}^{L}\left(\frac{\mu+1}{\mu}\right)^k\bar{\psi}_k\,|0\rangle = n(\mu)\sum_{k=1}^{L}[\mu]^k\bar{\psi}_k\,|0\rangle \end{aligned}\tag{69}$$

where we introduce concise notation

$$[\mu] = \frac{\mu+1}{\mu}, \quad \text{and} \quad n(\mu) = \frac{\mu^L}{\mu+1}. \tag{70}$$

5.2. 2-magnon
Using (67) and (68) we obtain

$$\begin{aligned} B(\lambda)\bar{\psi}_k\,|0\rangle = (\lambda+1)\lambda^{L-2}\sum_{m=0}^{k-2}[\lambda]^m\bar{\psi}_{m+1}\bar{\psi}_k\,|0\rangle + \\ +\frac{\lambda^{L-2}}{\lambda+1}\sum_{m=0}^{k-2}\sum_{j=1}^{L-k}[\lambda]^{j+m}\bar{\psi}_{m+1}\bar{\psi}_{k+j}\,|0\rangle + \frac{\lambda^L}{\lambda+1}[\lambda]^{k-1}\sum_{j=1}^{L-k}[\lambda]^j\bar{\psi}_k\bar{\psi}_{k+j}\,|0\rangle\,. \end{aligned}\tag{71}$$

We get for 2-magnon state using (71)

$$\begin{aligned} |\lambda,\mu\rangle \equiv B(\lambda)B(\mu)\,|0\rangle = n(\mu)\sum_{k=1}^{L}[\mu]^k B(\lambda)\bar{\psi}_k\,|0\rangle = n(\mu)n(\lambda)\Bigg[\sum_{k=2}^{L}\sum_{m=0}^{k-2}[\mu]^k[\lambda]^{m+2}\bar{\psi}_{m+1}\bar{\psi}_k+ \\ +\frac{1}{\lambda(\lambda+1)}\sum_{k=2}^{L-1}\sum_{m=0}^{k-2}\sum_{j=1}^{L-k}[\mu]^k[\lambda]^{j+m+1}\bar{\psi}_{m+1}\bar{\psi}_{k+j} + \sum_{k=1}^{L-1}\sum_{j=1}^{L-k}[\mu]^k[\lambda]^{k+j-1}\bar{\psi}_k\bar{\psi}_{k+j}\Bigg]\,|0\rangle = \\ = n(\mu)n(\lambda)\Bigg\{\sum_{1\le r<s\le L}\Bigg[[\mu]^s[\lambda]^{r+1} + [\mu]^r[\lambda]^{s-1}\Bigg]\bar{\psi}_r\bar{\psi}_s\,|0\rangle + \\ +\frac{1}{\lambda(\lambda+1)}\sum_{s=3}^{L}\sum_{r=1}^{s-2}\sum_{k=r+1}^{s-1}[\mu]^k[\lambda]^{s+r-k}\bar{\psi}_r\bar{\psi}_s\,|0\rangle\Bigg\} = \\ = n(\mu)n(\lambda)\sum_{1\le r<s\le L}\Bigg[[\mu]^s[\lambda]^{r+1} + [\mu]^r[\lambda]^{s-1} + \frac{1}{\lambda(\lambda+1)}\sum_{k=r+1}^{s-1}[\mu]^k[\lambda]^{s+r-k}\Bigg]\bar{\psi}_r\bar{\psi}_s\,|0\rangle\,. \end{aligned}\tag{72}$$

The finite sum in (72) can be calculated explicitly by means of geometric progression

$$\begin{aligned} \frac{1}{\lambda(\lambda+1)}\sum_{k=r+1}^{s-1}[\mu]^k[\lambda]^{s+r-k} = \\ = \frac{\mu}{\lambda(\lambda-\mu)}[\mu]^{r+1}[\lambda]^{s-1}\left(\left(\frac{[\mu]}{[\lambda]}\right)^{s-r-1} - 1\right) = \frac{\mu}{\lambda(\lambda-\mu)}\left([\mu]^s[\lambda]^r - [\mu]^{r+1}[\lambda]^{s-1}\right). \end{aligned}\tag{73}$$

