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Multiplet classification for SU(n,n)

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Abstract. In the present paper we review our project of systematic construction of invariant differential operators on the example of the non-compact algebras $su(n,n)$ for $n = 2, 3, 4$. We give explicitly the main multiplets of indecomposable elementary representations and some reduced multiplets. We give explicitly the minimal representations. Due to the recently established parabolic relations the multiplet classification results are valid also for the algebras $sl(2n, \mathbb{R})$ and when $n = 2k$ for the algebras $su^*(4k)$ with suitably chosen maximal parabolic subalgebras.

1. Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory. Thus, it is important for the applications in physics to study systematically such operators.

In a recent paper [1] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study. Since the most widely used algebras are the conformal algebras so(n,2) in n-dimensional Minkowski space-time we concentrated on a class that shares most of their properties. This class consists of:

$$so(n,2), \quad sp(n,\mathbb{R}), \quad su(n,n), \quad so^*(4n), \quad E_{7(-25)}$$

the corresponding analogs of Minkowski space-time V being:

$$\mathbb{R}^{n-1,1}, \quad \mathrm{Sym}(n,\mathbb{R}), \quad \mathrm{Herm}(n,\mathbb{C}), \quad \mathrm{Herm}(n,\mathbb{Q}), \quad \mathrm{Herm}(3,\mathbb{O})$$

involving the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{O}$.

In view of applications to physics, we proposed to call these algebras 'conformal Lie algebras', (or groups) [2].

We have started the study of the above class in the framework of the present approach in the cases: $so(n,2)$, $su(n,n)$, $sp(n,\mathbb{R})$, $E_{7(-25)}$, cf. [2, 3, 4, 5].

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Lately, we discovered an efficient way to extend our considerations beyond this class introducing the notion of 'parabolically related non-compact semisimple Lie algebras' [5].

• *Definition:* Let $\mathcal{G}, \mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them parabolically related if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ $(\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}})$.$\diamondsuit$

Certainly, there are many such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G}, \mathcal{G}'$ may be parabolically related with different parabolic subalgebras.

In the present paper we review our results on the case of $su(n,n)$, cf. [3, 6, 7]. Due to the parabolic relationships these would be valid also for $sl(2n,\mathbb{R})$, and if $n = 2k$ also for $su^*(4k)$.

2. Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an *Iwasawa decomposition* $G = KA_0N_0$, where A_0 is Abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a *minimal parabolic subgroup* of G. A *parabolic subgroup* $P = M'A'N'$ is any subgroup of G which contains a minimal parabolic subgroup.

Further, let $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$ denote the Lie algebras of G, K, P, M, A, N, resp.

For our purposes we need to restrict to *maximal parabolic subgroups* $P = MAN$, i.e. rank$A = 1$, resp. to *maximal parabolic subalgebras* $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with $\dim \mathcal{A} = 1$.

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, parameterized by a real number d, called the *conformal weight* or energy.

Further, let μ fix a discrete series representation D^μ of M on the Hilbert space V_μ, or the finite-dimensional (non-unitary) representation of M with the same Casimirs.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [8]. (These are called *generalized principal series representations* (or *limits thereof*) in [9].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\}$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the left regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g') \,, \quad g, g' \in G \,.$$

ERs are important due to the following fundamental result:
Theorem [10, 11]: Every irreducible admissible representation of G is equivalent to a subrepresentation of an ER.

• An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}^{\mathbb{C}}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [12].

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu\, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

• One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [13]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines (arrows) between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the BGG Verma module reducibility condition (for highest weight modules) is fulfilled [14]:

$$(\Lambda + \rho, \beta^\vee) \;=\; m \;, \quad \beta^\vee \equiv 2\beta/(\beta,\beta)$$

ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. When the above holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [15]. More explicitly, [12], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs). Then there exists [12] an intertwining differential operator

$$\mathcal{D}_{m,\beta} \;:\; \mathcal{C}_{\chi(\Lambda)} \;\longrightarrow\; \mathcal{C}_{\chi(\Lambda - m\beta)}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \;=\; \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-})$$

where $\widehat{\mathcal{G}^-}$ denotes the right action on the functions $\mathcal{F}$.

In most of these situations the invariant operator $\mathcal{D}_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $\mathcal{G}$ is realized. Thus, studying the equations with trivial RHS:

$$\mathcal{D}_{m,\beta}\, f \;=\; 0 \;, \qquad f \in \mathcal{C}_{\chi(\Lambda)} \;,$$

is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for $m = m_\beta = 1$, these equations are called conservation laws, and the elements $f \in \ker \mathcal{D}_{m,\beta}$ are called conserved currents.

Below in our exposition we shall use the so-called Dynkin labels:

$$m_i \;\equiv\; (\Lambda + \rho, \alpha_i^\vee) \;, \quad i = 1, \ldots, n,$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$.

We shall use also the so-called Harish-Chandra parameters [16]:

$$m_\beta \equiv (\Lambda + \rho, \beta) \;,$$

where β is any positive root of $\mathcal{G}^{\mathbb{C}}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms. (Clearly, both the Dynkin labels and Harish-Chandra parameters have their origin in the BGG reducibility condition.)

3. The Lie algebra $su(n,n)$ and parabolically related

Let $\mathcal{G} = su(n,n)$, $n \geq 2$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{C})_{\mathbb{R}}$. The number of ERs in the main multiplets is equal to [5]

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = \binom{2n}{n}$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi \;=\; \{n_1, \ldots, n_{n-1}, n_{n+1} \ldots, n_{2n-1};\; c\}\;, \quad n_j \in \mathbb{N}\;, \quad c = d - \tfrac{1}{2}n^2 \tag{1}$$

The restricted Weyl reflection is given by the Knapp–Stein integral operators [17]:

$$\begin{aligned} &G_{KS} : \mathcal{C}_{\chi} \longrightarrow \mathcal{C}_{\chi'}\;, \qquad \chi' \;=\; \{(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^*;\; -c\}\;, \\ &(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^* \doteq (n_{n+1}, \ldots, n_{2n-1}, n_1, \ldots, n_{n-1}) \end{aligned} \tag{2}$$

Multiplets

Below we give the multiplets for $su(n,n)$ for $n = 2, 3, 4$. They are valid also for $sl(2n,\mathbb{R})$ with $\mathcal{M}$-factor $sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$, and when $n = 2k$ these are multiplets also for the parabolically related algebra $su^*(4k)$ with $\mathcal{M}$-factor $su^*(2k) \oplus su^*(2k)$,

There are several types of multiplets: the main type, which contains maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations, and many reduced types of multiplets.

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $su(n,n)$, i.e., they will be labelled by the $2n-1$ positive Dynkin labels $m_i \in \mathbb{N}$.

4. Multiplets of SU(2,2), SL(4,$\mathbb{R}$) and SU*(4)

The main multiplet contains six ERs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned} \chi_0^{\pm} &= \{(m_1, m_3)^{\pm}; \pm\frac{(m_1 + 2m_2 + m_3)}{2}\} \\ \chi'^{\pm} &= \{(m_{12}, m_{23})^{\pm}; \pm\tfrac{1}{2}(m_1 + m_3)\} \\ \chi''^{\pm} &= \{(m_2, m_{13})^{\pm}; \pm\tfrac{1}{2}(m_3 - m_1)\} \end{aligned} \tag{3}$$

where we have used for the numbers $m_\beta = (\Lambda(\chi) + \rho, \beta)$ the same compact notation as for the roots β, and

$$(n_1, n_3)^- = (n_1, n_3)\;, \qquad (n_1, n_3)^+ = (n_1, n_3)^* = (n_3, n_1) \tag{4}$$

These multiplets were given first for $su^*(4)$ [18].

Obviously, the pairs in (3) are related by Knapp-Stein integral operators, i.e.,

$$G_{KS} : \mathcal{C}_{\chi^{\mp}} \longrightarrow \mathcal{C}_{\chi^{\pm}}$$

The multiplets are given explicitly in Fig. 1, where we use the notation: $\Lambda^{\pm} = \Lambda(\chi^{\pm})$. Each intertwining differential operator is represented by an arrow accompanied by a symbol i_{jk} encoding the root α_{jk} and the number $m_{\alpha_{jk}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential

operators which are non-composite are displayed, and that the data β, m_β, which is involved in the embedding $V^\Lambda \longrightarrow V^{\Lambda - m_\beta, \beta}$ turns out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 2n-1\}$, such that $m_\beta = m_i$. Hence the data α_{jk}, $m_{\alpha_{jk}}$ is represented by i_{jk} on the arrows.

