

An integrable Hierarchy on the Perfect Schrödinger Lie Algebra

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Abstract. We construct an integrable hierarchy of coupled integrable solitons equations on the non semisimple perfect Schrödinger Lie algebra $\mathfrak{sl}(2) \ltimes V(1)$.

1. Introduction

Since the celebrated discovery of Gelfand Dickey hierarchies [16] and their generalizations to any affine Kac–Moody Lie algebra made by Drinfeld and Sokolov in their famous paper [18], a deep and far reaching theory on the integrable systems defined on simple Lie algebras has been developed by many authors [12] [2] [3] [9] [10] [22] [13] [23] [20] [14] [24] [8] [37] [17] [25].

However in the last years have been discovered examples of integrable hierarchies of partial differential equations which live on non semisimple Lie algebras. Most of these new examples are due to Ma and his collaborators (see for example [26] [27] [28] [29] [31] [32], [33]) who have studied hierarchies on semidirect product of Lie algebras, but there exists also hierarchies defined on truncated current Lie algebras [6] [7] [4] or on Frobenius valued Lie algebras [39].

As far as we know the hierarchies of PDE which arise from such Lie algebras are systems of coupled equations. In particular if the Lie algebra is a semidirect product between a simple Lie algebra and an abelian subalgebra or a truncated current Lie algebra, the corresponding equations are systems where a (usually) known soliton equation is coupled with other ones, which can be sometimes regarded as perturbation of the first one. Such type of coupled PDE's as been for the first time considered in an important work by Ma and Fuchssteiner [30]. However also coupled equations which do not enter in this scheme have been studied in the literature [36].

The aim of this paper is to proceed further into this line of research presenting a new integrable hierarchy of PDE's living on the perfect but not semisimple Schrödinger Lie algebra $\mathfrak{sl}(2) \ltimes V(1)$. Beyond its intrinsic interest, these are new coupled soliton equations which contains the KdV equation as “first equation”, we hope that this case can be a first step to construct a theory of integrable systems defined on perfect Lie algebra at least in the case when the perfect Lie algebra \mathfrak{g} has a Levi decomposition: $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ where the nilradical part \mathfrak{r} is a irreducible module of the simple part \mathfrak{s} .

In the present work we investigate the Lax equation theory of such system which it is likely to be extended to more general cases of perfect Lie algebras.



The paper is organized as follows. In the second section, we briefly describe the Schrödinger Lie algebra and its properties. In the third Section we construct the integrable hierarchy using the lax operator formalism. In the fourth and last Section we determine the reduced coupled equations.

The author wishes to thank professor Laszlo Feher, for his interest in these new hierarchies defined on perfect Lie algebras and for sending me a deep and very general description of “generalized Drinfeld–Sokolov” hierarchies [21], which will followed here. The author whises also to thank the organizers and participants of the Integrable systems and quantum symmetries meeting for the fruitful and wonderful time spent in Prague.

2. The Schrödinger Lie Algebra

In this section we shall describe the Schrödinger Lie algebra $\mathfrak{s} = \mathfrak{sl}(2) \ltimes V(1)$ [5]. The Lie algebra \mathfrak{s} is a perfect Lie algebra i.e., it equals its commutator ideal $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, of dimension 5 spanned by the elements H, E, F, P, Q with Lie brackets

$$\begin{aligned} [H, E] &= 2E, & [H, F] &= -2F, & [E, F] &= H \\ [H, P] &= P & [H, Q] &= -Q \\ [E, P] &= 0 & [E, Q] &= P \\ [F, P] &= Q & [F, Q] &= 0 \\ [P, Q] &= 0. \end{aligned} \tag{1}$$

However The Lie algebra \mathfrak{s} is not semisimple, because it has a non trivial ideal spanned by the elements P, Q . In fact the Schrödinger Lie algebra is the easiest perfect Lie algebra (other than semisimple). Its Levi decomposition is

$$\mathfrak{s} \ltimes V(1)$$

with semisimple part $\mathfrak{s} = \mathfrak{sl}(2)$ and nilradical $V(1) = \{P, Q\}$. Observe that $V(1)$ is nothing else but the irreducible two dimensional $\mathfrak{sl}(2)$ –module with highest weight 1, which explains our notation. It may be worth to note here then in the literature the name Schrödinger Lie algebra is given also to the non trivial central extension of $\mathfrak{sl}(2) \ltimes V(1)$ [19].

The Lie algebra \mathfrak{s} can be embedded in $\mathfrak{sl}(3)$ as follows [35]

$$(H, E, F, P, Q) \mapsto \begin{pmatrix} H & E & 0 \\ F & -H & 0 \\ -Q & P & 0 \end{pmatrix}. \tag{2}$$

There exist non semisimple Lie algebras, which own an ad–invariant non degenerate bilinear form [6] [4], this is unfortunately not the case of \mathfrak{s} .