Substitution of (73) into (72) gives formula for 2-magnon

$$B(\lambda)B(\mu)\left|0\right\rangle = n(\mu)n(\lambda)\sum_{1\le r<s\le L}\Bigg[[\mu]^s[\lambda]^{r+1}+[\mu]^r[\lambda]^{s-1}+ \\ +\frac{\mu}{\lambda(\lambda-\mu)}\left([\mu]^s[\lambda]^r-[\mu]^{r+1}[\lambda]^{s-1}\right)\Bigg]\bar\psi_r\bar\psi_s\left|0\right\rangle = \\ = n(\mu)n(\lambda)\sum_{1\le r<s\le L}\left[[\lambda]^r[\mu]^s\frac{\lambda-\mu+1}{\lambda-\mu}+[\mu]^r[\lambda]^s\frac{\mu-\lambda+1}{\mu-\lambda}\right]\bar\psi_r\bar\psi_s\left|0\right\rangle. \tag{74}$$

5.3. 3-magnon

For 3-magnon we need at first

$$B(\nu)\bar\psi_r\bar\psi_s\left|0\right\rangle = (\nu+1-\nu N_1)X_{12\ldots L}(\nu)\bar\psi_1\bar\psi_r\bar\psi_s = \\ = \nu^{L-3}(\nu+1)^2\sum_{m=0}^{r-2}[\nu]^m\bar\psi_{m+1}\bar\psi_r\bar\psi_s\left|0\right\rangle + \nu^{L-3}\sum_{l=1}^{s-r-1}\sum_{m=0}^{r-2}[\nu]^{l+m}\bar\psi_{m+1}\bar\psi_{r+l}\bar\psi_s\left|0\right\rangle + \\ +\nu^{L-3}\sum_{j=1}^{L-s}\sum_{m=0}^{r-2}[\nu]^{j+m}\bar\psi_{m+1}\bar\psi_r\bar\psi_{s+j}\left|0\right\rangle + \\ +\nu^{L-3}(\nu+1)^{-2}\sum_{j=1}^{L-s}\sum_{l=1}^{s-r-1}\sum_{m=0}^{r-2}[\nu]^{j+l+m}\bar\psi_{m+1}\bar\psi_{r+l}\bar\psi_{s+j}\left|0\right\rangle + \\ +\nu^{L-s+r-1}(\nu+1)^{s-r-2}\sum_{j=1}^{L-s}\sum_{m=0}^{r-2}[\nu]^{j+m}\bar\psi_{m+1}\bar\psi_s\bar\psi_{s+j}\left|0\right\rangle + \\ +\nu^{L-s+2}(\nu+1)^{s-3}\sum_{j=1}^{L-s}[\nu]^j\bar\psi_r\bar\psi_s\bar\psi_{s+j}\left|0\right\rangle + \nu^{L-r}(\nu+1)^{r-1}\sum_{l=1}^{s-r-1}[\nu]^l\bar\psi_r\bar\psi_{r+l}\bar\psi_s\left|0\right\rangle + \\ +\nu^{L-r}(\nu+1)^{r-3}\sum_{j=1}^{L-s}\sum_{l=1}^{s-r-1}[\nu]^{j+l}\bar\psi_r\bar\psi_{r+l}\bar\psi_{s+j}\left|0\right\rangle. \tag{75}$$

3-magnon state is obtained from 2-magnon state (74)

$$\left|\nu,\mu,\lambda\right\rangle \equiv B(\nu)B(\mu)B(\lambda)\left|0\right\rangle = n(\mu)n(\lambda)\sum_{1\le r<s\le L}K_2(s,r)B(\nu)\bar\psi_r\bar\psi_s\left|0\right\rangle \tag{76}$$

where we denote for more comfort

$$K_2(s,r) = [\mu]^s[\lambda]^r\frac{\lambda-\mu+1}{\lambda-\mu}+[\mu]^r[\lambda]^s\frac{\mu-\lambda+1}{\mu-\lambda}. \tag{77}$$