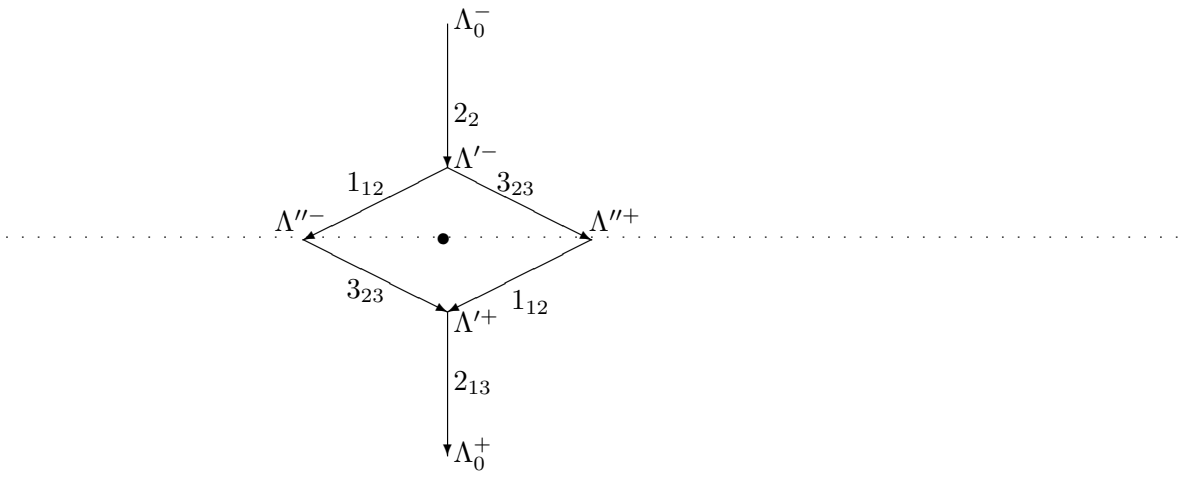


Fig. 1. Main multiplets for $su(2,2)$, $su^*(4)$ and $sl(4,\mathbb{R})$

The pairs $\Lambda^\pm$ are symmetric w.r.t. to the bullet in the middle of the figure - this represents the Weyl symmetry realized by the Knapp-Stein operators.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $\mathcal{G}$ with signature $\{m_1, m_2, m_3\}$ of dimension: $m_1 m_2 m_3 m_{12} m_{23} m_{13}/6$. The subspace $\mathcal{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator acting from χ^- to χ'^-. When all $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that case the conformal weight is zero: $d = 2 + c = 2 - \frac{1}{2}(m_1 + 2m_2 + m_3)_{|m_i=1} = 0$.

In the conjugate ER χ_0^+ there is a unitary subrepresentation in an infinite-dimensional subspace $\mathcal{D}$. It is annihilated by the operator G^-, and is the image of the operator G^+.

All the above is valid also for the algebras $sl(4,\mathbb{R}) \cong so(3,3)$ and $su^*(4) \cong so(5,1)$. However, the latter two do not have discrete series representations. On the other hand the algebra $su(2,2) \cong so(4,2)$ had discrete series representations and furthermore highest/lowest weight series representations.

Thus, in the case of $su(2,2)$ the ER χ_0^+ contains both the holomorphic discrete series representation and the conjugate anti-holomorphic discrete series. The direct sum of the latter two is realized in the invariant subspace $\mathcal{D}$ of the ER χ_0^+.

Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.
The conformal weight of the ER χ_0^+ has the restriction $d = 2 + c = 2 + \frac{1}{2}(m_1 + 2m_2 + m_3) \geq 4$.

Remark on SU(1,1)

As we mentioned the case $su(1,1)$ is well known - it was studied 60 years ago in the isomorphic form $sl(2,\mathbb{R})$ by Gelfand et al [19] and by Bargmann [20]. In the current setting it was given in [21]. Here we shall only mention that the multiplets contain two ERS/GVMs (cf. $\binom{2n}{n}_{n=1} = 2$), and we can take as their representatives the pair $\Lambda^{\pm}$ and all statements that fit the setting are true. In fact, the old results are prototypical for these pairs, which appear once for each algebra of the conformal type.$\diamondsuit$

Reduced multiplets.
There are three types of reduced multiplets, R_1, R_2, R_3. Each of them contains two ERS/GVMs and may be obtained from the main multiplet by setting formally $m_1 = 0$, $m_2 = 0$, $m_3 = 0$, resp. The signatures are

$$\begin{aligned} {}_1\chi^{\pm} &= \{(m_2, m_{23})^{\pm}; \pm\tfrac{1}{2}m_3\} \\ {}_2\chi^{\pm} &= \{(m_1, m_3)^{\pm}; \pm\tfrac{1}{2}(m_1+m_3)\} \\ {}_3\chi^{\pm} &= \{(m_{12}, m_2)^{\pm}; \pm\tfrac{1}{2}m_1\} \end{aligned} \tag{5}$$

The above is valid for the parabolically related algebras $su(2,2), su^*(4), sl(4,\mathbb{R})$. For $su(2,2)$ the ER ${}_2\chi^+$ contains the limits of the (anti)holomorphic discrete series representations. Its conformal weight has the restriction $d = 2 + \frac{1}{2}(m_1+m_3) \geq 3$.

Actually, types R_1, R_3 are conjugated under the * operation (that is not the Weyl symmetry since the sign of c is not changed).

Finally, there is the reduced multiplet R_{13} containing a single representation

$$\chi^s = \{(m,m); 0\}$$

This multiplet may be omitted from this classification since it contains no operators, but its importance was understood in the framework of conformal supersymmetry, i.e., in the multiplet classification for the superconformal algebra $su(2,2/N)$ given in [22]. It turns out that the infinite multiplets of $su(2,2/N)$ have as building blocks all mentioned above multiplets of $su(2,2)$ - sextets, doublets and singlets.

5. Multiplets of SU(3,3) and SL(6,$\mathbb{R}$)

The main multiplet contains 20 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned} \chi_0^{\pm} &= \{(m_1,m_2,m_4,m_5)^{\pm}; \pm m_\rho\} \\ \chi_a^{\pm} &= \{(m_1,m_{23},m_{34},m_5)^{\pm}; \pm(m_\rho - m_3)\} \\ \chi_b^{\pm} &= \{(m_{12},m_3,m_{24},m_5)^{\pm}; \pm(m_\rho - m_{23})\} \\ \chi_{b'}^{\pm} &= \{(m_1,m_{24},m_3,m_{45})^{\pm}; \pm(m_\rho - m_{34})\} \\ \chi_c^{\pm} &= \{(m_2,m_3,m_{14},m_5)^{\pm}; \pm(m_\rho - m_{13})\} \\ \chi_{c'}^{\pm} &= \{(m_{12},m_{34},m_{23},m_{45})^{\pm}; \pm(m_\rho - m_{24})\} \\ \chi_{c''}^{\pm} &= \{(m_1,m_{25},m_3,m_4)^{\pm}; \pm(m_\rho - m_{35})\} \\ \chi_d^{\pm} &= \{(m_2,m_{34},m_{13},m_{45})^{\pm}; \pm(m_\rho - m_{14})\} \\ \chi_{d'}^{\pm} &= \{(m_{12},m_{35},m_{23},m_4)^{\pm}; \pm(m_\rho - m_{25})\} \\ \chi_e^{\pm} &= \{(m_2,m_{35},m_{13},m_4)^{\pm}; \pm(m_\rho - m_{15})\} \end{aligned} \tag{6}$$

where $m_\rho = \frac{1}{2}(m_1 + 2m_2 + 3m_3 + 2m_4 + m_5)$. They are given in Fig. 2.

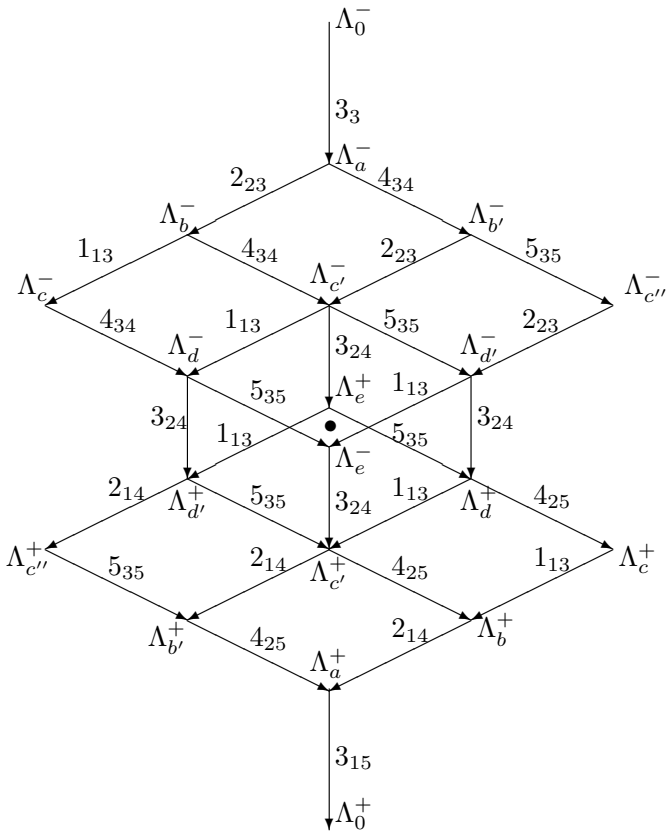


Fig. 2. Main multiplets for $su(3,3)$ and $sl(6,\mathbb{R})$

All general facts that were stated in the $SU(2,2)$ case are valid also here, in particular, the special role of the pair $\chi_0^{\pm}$. The finite-dimensional irreps $\mathcal{E}$ of $su(3,3)$ or $sl(6,\mathbb{R})$ are sitting in the ERs χ_0^- and have dimension as the UIRs of $SU(6)$.