Proposition 2.1 For any ad-invariant bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{s} we have $\langle P, \mathfrak{s} \rangle = 0$.

Proof Indeed since $\langle \cdot, \cdot \rangle$ is ad-invariant we have

$$\langle P, H \rangle = \langle [H, P], H \rangle = -\langle P, [H, H] \rangle = 0$$

$$\langle P, E \rangle = \langle [E, Q], E \rangle = -\langle Q, [E, E] \rangle = 0$$

$$\langle P, F \rangle = -\frac{1}{2}\langle P, [H, F] \rangle = \frac{1}{2}\langle [H, P], F \rangle = \frac{1}{2}\langle P, F \rangle \implies \langle P, F \rangle = 0$$

$$\langle P, P \rangle = \langle [H, P], P \rangle = \langle H, [P, P] \rangle = 0$$

$$\langle P, Q \rangle = \langle P, [F, P] \rangle = \langle [P, P], F \rangle = 0.$$

□

This last fact does not really affect the construction of the integrable hierarchy through the Lax operator but makes the bihamiltonian approach rather “uncomfortable”, as we shall discuss briefly later.

Let $\mathcal{L}(\mathfrak{s})$ be the loop algebra given by all the formal Laurent series in a parameter λ with values in \mathfrak{s} :

$$\mathcal{L}(\mathfrak{s}) = \mathfrak{s} \otimes C[\lambda] = \left\{ \sum_{k \leq N} X_k \lambda^k \mid X_k \in \mathfrak{s} \right\} \quad (3)$$

Now let Λ be the element of $\mathcal{L}(\mathfrak{s})$:

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 \\ \lambda & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}. \quad (4)$$

It is easy to check that

$$\text{Ker}(\Lambda) = \begin{pmatrix} 0 & v & 0 \\ v & 0 & 0 \\ v & 0 & 0 \end{pmatrix}, \quad (5)$$

while

$$\text{Im}(\Lambda) = \begin{pmatrix} w_1 & w_2 & 0 \\ -w_2 & -w_1 & 0 \\ w_3 & w_4 & 0 \end{pmatrix}. \quad (6)$$

Hence

$$\text{Ker}(\Lambda) \cap \text{Im}(\Lambda) = \{0\} \quad \text{Ker}(\Lambda) \oplus \text{Im}(\Lambda) = \mathcal{L}(\mathfrak{s}). \quad (7)$$

From equation (5) we have that the annihilator $\mathcal{L}(\mathfrak{s})_\Lambda$ of Λ in $\mathcal{L}(\mathfrak{s})$, $\mathcal{L}(\mathfrak{s})_\Lambda = \{X \in \mathcal{L} \mid [\Lambda, X] = 0\} = \text{Ker}(\Lambda)$ is

$$\mathcal{L}(\mathfrak{s})_\Lambda = \{c_k \Lambda^k \mid c_k \in \mathbb{C} \ k \in \mathbb{Z}\}.$$

This corresponds to the fact that in the case of the Drinfeld Sokolov hierarchies the annihilator of Λ is a Cartan subalgebra. On $\mathcal{L}(\mathfrak{s})$ we can introduce a \mathbb{Z} -gradation by setting:

$$\deg(H) = 0, \quad \deg(E) = \deg(P) = 1, \quad \deg(F) = \deg(Q) = -1 \quad \deg(\lambda) = 2.$$

With respect to this gradation Λ is a homogeneous element with grade 1. We denote by $\mathcal{L}(\mathfrak{s})^k$ the subset given by of all elements of $\mathcal{L}(\mathfrak{s})$ of degree k . If $X = \sum_{k \leq p} X^k \in \mathcal{L}(\mathfrak{s})$ we denote by $(X)_+ = \sum_{k \geq 0} X^k$ its projection on the space spanned by the positive homogeneous elements.

3. The Integrable Hierarchy

Let us construct commuting differential polynomial vector fields on the “phase space”

$$\mathcal{M} = C^\infty(S^1, \mathcal{L}(\mathfrak{s})) \quad (8)$$

We begin by defining the Lax operator L of our hierarchy: the differential:

$$L = \partial_x + \Lambda + Q(x) = \partial_x + \begin{pmatrix} 0 & 1 & 0 \\ \lambda & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix} + \begin{pmatrix} p_1 & 0 & 0 \\ q_1 & -p_1 & 0 \\ q_2 & p_2 & 0 \end{pmatrix} \quad (9)$$

where with ∂_x is denoted the derivative along the coordinate x on S^1 . Observe that the elements of Q have degree between -1 and 0 i.e, less then $\deg(\Lambda) = 1$.