Using(75), we get

$$
\begin{aligned}
|\nu,\mu,\lambda\rangle &= n(\nu)n(\mu)n(\lambda)\sum_{1\le q<r<s\le L}\Bigg[[\nu]^{q+2}K_2(s,r)+\frac{1}{\nu^2}\sum_{l=1}^{r-q-1}[\nu]^{l+q}K_2(s,r-l)+\\
&+\frac{1}{\nu^2}\sum_{j=1}^{s-r-1}[\nu]^{j+q}K_2(s-j,r)+\frac{1}{\nu^2(\nu+1)^2}\sum_{j=1}^{s-r-1}\sum_{l=1}^{r-q-1}[\nu]^{j+l+q}K_2(s-j,r-l)+\\
&+\frac{1}{(\nu+1)^2}\sum_{l=q+1}^{r-1}[\nu]^{s-l+q}K_2(r,l)+[\nu]^{s-2}K_2(r,q)+[\nu]^{r}K_2(s,q)+\\
&+\frac{1}{(\nu+1)^2}\sum_{l=r+1}^{s-1}[\nu]^{s+r-l}K_2(l,q)\Bigg]\bar{\psi}_q\bar{\psi}_r\bar{\psi}_s|0\rangle = n(\nu)n(\mu)n(\lambda)\times\\
&\sum_{1\le q<r<s\le L}\sum_{\sigma\in S_3}\sigma\Big([\nu]^q[\mu]^r[\lambda]^s\frac{\nu-\mu+1}{\nu-\mu}\cdot\frac{\nu-\lambda+1}{\nu-\lambda}\cdot\frac{\mu-\lambda+1}{\mu-\lambda}\Big)\bar{\psi}_q\bar{\psi}_r\bar{\psi}_s|0\rangle \qquad (78)
\end{aligned}
$$

where S_3 denotes symmetric group.

5.4. M-magnon

From results (69), (74) and (78) we can conjecture that general M-magnon state is of the form

$$
\begin{aligned}
&|\lambda_1,\ldots,\lambda_M\rangle \equiv B(\lambda_1)\cdots B(\lambda_M)|0\rangle =\\
&=\Big(\prod_{i=1}^{M}n(\lambda_i)\Big)\sum_{1\le k_1<\ldots<k_M\le L}\;\sum_{\sigma_\lambda\in S_M}\sigma_\lambda\Big(\prod_{i<j}^{M}\frac{\lambda_i-\lambda_j+1}{\lambda_i-\lambda_j}\prod_{i=1}^{M}[\lambda_i]^{k_i}\Big)\bar{\psi}_{k_1}\cdots\bar{\psi}_{k_M}|0\rangle \qquad (79)
\end{aligned}
$$

where σ_λ is a permutation of the parameters $\{\lambda_1,\ldots,\lambda_M\}$ and $\sum_{\sigma_\lambda\in S_M}$ is the sum over all such permutations. We are going to provide a proof of this conjecture in more general form below.

6. N-component model

In the literature, cf. [2, 5], the so-called two-component model appears. It was introduced to avoid problems with computation of correlation functions for local operators attached to some node x of the chain in the Yang-Baxter algebra generated by (13).

The chain $[1,\ldots,L]$ is divided into two components $[1,\ldots,x]$ and $[x+1,\ldots,L]$. Then, the Hilbert space is splitted into two parts $\mathscr{H}=\mathscr{H}_1\otimes\mathscr{H}_2$ where $\mathscr{H}_1=h_1\otimes\cdots\otimes h_x$ and $\mathscr{H}_2=h_{x+1}\otimes\cdots\otimes h_L$. The pseudovacuum $|0\rangle\in\mathscr{H}$ is of the form $|0\rangle=|0\rangle_1\otimes|0\rangle_2$ where $|0\rangle_1\in\mathscr{H}_1$ and $|0\rangle_2\in\mathscr{H}_2$. We define on $V_a\otimes\mathscr{H}_1\otimes\mathscr{H}_2$ a monodromy matrix for each component

$$T_1(\lambda)=L_{a,1}(\lambda)\cdots L_{a,x}(\lambda)=\begin{pmatrix}A_1(\lambda) & B_1(\lambda)\\ C_1(\lambda) & D_1(\lambda)\end{pmatrix}, \qquad (80)$$

resp.