Reduced multiplets.
There are five types of reduced multiplets, R_a^3, $a = 1,\dots,5$, which may be obtained from the main multiplet by setting formally $m_a = 0$. Multiplets of type R_4^3, R_5^3, are conjugate to the multiplets of type R_2^3, R_1^3, resp., and are not shown.

The reduced multiplets of type R_3^3 contain 14 ERs/GVMs with signatures:

$$\begin{array}{rcl}
\chi_0^{\pm} & = & \{\,(m_1,m_2,m_4,m_5)^{\pm}\,;\,\pm m_\rho\,\} \\
\chi_b^{\pm} & = & \{\,(m_{12},0,m_{24},m_5)^{\pm}\,;\,\pm(m_\rho-m_2)\,\} \\
\chi_{b'}^{\pm} & = & \{\,(m_1,m_{24},0,m_{45})^{\pm}\,;\,\pm(m_\rho-m_4)\,\} \\
\chi_c^{\pm} & = & \{\,(m_2,0,m_{14},m_5)^{\pm}\,;\,\pm(m_\rho-m_{12})\,\} \\
\chi_{c''}^{\pm} & = & \{\,(m_1,m_{25},0,m_4)^{\pm}\,;\,\pm(m_\rho-m_{45})\,\} \\
\chi_d^{\pm} & = & \{\,(m_2,m_4,m_{12},m_{45})^{\pm}\,;\,\pm(m_\rho-m_{12,4})=\mp(m_\rho-m_{2,45})\,\} \\
\chi_e^{\pm} & = & \{\,(m_2,m_{45},m_{12},m_4)^{\pm}\,;\,\pm(m_\rho-m_{2,4})=\pm\tfrac{1}{2}(m_1+m_5)\,\}\ ,
\end{array} \tag{7}$$

here $m_\rho = \frac{1}{2}(m_1+2m_2+2m_4+m_5)$. These multiplets are given in Fig. 3. They may be called the main type of reduced multiplets since for $su(3,3)$ in χ_0^+ are contained the limits of the (anti)holomorphic discrete series.

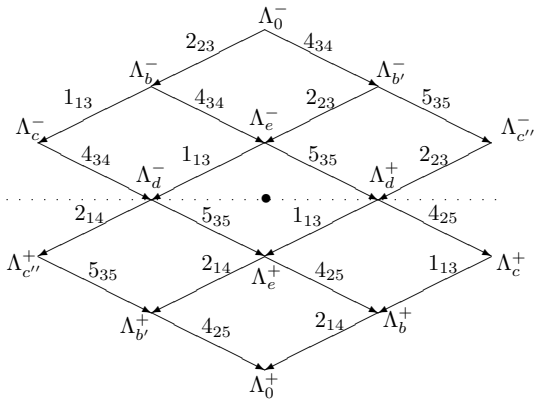


Fig. 3. Reduced multiplets of type R_3^3 for $su(3,3)$ and $sl(6,\mathbb{R})$

The reduced multiplets of type R_2^3, resp., R_1^3, contain 14 ERs/GVMs each. These multiplets are given in Fig. 4, resp., Fig. 5 :

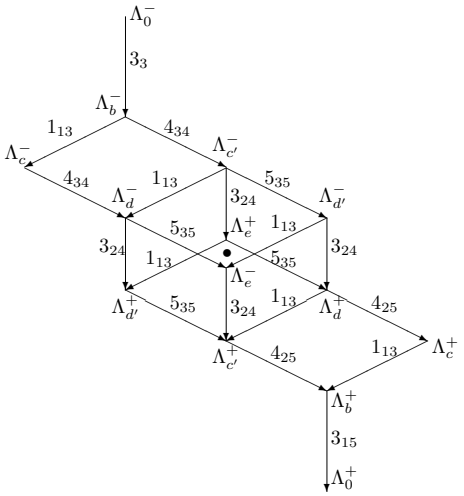


Fig. 4. Reduced multiplets of type R_2^3 for $su(3,3)$ and $sl(6,\mathbb{R})$

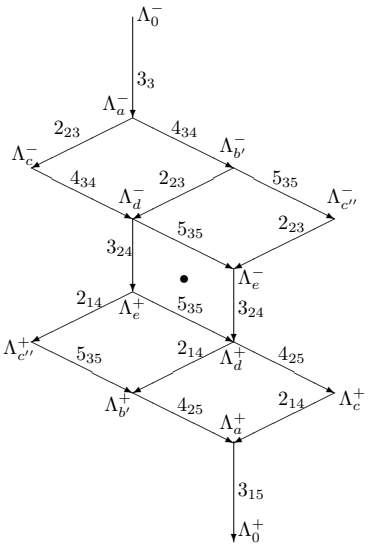


Fig. 5. Reduced multiplets of type R_1^3 for $su(3,3)$ and $sl(6,\mathbb{R})$

Further reduction of multiplets

There are further reductions of the multiplets denoted by R_{ab}^3, $a,b=1,\ldots,5$, $a<b$, which may be obtained from the main multiplet by setting formally $m_a = m_b = 0$. From these ten reductions four (for $(a,b)=(1,2),(2,3),(3,4),(4,5)$) do not contain representations of physical interest, i.e., induced from finite-dimensional irreps of the $\mathcal{M}$ subalgebra. From the others R_{35}^3 and R_{25}^3 are conjugated to R_{13}^3 and R_{14}^3, resp., as explained above. Thus, we present only four types of multiplets.

The reduced multiplets of type R_{13}^3 contain 10 ERs/GVMs with signatures:

$$\begin{aligned}
\chi_a^{\pm} &= \{\,(0,m_2,m_4,m_5)^{\pm}\,;\,\pm m_\rho\,\} \\
\chi_b^{\pm} &= \{\,(m_2,0,m_{2,4},m_5)^{\pm}\,;\,\pm(m_\rho-m_2)\,\} \\
\chi_{b'}^{\pm} &= \{\,(0,m_{2,4},0,m_{45})^{\pm}\,;\,\pm(m_\rho-m_4)\,\} \\
\chi_c^{\pm} &= \{\,(0,m_{2,45},0,m_4)^{\pm}\,;\,\pm(m_\rho-m_{45})\,\} \\
\chi_d^{\pm} &= \{\,(m_2,m_4,m_2,m_{45})^{\pm}\,;\,\pm(m_\rho-m_{2,4})=\pm\tfrac{1}{2}m_5\,\}\ ,
\end{aligned} \tag{8}$$

here $m_\rho = m_2 + m_4 + \frac{1}{2}m_5$. The multiplets are given in Fig. 6.

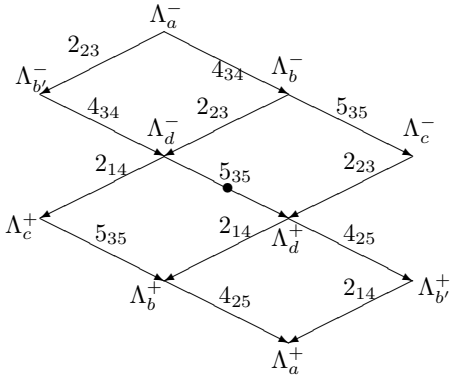


Fig. 6. Reduced multiplets of type R_{13}^3 for $su(3,3)$ and $sl(6,\mathbb{R})$

Note that the differential operator (of order m_5) from χ_d^- to χ_d^+ is a degeneration of an integral Knapp-Stein operator.

The reduced multiplets of type R^3_{15} contain 10 ERs/GVMs with signatures:

$$\begin{array}{rcl}
\chi_0^{\pm} & = & \{\,(0,m_2,m_4,0)^{\pm}\,;\,\pm m_\rho\,\} \\
\chi_a^{\pm} & = & \{\,(0,m_{23},m_{34},0)^{\pm}\,;\,\pm(m_\rho-m_3)\,\} \\
\chi_b^{\pm} & = & \{\,(m_2,m_3,m_{24},0)^{\pm}\,;\,\pm(m_\rho-m_{23})\,\} \\
\chi_{b'}^{\pm} & = & \{\,(0,m_{24},m_3,m_4)^{\pm}\,;\,\pm(m_\rho-m_{34})\,\} \\
\chi_d^{\pm} & = & \{\,(m_2,m_{34},m_{23},m_4)^{\pm}\,;\,\pm(m_\rho-m_{24})=\pm\tfrac{1}{2}m_3\,\}\ ,
\end{array} \tag{9}$$

here $m_\rho = m_2 + \frac{3}{2}m_3 + m_4$. The multiplets are given in Fig. 7. Here the differential operator (of order m_3) from χ_d^- to χ_d^+ is a degeneration of an integral Knapp-Stein operator.