$$T_2(\lambda)=L_{a,x+1}(\lambda)\cdots L_{a,L}(\lambda)=\begin{pmatrix}A_2(\lambda) & B_2(\lambda)\\ C_2(\lambda) & D_2(\lambda)\end{pmatrix}. \qquad (81)$$

Each of these monodromy matrices satisfies exactly the same commutation relations (14) as original undivided monodromy matrix (12). Moreover, we have

$$A_j(\lambda)|0\rangle_j=\alpha_j(\lambda)|0\rangle_j, \qquad D_j(\lambda)|0\rangle_j=\delta_j(\lambda)|0\rangle_j, \qquad C_j(\lambda)|0\rangle_j=0, \qquad (82)$$

and operators corresponding to different components mutually commute. From construction, we see that

$$\alpha(\lambda) = \alpha_1(\lambda)\alpha_2(\lambda), \qquad \delta(\lambda) = \delta_1(\lambda)\delta_2(\lambda). \tag{83}$$

The full monodromy matrix $T(\lambda)$ for the complete chain $[1,\ldots,L]$ is

$$T(\lambda) = T_1(\lambda)\,T_2(\lambda) = \begin{pmatrix} A_1(\lambda)A_2(\lambda) + B_1(\lambda)C_2(\lambda) & A_1(\lambda)B_2(\lambda) + B_1(\lambda)D_2(\lambda) \\ C_1(\lambda)A_2(\lambda) + D_1(\lambda)C_2(\lambda) & C_1(\lambda)B_2(\lambda) + D_1(\lambda)D_2(\lambda) \end{pmatrix}, \tag{84}$$

and the M-magnon state is represented in the form

$$|\lambda_1,\ldots,\lambda_M\rangle = \prod_{k=1}^{M} B(\lambda_k)\,|0\rangle = \prod_{k=1}^{M}\Big(A_1(\lambda_k)B_2(\lambda_k) + B_1(\lambda_k)D_2(\lambda_k)\Big)\,|0\rangle_1 \otimes |0\rangle_2\ . \tag{85}$$

Izergin and Korepin [5] state that Bethe vectors of the full model can be expressed in terms of Bethe vectors of its components. To obtain this expression we should commute in (85) all operators $A_1(\lambda_k)$ and $D_2(\lambda_k)$ to the right with the help of (26) and (27) and then use (82).

Proposition 1. *An arbitrary Bethe vector corresponding to the full system can be expressed in terms of the Bethe vectors of the first and second component. Let $I = \{\lambda_1,\ldots,\lambda_M\}$ be a finite set of spectral parameters. To concise notation below we will consider the set I as a finite set of indices $I = \{1,\ldots,M\}$, then*

$$\prod_{k\in I} B(\lambda_k)\,|0\rangle =$$

$$\sum_{I_1\cup I_2}\prod_{k_1\in I_1}\Big(\delta_2(\lambda_{k_1})B_1(\lambda_{k_1})\Big)\prod_{k_2\in I_2}\Big(\alpha_1(\lambda_{k_2})B_2(\lambda_{k_2})\Big)\,|0\rangle_1\otimes|0\rangle_2\prod_{k_1\in I_1}\prod_{k_2\in I_2} f(\lambda_{k_1},\lambda_{k_2}) \tag{86}$$

where $f(\lambda_{k_1},\lambda_{k_2})$ is defined in (30) and the summation is performed over all divisions of index set I into two disjoint subsets I_1 and I_2 where $I = I_1\cup I_2$.

For detailed proof, please, see e.g. [2]. This result can be straightforwardly generalized to arbitrary number of components $N \le L$.

Proposition 2. *An arbitrary Bethe vector of the full system can be expressed in terms of the Bethe vectors of its components. For $N \le L$ components the Bethe vector is of the form*

$$\prod_{k\in I} B(\lambda_k)\,|0\rangle = \sum_{I_1\cup I_2\cup\cdots\cup I_N}\prod_{k_1\in I_1}\prod_{k_2\in I_2}\cdots\prod_{k_N\in I_N}\prod_{1\le i<j\le N}\Big(\alpha_i(\lambda_{k_j})\delta_j(\lambda_{k_i})f(\lambda_{k_i},\lambda_{k_j})\Big)$$
$$\times B_1(\lambda_{k_1})\,|0\rangle_1\otimes B_2(\lambda_{k_2})\,|0\rangle_2\otimes\cdots\otimes B_N(\lambda_{k_N})\,|0\rangle_N \tag{87}$$

where summation is performed over all divisions of the set I into its N mutually disjoint subsets $I_1, I_2, \ldots, I_N$.

The proof is simply performed using (86) by induction on number of components N. More details will be included in [3] where more general inhomogeneous XXZ-chain is concerned.

6.1. Bethe vectors explicitly

By assumption we have a chain of length L. Let us divide it in L components, i.e. into L 1-chains. Using proposition 2 we get for M-magnon (Bethe vector) with $M \le L$:

$$\prod_{k=1}^{M} B(\lambda_k)\,|0\rangle = \sum_{I_1\cup I_2\cup\cdots\cup I_L}\prod_{k_1\in I_1}\prod_{k_2\in I_2}\cdots\prod_{k_L\in I_L}\prod_{1\le i<j\le L}\Big(\alpha_i(\lambda_{k_j})\delta_j(\lambda_{k_i})f(\lambda_{k_i},\lambda_{k_j})\Big)$$
$$\times B_1(\lambda_{k_1})\,|0\rangle_1\otimes B_2(\lambda_{k_2})\,|0\rangle_2\otimes\cdots\otimes B_L(\lambda_{k_L})\,|0\rangle_L\,. \tag{88}$$

It holdsd for 1-chain, which is a chain with Hilbert space $h = \mathbb{C}^2$, that

$$B(\lambda)B(\mu)\,|0\rangle = 0. \tag{89}$$

Therefore, the sum over all divisions of $\{1, \dots, M\}$ into L subsets contains just divisions into subsets containing at most one element, i.e. $|I_j| = 0, 1$. Moreover, only M of them is nonempty, let us denote them $I_{n_1}, I_{n_2}, \dots, I_{n_M}$. We have to sum over all possible combinations of such sets, i.e. over all M-tuples $n_1 < n_2 < \cdots < n_M$; and then, to sum over all distributions of parameters $\lambda_1, \lambda_2, \dots, \lambda_M$ into the sets $I_{n_1}, \dots, I_{n_M}$.

After all, we get

$$\prod_{k=1}^{M} B(\lambda_k)\,|0\rangle = \sum_{1\le n_1<n_2<\cdots<n_M\le L} \sum_{\sigma_\lambda\in S_M} \sigma_\lambda \Bigg(\prod_{j=1}^{M} \Big(\prod_{i=1}^{n_j-1} \alpha_i(\lambda_j) \prod_{i=n_j+1}^{L} \delta_i(\lambda_j) \prod_{i=1}^{j-1} f(\lambda_i,\lambda_j) \Big) \\ \times B_{n_1}(\lambda_1) B_{n_2}(\lambda_2) \cdots B_{n_M}(\lambda_M) \Bigg) |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_L . \tag{90}$$

Moreover, it holds for 1-chain that $B(\lambda) = B$ is parameter independent and eigenvalues $\alpha_i(\lambda) = a(\lambda)$, $\delta_i(\lambda) = d(\lambda)$ are still the same for all components $i = 1, \dots, L$, where $a(\lambda) = \lambda + 1$ and $d(\lambda) = \lambda$. We get

$$\prod_{k=1}^{M} B(\lambda_k)\,|0\rangle = \sum_{1\le n_1<n_2<\cdots<n_M\le L} \sum_{\sigma\in S_M} \sigma_\lambda \Bigg(\prod_{j=1}^{M} a(\lambda_j)^{n_j-1} d(\lambda_j)^{L-n_j} \prod_{i=1}^{j-1} f(\lambda_i,\lambda_j) \Bigg) \\ \times B_{n_1} B_{n_2} \cdots B_{n_M} |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_L = \\ = \prod_{j=1}^{M} \frac{d(\lambda_j)^L}{a(\lambda_j)} \sum_{1\le n_1<n_2<\cdots<n_M\le L} \sum_{\sigma\in S_M} \sigma_\lambda \Bigg(\prod_{1\le i<j\le M} f(\lambda_i,\lambda_j) \prod_{j=1}^{M} \left(\frac{a(\lambda_j)}{d(\lambda_j)} \right)^{n_j} \Bigg) \\ \times B_{n_1} B_{n_2} \cdots B_{n_M} |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_L . \tag{91}$$

Again, the reader is refered for more details to [3].