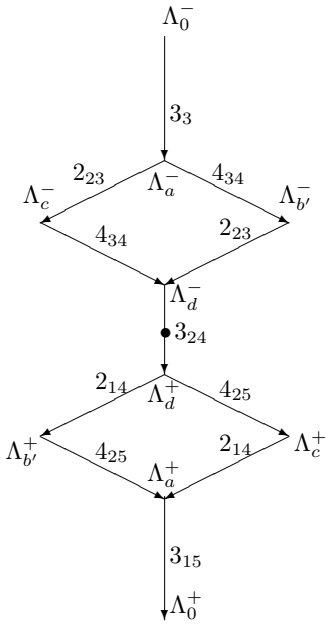


Fig. 7. Reduced multiplets of type R^3_{15} for $su(3,3)$ and $sl(6,\mathbb{R})$

The reduced multiplets of type R^3_{14}, R^3_{24}, contain 10 ERs/GVMs each, the corresponding multiplets being given below:

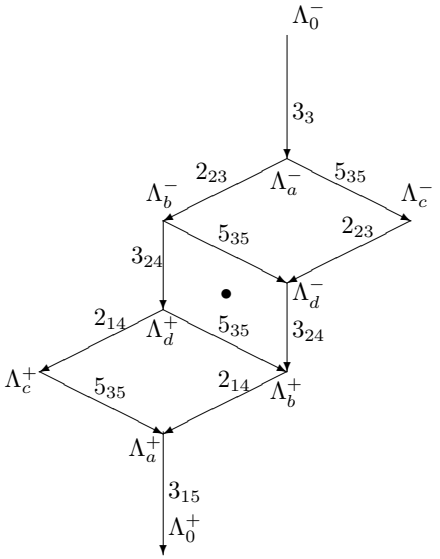


Fig. 8. Reduced multiplets of type R^3_{14} for $su(3,3)$ and $sl(6,\mathbb{R})$

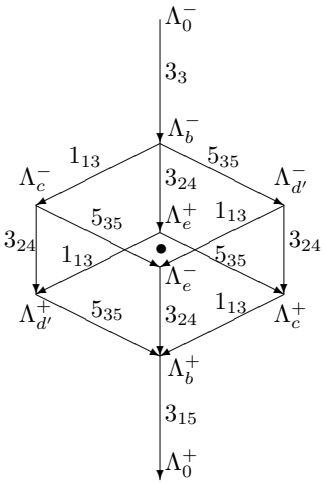


Fig. 9. Reduced multiplets of type R^3_{24} for $su(3,3)$ and $sl(6,\mathbb{R})$

Last reduction of multiplets

There are further reductions of the multiplets - triple and quadruple, but only one triple reduction contains representations of physical interest. Namely, this is the multiplet R^3_{135}, which may be obtained from the main multiplet by setting formally $m_1 = m_3 = m_5 = 0$. It contains 7 ERs/GVMs with signatures:

$$\begin{aligned}
\chi_a^{\pm} &= \{\,(0, m_2, m_4, 0)^{\pm}\,;\, \pm m_\rho = \pm m_{2,4}\,\} \\
\chi_b^{\pm} &= \{\,(0, m_{2,4}, 0, m_4)^{\pm}\,;\, \pm m_2\,\} \\
\chi_{b'}^{\pm} &= \{\,(m_2, 0, m_{2,4}, 0)^{\pm}\,;\, \pm m_4\,\} \\
\chi_d &= \{\,(m_2, m_4, m_2, m_4)\,;\, 0\,\}
\end{aligned} \tag{10}$$

The multiplets are given below:

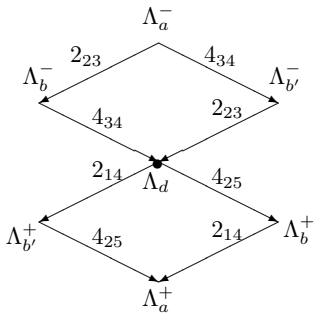


Fig. 10. Reduced multiplets of type R^3_{135} for $su(3,3)$ and $sl(6,\mathbb{R})$

The representation χ^d is a singlet, not in a pair, since it has zero weight c, and the $\mathcal{M}$ entries are self-conjugate. It is placed in the middle of the figure as the bullet. That ER contains the *minimal irreps* characterized by two positive integers which are denoted in this context as $m_2\,, m_4$. Each such irrep is the kernel of the two invariant differential operators $\mathcal{D}_{14}^{m_2}$ and $\mathcal{D}_{25}^{m_4}$, which are of order m_2 , m_4 , resp., corresponding to the noncompact roots α_{14} , α_{25} , resp.

6. Multiplets of SU(4,4), SL(8,$\mathbb{R}$) and SU*(8)

The main multiplet R^4 contains 70 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^{\pm} &= \{\,(m_1,m_2,m_3,m_5,m_6,m_7)^{\pm}\,;\,\pm m_\rho\,\} \qquad (11)\\
\chi_{00}^{\pm} &= \{\,(m_1,m_2,m_{34},m_{45},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_4)\,\}\\
\chi_{10}^{\pm} &= \{\,(m_1,m_{23},m_4,m_{35},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_{34})\,\}\\
\chi_{01}^{\pm} &= \{\,(m_1,m_2,m_{35},m_4,m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{45})\,\}\\
\chi_{20}^{\pm} &= \{\,(m_{12},m_3,m_4,m_{25},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_{24})\,\}\\
\chi_{11}^{\pm} &= \{\,(m_1,m_{23},m_{45},m_{34},m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{35})\,\}\\
\chi_{02}^{\pm} &= \{\,(m_1,m_2,m_{36},m_4,m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{46})\,\}\\
\chi_{30}^{\pm} &= \{\,(m_2,m_3,m_4,m_{15},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_{14})\,\}\\
\chi_{21}^{\pm} &= \{\,(m_{12},m_3,m_{45},m_{24},m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{25})\,\}\\
\chi_{12}^{\pm} &= \{\,(m_1,m_{23},m_{46},m_{34},m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{36})\,\}\\
\chi_{03}^{\pm} &= \{\,(m_1,m_2,m_{37},m_4,m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{47})\,\}\\
\chi_{31}^{\pm} &= \{\,(m_2,m_3,m_{45},m_{14},m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{15})\,\}\\
\chi_{22}^{\pm} &= \{\,(m_{12},m_3,m_{46},m_{24},m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{26})\,\}\\
\chi_{13}^{\pm} &= \{\,(m_1,m_{23},m_{47},m_{34},m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{37})\,\}\\
\chi_{32}^{\pm} &= \{\,(m_2,m_3,m_{46},m_{14},m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{16})\,\}\\
\chi_{23}^{\pm} &= \{\,(m_{12},m_3,m_{47},m_{24},m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{27})\,\}\\
\chi_{33}^{\pm} &= \{\,(m_2,m_3,m_{47},m_{14},m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{17})\,\}\\
\chi_{00}^{\prime\pm} &= \{\,(m_1,m_{24},m_5,m_3,m_{46},m_7)^{\pm}\,;\,\pm(m_\rho-m_{35}-m_4)\,\}\\
\chi_{10}^{\prime\pm} &= \{\,(m_{12},m_{34},m_5,m_{23},m_{46},m_7)^{\pm}\,;\,\pm(m_\rho-m_{25}-m_4)\,\}\\
\chi_{01}^{\prime\pm} &= \{\,(m_1,m_{24},m_{56},m_3,m_{45},m_{67})^{\pm}\,;\,\pm(m_\rho-m_{36}-m_4)\,\}\\
\chi_{20}^{\prime\pm} &= \{\,(m_2,m_{34},m_5,m_{13},m_{46},m_7)^{\pm}\,;\,\pm(m_\rho-m_{15}-m_4)\,\}\\
\chi_{11}^{\prime\pm} &= \{\,(m_{12},m_{34},m_{56},m_{23},m_{45},m_{67})^{\pm}\,;\,\pm(m_\rho-m_{26}-m_4)\,\}\\
\chi_{02}^{\prime\pm} &= \{\,(m_1,m_{24},m_{57},m_3,m_{45},m_6)^{\pm}\,;\,\pm(m_\rho-m_{37}-m_4)\,\}\\
\chi_{20}^{\prime\prime\pm} &= \{\,(m_{13},m_4,m_5,m_2,m_{36},m_7)^{\pm}\,;\,\pm(m_\rho-m_{25}-m_{34})\,\}\\
\chi_{21}^{\prime\prime\pm} &= \{\,(m_{13},m_4,m_{56},m_2,m_{35},m_{67})^{\pm}\,;\,\pm(m_\rho-m_{26}-m_{34})\,\}\\
\chi_{12}^{\prime\prime\pm} &= \{\,(m_{12},m_{35},m_6,m_{23},m_4,m_{57})^{\pm}\,;\,\pm(m_\rho-m_{26}-m_{45})\,\}\\
\chi_{02}^{\prime\prime\pm} &= \{\,(m_1,m_{25},m_6,m_3,m_4,m_{57})^{\pm}\,;\,\pm(m_\rho-m_{36}-m_{45})\,\}
\end{aligned}
$$

$$
\begin{aligned}
\chi'^{\pm}_{30} &= \{\,(m_{23}, m_4, m_5, m_{12}, m_{36}, m_7)^{\pm}\,;\, \pm(m_\rho - m_{15} - m_{34})\,\} \\
\chi'^{\pm}_{21} &= \{\,(m_2, m_{34}, m_{56}, m_{13}, m_{45}, m_{67})^{\pm}\,;\, \pm(m_\rho - m_{16} - m_4)\,\} \\
\chi'^{\pm}_{12} &= \{\,(m_{12}, m_{34}, m_{57}, m_{23}, m_{45}, m_6)^{\pm}\,;\, \pm(m_\rho - m_{27} - m_4)\,\} \\
\chi'^{\pm}_{03} &= \{\,(m_1, m_{25}, m_{67}, m_3, m_4, m_{56})^{\pm}\,;\, \pm(m_\rho - m_{37} - m_{45})\,\} \\
\chi'^{\pm}_{40} &= \{\,(m_3, m_4, m_5, m_1, m_{26}, m_7)^{\pm}\,;\, \pm(m_\rho - m_{15} - m_{24})\,\} \\
\chi'^{\pm}_{31} &= \{\,(m_{23}, m_4, m_{56}, m_{12}, m_{35}, m_{67})^{\pm}\,;\, \pm(m_\rho - m_{16} - m_{34})\,\} \\
\chi'^{\pm}_{22} &= \{\,(m_2, m_{34}, m_{57}, m_{13}, m_{45}, m_6)^{\pm}\,;\, \pm(m_\rho - m_{17} - m_4)\,\} \\
\chi''^{\pm}_{22} &= \{\,(m_{13}, m_4, m_{57}, m_2, m_{35}, m_6)^{\pm}\,;\, \pm(m_\rho - m_{27} - m_{34})\,\}\ ,
\end{aligned}
$$

where $m_\rho = \frac{1}{2}(m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + m_7)$. The multiplets are given explicitly in Fig. 11 (first in [3]).