Let us return to (79). To prove its validity, it is sufficient to realize that $\bar\psi_{k_1}\bar\psi_{k_2}\dots\bar\psi_{k_M}\,|0\rangle = B_{k_1}B_{k_2}\dots B_{k_M}\,|0\rangle$ for $k_1 < k_2 < \cdots < k_M$.

7. Some remarks on inhomogeneous XXX-chain

We start from inhomogeneous monodromy matrix

$$T_a^{\vec\xi}(\lambda) = L_{a,1}(\lambda+\xi_1) L_{a,2}(\lambda+\xi_2) \cdots L_{a,L}(\lambda+\xi_L) \tag{92}$$

where $L_{a,j}(\lambda)$ are Lax operators defined in (4) and $\vec\xi = (\xi_1 \dots, \xi_L)$ is an inhomogenity vector. Expressing $T_a^{\vec\xi}(\lambda)$ in the auxiliary space V_a, we get

$$T_a^{\vec\xi}(\lambda) = \begin{pmatrix} A^{\vec\xi}(\lambda) & B^{\vec\xi}(\lambda) \\ C^{\vec\xi}(\lambda) & D^{\vec\xi}(\lambda) \end{pmatrix} \tag{93}$$

where, again, operators $A^{\vec\xi}(\lambda)$, $B^{\vec\xi}(\lambda)$, $C^{\vec\xi}(\lambda)$ and $D^{\vec\xi}(\lambda)$ are acting in $\mathscr{H} = h_1 \otimes \cdots \otimes h_L$. Acting on pseudovacuum vector $|0\rangle \in \mathscr{H}$ we get

$$A^{\vec\xi}(\lambda)\,|0\rangle = \alpha^{\vec\xi}(\lambda)\,|0\rangle\,, \tag{94}$$

$$D^{\vec\xi}(\lambda)\,|0\rangle = \delta^{\vec\xi}(\lambda)\,|0\rangle\,, \tag{95}$$

$$C^{\vec\xi}(\lambda)\,|0\rangle = 0 \tag{96}$$

where

$$\alpha^{\vec{\xi}}(\lambda) = a(\lambda+\xi_1)a(\lambda+\xi_2)\cdots a(\lambda+\xi_L), \tag{97}$$

$$\delta^{\vec{\xi}}(\lambda) = d(\lambda+\xi_1)d(\lambda+\xi_2)\cdots d(\lambda+\xi_L). \tag{98}$$

Here, functions $a(\lambda) = \lambda + 1$ and $d(\lambda) = \lambda$.

For inhomogeneous version, we can introduce N-component model as well as for homogeneous Bethe ansatz. For 2-compotent model, for example, we have

$$T_a^{\vec{\xi}}(\lambda) = \underbrace{L_{a,1}(\lambda+\xi_1)\cdots L_{a,x}(\lambda+\xi_x)}_{\text{1st component}}\underbrace{L_{a,x+1}(\lambda+\xi_{x+1})\cdots L_{a,L}(\lambda+\xi_L)}_{\text{2nd component}} = T_a^{\vec{\xi}_1}(\lambda)T_a^{\vec{\xi}_2}(\lambda) \tag{99}$$

where $\vec{\xi}_1 = (\xi_1,\dots,\xi_x)$ resp. $\vec{\xi}_2 = (\xi_{x+1},\dots,\xi_L)$. We have

$$A^{\vec{\xi}}(\lambda)\left|0\right\rangle = \alpha_1^{\vec{\xi}_1}(\lambda)\alpha_2^{\vec{\xi}_2}(\lambda)\left|0\right\rangle, \qquad D^{\vec{\xi}}(\lambda)\left|0\right\rangle = \delta_1^{\vec{\xi}_1}(\lambda)\delta_2^{\vec{\xi}_2}(\lambda)\left|0\right\rangle. \tag{100}$$