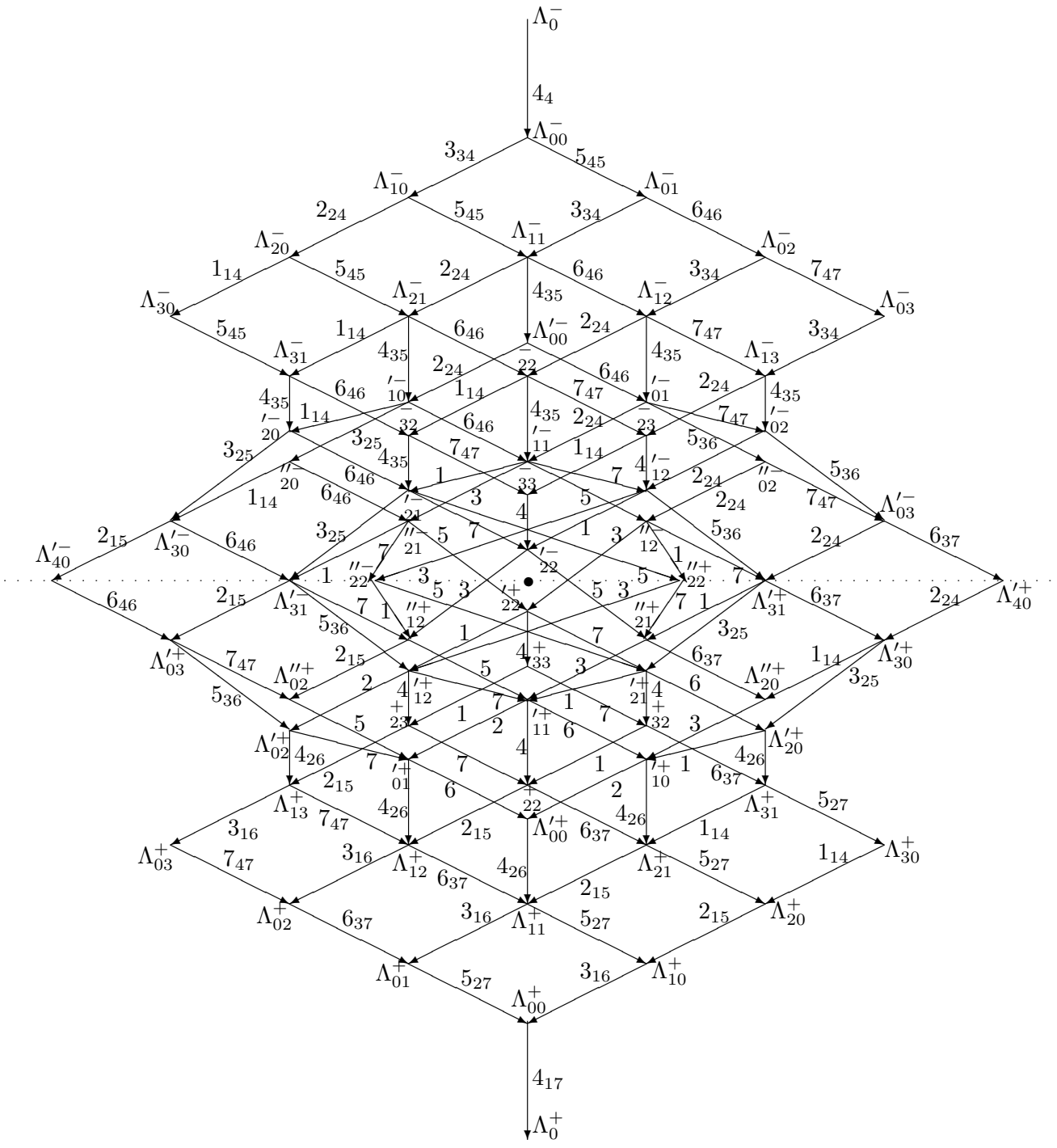


Fig. 11. Main multiplets R^4 for $su(4,4)$, $su^*(8)$ and $sl(8,\mathbb{R})$

Main reduced multiplets

There are nine physically relevant and essentially different reductions of multiplets denoted by R^4_3, R^4_3, R^4_2, R^4_1. Each of them contains 50 ERs/GVMs [7]. Here we present only the reduced multiplets R^4_4. Their 50 ERs/GVMs has signatures that can be given in the following pair-wise

manner:

$$
\begin{aligned}
\chi_{0}^{\pm} &= \{\,(m_1,m_2,m_3,m_5,m_6,m_7)^{\pm}\,;\,\pm m_\rho\,\} \qquad (12)\\
\chi_{10}^{\pm} &= \{\,(m_1,m_{23},0,m_{3,5},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_3)\,\}\\
\chi_{01}^{\pm} &= \{\,(m_1,m_2,m_{3,5},0,m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_5)\,\}\\
\chi_{20}^{\pm} &= \{\,(m_{12},m_3,0,m_{23,5},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_{23})\,\}\\
\chi_{11}^{\pm} &= \{\,(m_1,m_{23},m_5,m_3,m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{3,5})\,\}\\
\chi_{02}^{\pm} &= \{\,(m_1,m_2,m_{3,56},0,m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{56})\,\}\\
\chi_{30}^{\pm} &= \{\,(m_2,m_3,0,m_{13,5},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_{13})\,\}\\
\chi_{21}^{\pm} &= \{\,(m_{12},m_3,m_5,m_{23},m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{23,5})\,\}\\
\chi_{12}^{\pm} &= \{\,(m_1,m_{23},m_{56},m_3,m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{3,56})\,\}\\
\chi_{03}^{\pm} &= \{\,(m_1,m_2,m_{3,57},0,m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{57})\,\}\\
\chi_{31}^{\pm} &= \{\,(m_2,m_3,m_5,m_{13},m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{13,5})\,\}\\
\chi_{22}^{\pm} &= \{\,(m_{12},m_3,m_{56},m_{23},m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{23,56})\,\}\\
\chi_{13}^{\pm} &= \{\,(m_1,m_{23},m_{57},m_3,m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{3,57})\,\}\\
\chi_{32}^{\pm} &= \{\,(m_2,m_3,m_{56},m_{13},m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{13,56})\,\}\\
\chi_{23}^{\pm} &= \{\,(m_{12},m_3,m_{57},m_{23},m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{23,57})\,\}\\
\chi_{33}^{\pm} &= \{\,(m_2,m_3,m_{57},m_{13},m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{13,57})\,\}\\
\chi_{20}^{\prime\prime\pm} &= \{\,(m_{13},0,m_5,m_2,m_{3,56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{23,5}-m_3)\,\}\\
\chi_{21}^{\prime\prime\pm} &= \{\,(m_{13},0,m_{56},m_2,m_{3,5},m_{67})^{\pm}\,;\,\pm(m_\rho-m_{23,56}-m_3)\,\}\\
\chi_{12}^{\prime\prime\pm} &= \{\,(m_{12},m_{3,5},m_6,m_{23},0,m_{57})^{\pm}\,;\,\pm(m_\rho-m_{23,56}-m_5)\,\}\\
\chi_{02}^{\prime\prime\pm} &= \{\,(m_1,m_{23,5},m_6,m_3,0,m_{57})^{\pm}\,;\,\pm(m_\rho-m_{3,56}-m_5)\,\}\\
\chi_{30}^{\prime\pm} &= \{\,(m_{23},0,m_5,m_{12},m_{3,56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{13,5}-m_3)\,\}\\
\chi_{03}^{\prime\pm} &= \{\,(m_1,m_{23,5},m_{67},m_3,0,m_{56})^{\pm}\,;\,\pm(m_\rho-m_{3,57}-m_5)\,\}\\
\chi_{40}^{\prime\pm} &= \{\,(m_3,0,m_5,m_1,m_{23,56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{13,5}-m_{23})\,\}\\
\chi_{31}^{\prime\pm} &= \{\,(m_{23},0,m_{56},m_{12},m_{3,5},m_{67})^{\pm}\,;\,\pm(m_\rho-m_{13,56}-m_3)\,\}\\
\chi_{22}^{\prime\prime\pm} &= \{\,(m_{13},0,m_{57},m_2,m_{3,5},m_6)^{\pm}\,;\,\pm(m_\rho-m_{23,57}-m_3)\,\}\,,
\end{aligned}
$$

here $m_\rho=\frac{1}{2}(m_1+2m_2+3m_3+3m_5+2m_6+m_7)$. This is a very important type of reduced multiplets since for $su(4,4)$ in χ_0^+ are contained the limits of the (anti)holomorphic discrete series. The multiplets are given in Fig. 12.

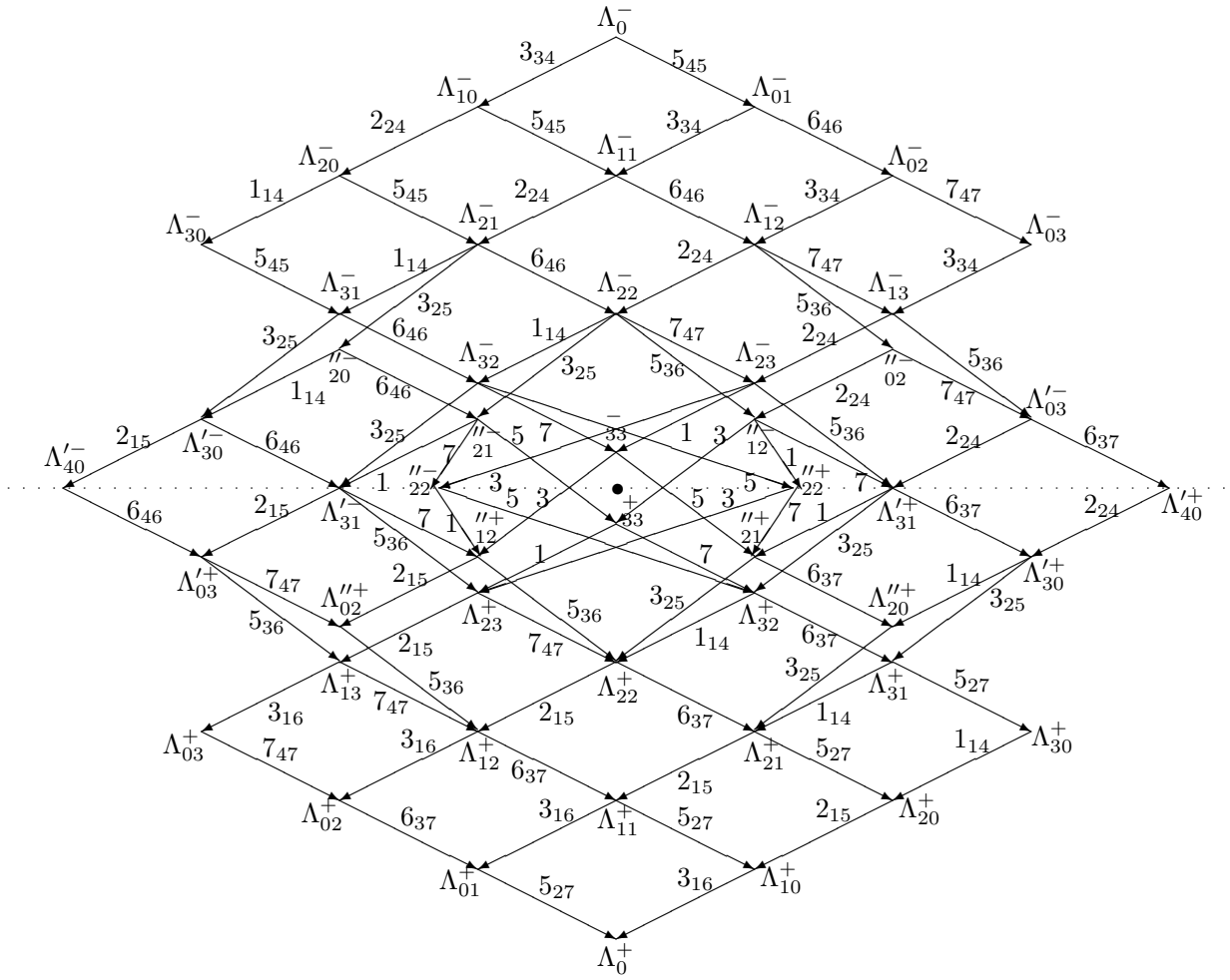


Fig. 12. Reduced multiplets R^4_4 for $su(4,4)$, $su^*(8)$ and $sl(8,\mathbb{R})$

Further reduction of multiplets

There are nine physically relevant and essentially different further reductions of multiplets denoted by R^3_{ab}, $(a,b) = (13),(14),(15),(16),(17),(24),(25),(26),(35)$. They contain 36 ERs/GVMs each and were given in [7]. Here we give only type R^4_{13} :

$$
\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,m_5,m_6,m_7)^{\pm}\,;\,\pm m_\rho)\,\} \\
\chi^{\pm}_{10} &= \{\,(0,m_2,m_4,m_{45},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_4)\,\} \\
\chi^{\pm}_{20} &= \{\,(m_2,0,m_4,m_{2,45},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_{2,4})\,\} \\
\chi^{\pm}_{11} &= \{\,(0,m_2,m_{45},m_4,m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{45})\,\} \\
\chi^{\pm}_{21} &= \{\,(m_2,0,m_{45},m_{2,4},m_{56},m_7)^{\pm}\,;\,\pm(m_\rho-m_{2,45})\,\} \\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{46},m_4,m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{46})\,\} \\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{46},m_{2,4},m_5,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{2,46})\,\} \\
\chi^{\pm}_{13} &= \{\,(0,m_2,m_{47},m_4,m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{47})\,\} \\
\chi^{\pm}_{23} &= \{\,(m_2,0,m_{47},m_{2,4},m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{2,47})\,\} \\
\chi'^{\pm}_{00} &= \{\,(0,m_{2,4},m_5,0,m_{46},m_7)^{\pm}\,;\,\pm(m_\rho-m_5-2m_4)\,\} \\
\chi'^{\pm}_{10} &= \{\,(m_2,m_4,m_5,m_2,m_{46},m_7)^{\pm}\,;\,\pm(m_\rho-m_{2,5}-2m_4)\,\} \\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_{56},0,m_{45},m_{67})^{\pm}\,;\,\pm(m_\rho-m_{56}-2m_4)\,\}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\chi'^{\pm}_{11} &= \{\,(m_2, m_4, m_{56}, m_2, m_{45}, m_{67})^{\pm}\,;\, \pm(m_\rho - m_{2,56} - 2m_4)\,\}\\
\chi'^{\pm}_{02} &= \{\,(0, m_{2,4}, m_{57}, 0, m_{45}, m_6)^{\pm}\,;\, \pm(m_\rho - m_{57} - 2m_4)\,\}\\
\chi''^{\pm}_{02} &= \{\,(0, m_{2,45}, m_6, 0, m_4, m_{57})^{\pm}\,;\, \pm(m_\rho - m_6 - 2m_{45})\,\}\\
\chi'^{\pm}_{12} &= \{\,(m_2, m_4, m_{57}, m_2, m_{45}, m_6)^{\pm}\,;\, \pm(m_\rho - m_{2,57} - 2m_4)\,\}\\
\chi'^{\pm}_{03} &= \{\,(0, m_{2,45}, m_{67}, 0, m_4, m_{56})^{\pm}\,;\, \pm(m_\rho - m_{67} - 2m_{45})\,\}\\
\chi'^{\pm}_{40} &= \{\,(0, m_4, m_5, 0, m_{2,46}, m_7)^{\pm}\,;\, \pm(m_\rho - m_5 - 2m_{2,4})\,\}\ ,
\end{aligned}
$$

here $m_\rho = \frac{1}{2}(2m_2 + 4m_4 + 3m_5 + 2m_6 + m_7)$. Their diagram is given in Fig. 13.