The very important property of inhomogeneous chain is, that its operators $A^{\vec{\xi}}(\lambda)$, $B^{\vec{\xi}}(\lambda)$, $C^{\vec{\xi}}(\lambda)$ and $D^{\vec{\xi}}(\lambda)$ satisfy the same fundamental commutation relations as homogeneous chain (23)-(30). Therefore, analogy of propositions 1 and 2 can be easily formulated.

Proposition 3. *Let $N \le L$. An arbitrary Bethe vector of the full system can be expressed in terms of Bethe vectors of its N components*

$$\begin{aligned}\prod_{k\in I} B^{\vec{\xi}}(\lambda_k)\left|0\right\rangle = \sum_{I_1\cup\cdots\cup I_N}\prod_{k_1\in I_1}\cdots\prod_{k_N\in I_N}\prod_{1\le i<j\le N}\left(\alpha_i^{\vec{\xi}_i}(\lambda_{k_j})\delta_j^{\vec{\xi}_j}(\lambda_{k_i})f(\lambda_{k_i},\lambda_{k_j})\right)&\\ \times B_1^{\vec{\xi}_1}(\lambda_{k_1})B_2^{\vec{\xi}_2}(\lambda_{k_2})\cdots B_N^{\vec{\xi}_N}(\lambda_{k_N})\left|0\right\rangle.&\end{aligned} \tag{101}$$

To get explicit formula for Bethe vectors we have to divide the chain into L components of length 1 as we did in the last section. We get for M-magnon

$$\begin{aligned}&\prod_{k=1}^{M} B^{\vec{\xi}}(\lambda_k)\left|0\right\rangle =\\ &= \sum_{1\le n_1<\cdots<n_M\le L}\ \sum_{\sigma_\lambda\in S_M}\sigma_\lambda\left(\prod_{j=1}^{M}\Big(\prod_{i=1}^{n_j-1}\alpha_i^{\xi_i}(\lambda_j)\prod_{i=n_j+1}^{L}\delta_i^{\xi_i}(\lambda_j)\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\Big)\right.\\ &\qquad\left.\times B_{n_1}^{\xi_{n_1}}(\lambda_1)\cdots B_{n_M}^{\xi_{n_M}}(\lambda_M)\right)\left|0\right\rangle =\\ &= \sum_{1\le n_1<\cdots<n_M\le L}\ \sum_{\sigma_\lambda\in S_M}\sigma_\lambda\left(\prod_{j=1}^{M}\Big(\prod_{i=1}^{n_j-1}\alpha_i^{\xi_i}(\lambda_j)\prod_{i=n_j+1}^{L}\delta_i^{\xi_i}(\lambda_j)\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\Big)\right)B_{n_1}\cdots B_{n_M}\left|0\right\rangle =\\ &= \prod_{j=1}^{M}\prod_{i=1}^{L}d(\lambda_j+\xi_i)\sum_{1\le n_1<\cdots<n_M\le L}\ \sum_{\sigma_\lambda\in S_M}\sigma_\lambda\left(\prod_{j=1}^{M}\frac{1}{a(\lambda_j+\xi_{n_j})}\prod_{i=1}^{n_j}\frac{a(\lambda_j+\xi_i)}{d(\lambda_j+\xi_i)}\prod_{i=1}^{j-1}f(\lambda_i,\lambda_j)\right)\\ &\qquad\times B_{n_1}\cdots B_{n_M}\left|0\right\rangle.\end{aligned} \tag{102}$$

where, again, B-operators $B_{n_j}^{\xi_{n_j}}(\lambda) = B_{n_j}$ are parameter independent for 1-chains. For more details, see [3].

8. Final remarks

We showed in this text explicit expressions for Bethe vectors of XXX-spin chain, both in fermionic and usual representation. We discussed also inhomogeneous version. We refer the reader for more details to [3] where we plan to discuss Bethe vectors for XXZ-chain in more detailed form.

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