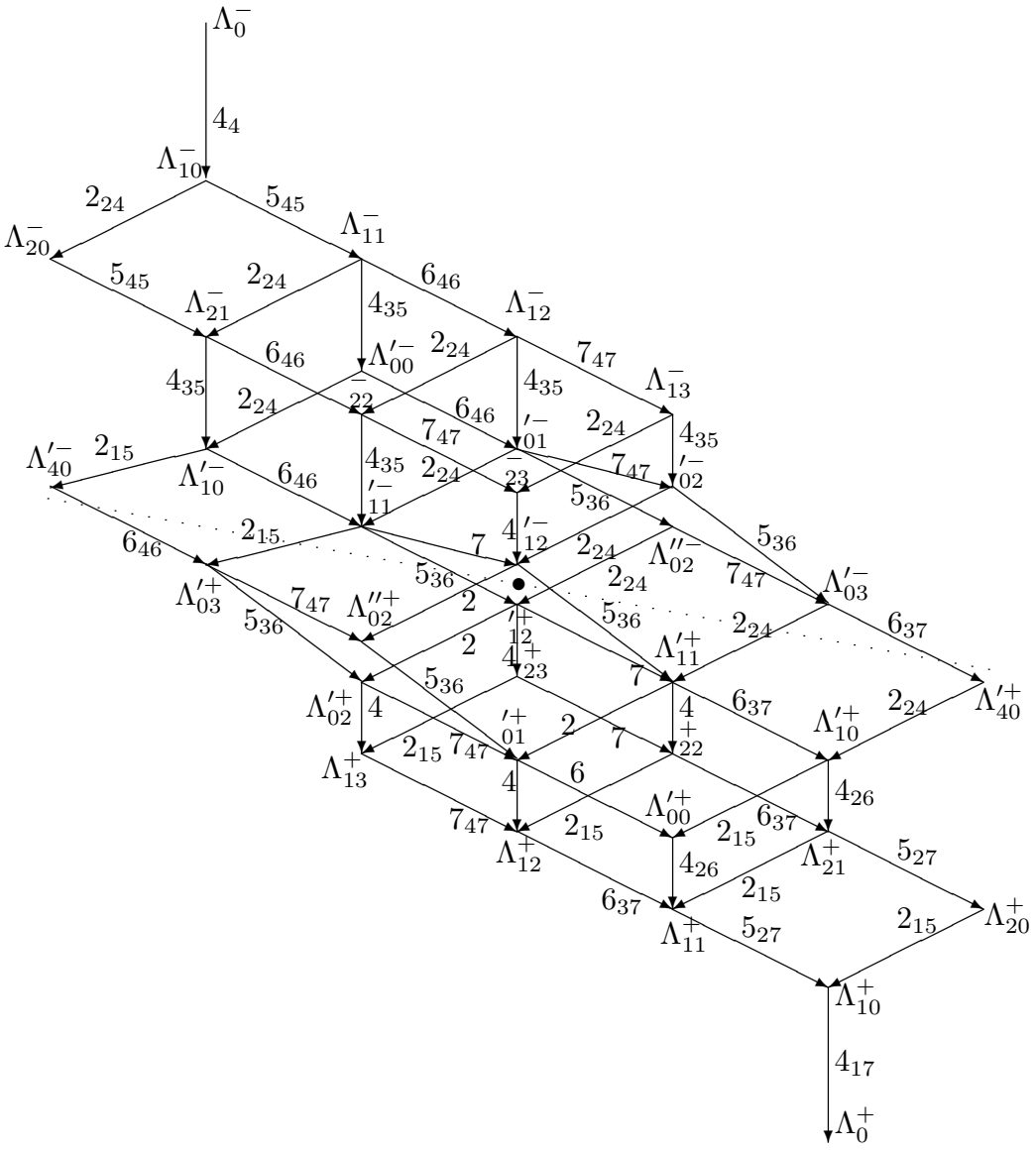


Fig. 13. Reduced multiplets R^4_{13} for $su(4,4)$, $su^*(8)$ and $sl(8,\mathbb{R})$

Yet further reduction of multiplets

There are six physically relevant and essentially different further reductions of multiplets

denoted by R^3_{abc}, $(a,b,c) = (1,3,5),(1,3,6),(1,3,7),(1,4,6),(1,4,7),(2,4,6)$. They contain 26 ERs/GVMs each and were given in [7]. Here we give only two types. First we give R^3_{135} :

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,0,m_6,m_7)^{\pm}\,;\,\pm m_\rho\,\} \qquad (14)\\
\chi^{\pm}_{11} &= \{\,(0,m_2,m_4,m_4,m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_4)\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2,0,m_4,m_{2,4},m_6,m_7)^{\pm}\,;\,\pm(m_\rho-m_{2,4})\,\}\\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{4,6},m_4,0,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{4,6})\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{4,6},m_{2,4},0,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{2,4,6})\,\}\\
\chi^{\pm}_{13} &= \{\,(0,m_2,m_{4,67},m_4,0,m_6)^{\pm}\,;\,\pm(m_\rho-m_{4,67})\,\}\\
\chi^{\pm}_{23} &= \{\,(m_2,0,m_{4,67},m_{2,4},0,m_6)^{\pm}\,;\,\pm(m_\rho-m_{2,4,67})\,\}\\
\chi'^{\pm}_{00} &= \{\,(0,m_{2,4},0,0,m_{4,6},m_7)^{\pm}\,;\,\pm(m_\rho-2m_4)\,\}\\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_6,0,m_4,m_{67})^{\pm}\,;\,\pm(m_\rho-m_6-2m_4)\,\}\\
\chi'^{\pm}_{02} &= \{\,(0,m_{2,4},m_{67},0,m_4,m_6)^{\pm}\,;\,\pm(m_\rho-m_{67}-2m_4)\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_2,m_4,0,m_2,m_{4,6},m_7)^{\pm}\,;\,\pm(m_\rho-m_2-2m_4)\,\}\\
\chi'^{\pm}_{40} &= \{\,(0,m_4,0,0,m_{26},m_7)^{\pm}\,;\,\pm(m_\rho-2m_{2,4})\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_2,m_4,m_6,m_2,m_4,m_{67})^{\pm}\,;\,\pm(m_\rho-m_{2,6}-2m_4)=\pm\tfrac{1}{2}m_7\,\}\ ,
\end{aligned}$$

here $m_\rho = m_2 + 2m_4 + m_6 + \frac{1}{2}m_7$. The multiplets are given in Fig. 3-135. Note that the differential operator (of order m_7) from χ^-_{31} to χ^+_{31} is a degeneration of an integral Knapp-Stein operator.

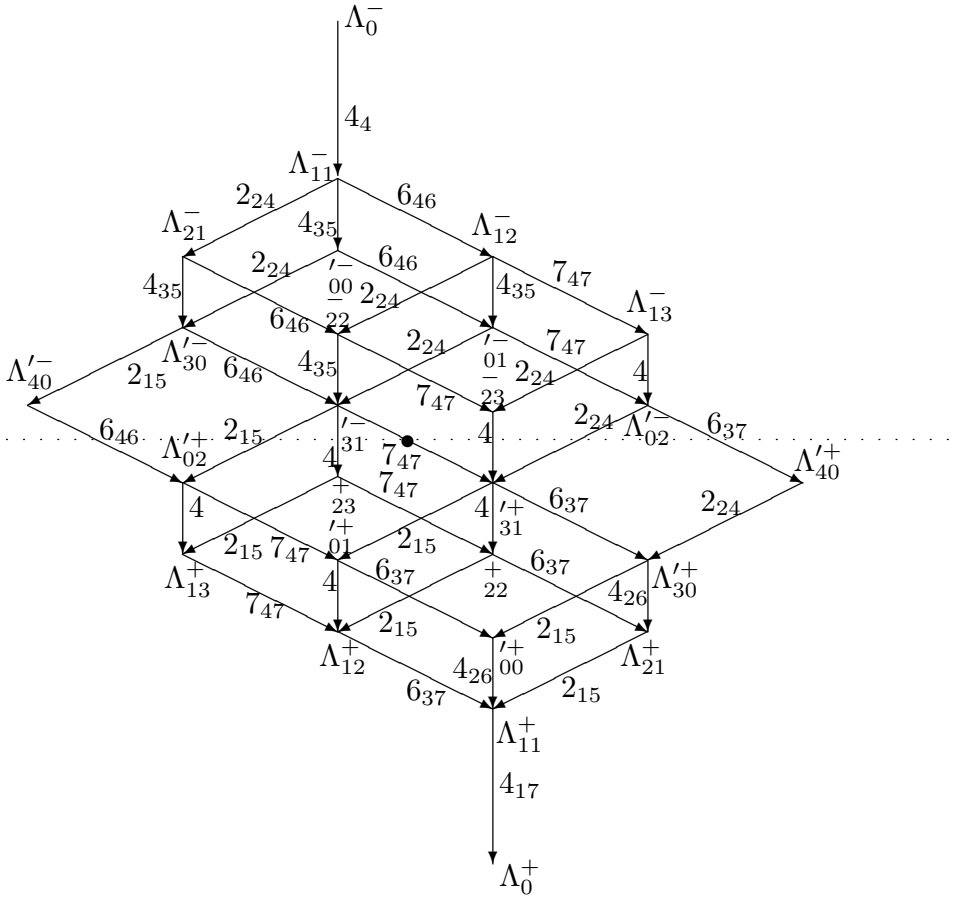


Fig. 14. Reduced multiplets of type R^4_{135} for $su(4,4)$, $su^*(8)$ and $sl(8,\mathbb{R})$

Then we give type R^4_{137} :

$$
\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,m_5,m_6,0)^{\pm}\,;\,\pm m_\rho\,\} \qquad (15)\\
\chi^{\pm}_{10} &= \{\,(0,m_2,m_4,m_{45},m_6,0)^{\pm}\,;\,\pm(m_\rho-m_4)\,\}\\
\chi^{\pm}_{20} &= \{\,(m_2,0,m_4,m_{2,45},m_6,0)^{\pm}\,;\,\pm(m_\rho-m_{2,4})\,\}\\
\chi^{\pm}_{11} &= \{\,(0,m_2,m_{45},m_4,m_{56},0)^{\pm}\,;\,\pm(m_\rho-m_{45})\,\}\\
\chi^{\pm}_{21} &= \{\,(m_2,0,m_{45},m_{2,4},m_{56},0)^{\pm}\,;\,\pm(m_\rho-m_{2,45})\,\}\\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{46},m_4,m_5,m_6)^{\pm}\,;\,\pm(m_\rho-m_{46})\,\}\\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{46},m_{2,4},m_5,m_6)^{\pm}\,;\,(m_\rho-m_{2,46})\,\}\\
\chi'^{\pm}_{00} &= \{\,(0,m_{2,4},m_5,0,m_{46},0)^{\pm}\,;\,\pm(m_\rho-m_5-2m_4)\,\}\\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_{56},0,m_{45},m_6)^{\pm}\,;\,\pm(m_\rho-m_{56}-2m_4)\,\}\\
\chi'^{\pm}_{30} &= \{\,(m_2,m_4,m_5,m_2,m_{46},0)^{\pm}\,;\,\pm(m_\rho-m_{2,5}-2m_4)\,\}\\
\chi'^{\pm}_{03} &= \{\,(0,m_{2,45},m_6,0,m_4,m_{56})^{\pm}\,;\,\pm(m_\rho-m_6-2m_{45})\,\}\\
\chi'^{\pm}_{40} &= \{\,(0,m_4,m_5,0,m_{2,46},0)^{\pm}\,;\,\pm(m_\rho-m_5-2m_{2,4})\,\}\\
\chi'^{\pm}_{31} &= \{\,(m_2,m_4,m_{56},m_2,m_{45},m_6)^{\pm}\,;\,(m_\rho-m_{2,56}-2m_4)=\pm\tfrac{1}{2}m_5\,\}\ ,
\end{aligned}
$$

here $m_\rho = m_2+2m_4+\frac{3}{2}m_5+m_6$. The multiplets are given in Fig. 15. Note that the differential operator (of order m_5) from χ^-_{31} to χ^+_{31} is a degeneration of an integral Knapp-Stein operator.

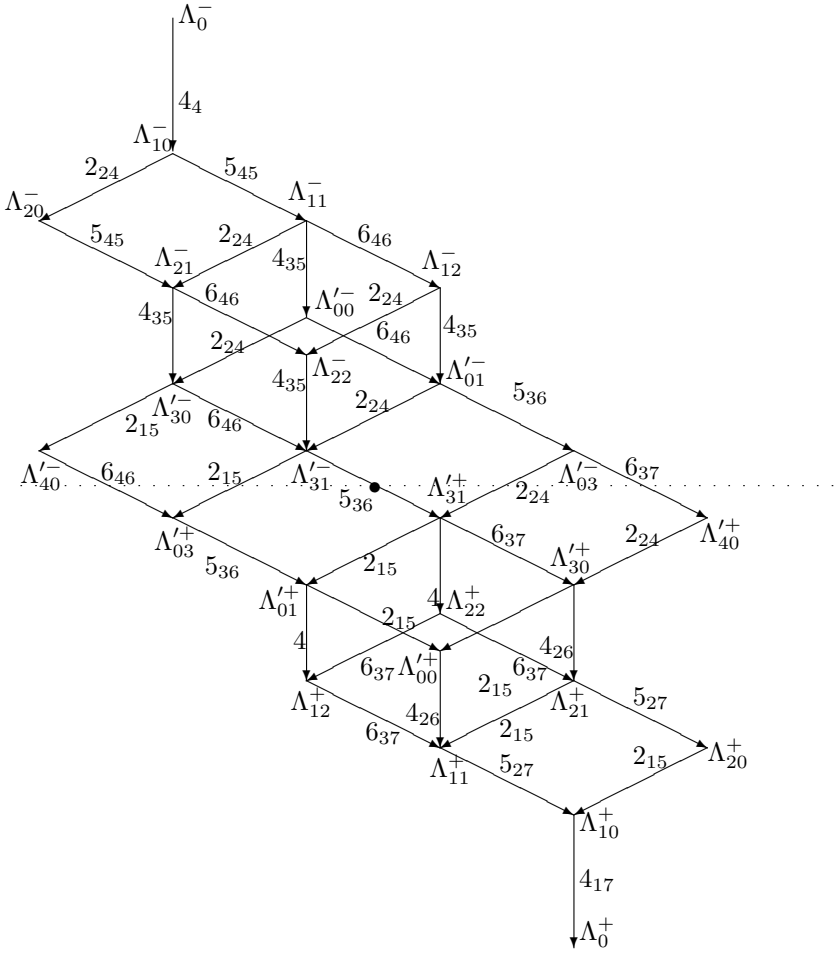


Fig. 15. Reduced multiplets of type R^4_{137} for $su(4,4)$, $su^*(8)$ and $sl(8,\mathbb{R})$

Last reduction of multiplets

There are further reductions of the multiplets - quadruple, etc., but only one quadruple reduction contains representations of physical interest. Namely, this is the multiplet R^4_{1357}, which may be obtained from the main multiplet by setting formally $m_1 = m_3 = m_5 = m_7 = 0$. These multiplets contain 19 ERs/GVMs whose signatures can be given in the following manner:

$$\begin{aligned}
\chi^{\pm}_{0} &= \{\,(0,m_2,0,0,m_6,0)^{\pm}\,;\,\pm m_\rho = \pm(m_2+2m_4+m_6)\,\} \\
\chi^{\pm}_{11} &= \{\,(0,m_2,m_4,m_4,m_6,0)^{\pm}\,;\,\pm(m_\rho-m_4) = \pm m_{2,4,6}\,\} \\
\chi^{\pm}_{21} &= \{\,(m_2,0,m_4,m_{2,4},m_6,0)^{\pm}\,;\,\pm(m_\rho-m_{2,4}) = \pm m_{4,6}\,\} \\
\chi^{\pm}_{12} &= \{\,(0,m_2,m_{46},m_4,0,m_6)^{\pm}\,;\,\pm(m_\rho-m_{4,6}) = \pm m_{2,4}\,\} \\
\chi^{\pm}_{22} &= \{\,(m_2,0,m_{46},m_{2,4},0,m_6)^{\pm}\,;\,\pm(m_\rho-m_{2,4,6}) = \pm m_4\,\} \\
\chi'^{\pm}_{00} &= \{\,(0,m_{2,4},0,0,m_{46},0)^{\pm}\,;\,\pm(m_\rho-2m_4) = \pm m_{2,6}\,\} \\
\chi'^{\pm}_{01} &= \{\,(0,m_{2,4},m_6,0,m_4,m_6)^{\pm}\,;\,\pm(m_\rho-m_6-2m_4) = \pm m_2\,\} \\
\chi'^{\pm}_{30} &= \{\,(m_2,m_4,0,m_2,m_{46},0)^{\pm}\,;\,\pm(m_\rho-m_2-2m_4) = \pm m_6\,\} \\
\chi'^{\pm}_{40} &= \{\,(0,m_4,0,0,m_{2,46},0)^{\pm}\,;\,\pm(m_\rho-2m_{2,4}) = \pm(m_6-m_2)\,\} \\
\chi_{31} &= \{\,(m_2,m_4,m_6,m_2,m_4,m_6)\,;\,0\,\}
\end{aligned} \tag{16}$$

The multiplets are given in Fig. 3-1357:

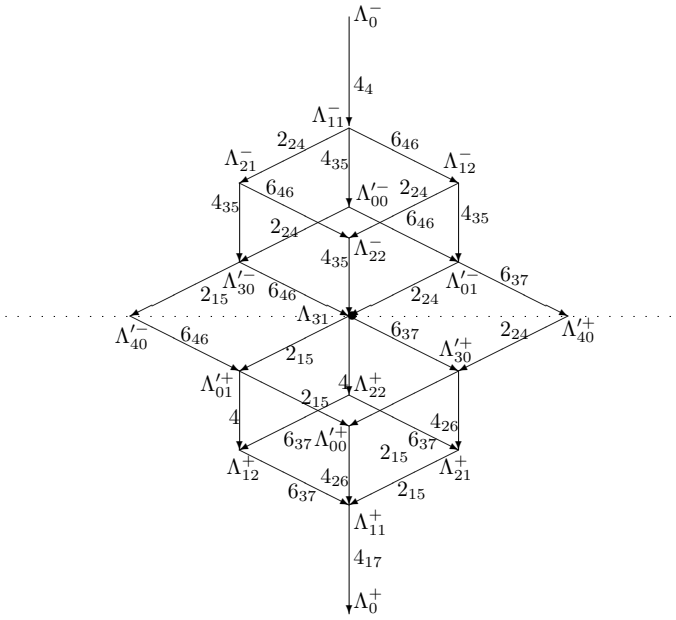


Fig. 16. Reduced multiplets of type R^4_{1357} for $su(4,4)$, $su^*(8)$ and $sl(8,\mathbb{R})$

Note that the ER χ_{31} is not in a pair – it has $c = 0$ and its $\mathcal{M}$ signature is self conjugated. It is placed in the middle of the figure as the bullet. That ER contains the *minimal irreps* characterized by three positive integers which are denoted in this context as m_2, m_4, m_6. Each such irrep is the kernel of the three invariant differential operators $\mathcal{D}^{m_2}_{15}$, $\mathcal{D}^{m_4}_{26}$, $\mathcal{D}^{m_6}_{37}$, which are of order m_2, m_4, m_6, resp., and correspond to the noncompact roots α_{15}, α_{26}, α_{37}, resp.

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