Following [21] the next step is

Lemma 3.1 (*dressing lemma*). *For $Q(x)$ defined as in (9) consider the equation*

$$L := \partial_x + \Lambda + J(x) \mapsto e^{ad(F)}(L) = \partial_x + \Lambda + H(x) \quad (10)$$

where $F(x) = \sum_{k < 0} F^k$, $H(x) = \sum_{k < 0} H^k \in \oplus_{k < 0} \mathcal{L}(\mathfrak{s})^k$ and $H(x) \in \text{Ker}(ad\Lambda)$. Then there exists a solution $F(x)$, $H(x)$ of (10) for any given $J(x)$. In terms of a particular solution, given by $F_0(x)$ $H(x)$ the general solution $F(x)$ is determined by

$$e^{adF} = e^{adK} e^{adF_0}, \quad (11)$$

where $K(x) \in \oplus_{j \leq 0} \mathcal{L}(\mathfrak{s})_\Lambda^j$ is arbitrary. One has a unique solution $F_J(x)$ satisfying $F_J(x) \in \text{Im}(ad\Lambda)^{<0}$, and the components of this solution are differential polynomials in $J(x)$.

Proof Following the lines of Drinfeld-Sokolov [18], the existence is proved by induction based on the gradation and the use of the decomposition in (7) in each step. The uniqueness property is straightforward to see. The proof also shows that the components of the unique solution $F_J(x) \in \text{Im}(ad\Lambda)$ are differential polynomials in $J(x)$. □

Set now

$$\mathcal{H}_\Lambda^+ = \mathcal{L}(\mathfrak{s})_\Lambda \cap \bigoplus_{k \geq 0} \mathcal{L}(\mathfrak{s})^k.$$

For any $b \in \mathcal{H}_\Lambda^+$ define

$$B_b(J) = e^{ad_{F_J}(b)}. \quad (12)$$

Then we can give

Definition 3.2 For any $b \in \mathcal{H}_\Lambda^+$ let $\frac{\partial}{\partial t_b}$ the vector field

$$\frac{\partial}{\partial t_b} J(x) = [B_b(J), L].$$

With the same arguments used in [18] [2] [21] one can prove

Theorem 3.3 (i) The vector fields on \mathcal{M} defined by (12) pairwise commute with each other with respect to the natural Lie derivative,

$$\left[\frac{\partial}{\partial t_a}, \frac{\partial}{\partial t_b} \right] \quad a, b \in \mathcal{H}_\Lambda^+$$

(ii) The quantities H^k of Lemma 3.1 are the conserved densities for the hierarchy.

Actually for the still easy case of the Schrödinger Lie algebra it is possible to construct directly the equations of the hierarchy without referring to the dressing method, which however seems to be very useful to prove Theorem 3.3.

We have indeed

Theorem 3.4 For L given as in (9) there exist a solution $V(x) = \sum_{k \leq 1} V_{-k} \lambda^k$ $V_k \in C^\infty(S^1, \mathfrak{s})$ of the “Casimir” equation

$$[L, V] = 0. \quad (13)$$

Proof Explicitly equation (13) is

$$\left[\partial_x + \Lambda + Q(x), \sum_{k \leq 1} V_{-k} \lambda^k \right] = 0$$

equating the coefficient with the same power in λ we obtain

$$\begin{aligned} [A, V_{-1}] &= 0 \\ (V_k)_x + [Q(x) + B, V_k] &= [V_{k+1}, A] \quad k \geq -1 \end{aligned} \quad (14)$$

where

$$A = F - Q = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B = E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (15)$$

Let us first observe that

$$\text{Im}(\text{ad}A) = \begin{pmatrix} v_1 & 0 & 0 \\ v_2 & -v_1 & 0 \\ v_3 & v_1 & 0 \end{pmatrix}. \quad (16)$$

The first equation in (14); $[A, V_{-1}] = 0$ implies that

$$V_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \gamma_{-1} & 0 & 0 \\ \delta_{-1} & 0 & 0 \end{pmatrix}$$

with $\gamma_{-1}(x)$ and $\delta_{-1}(x)$ arbitrary functions. Now since

$$(V_{-1})_x + [Q(x) + B, V_{-1}] = \begin{pmatrix} \gamma_{-1} & 0 & 0 \\ (\gamma_{-1})_x - 2p_1\gamma_{-1} & -\gamma_{-1} & 0 \\ (\delta_{-1})_x + p_2\gamma_{-1} - p_1\delta_{-1} & \delta_{-1} & 0 \end{pmatrix} \in \text{Im}(\text{ad}A)$$

we can choose at this step $\gamma_{-1} = \delta_{-1} = 1$. The equation $(V_{-1})_x + [Q(x) + B, V_{-1}] = [V_0, A]$ has then solution

$$V_0 = \begin{pmatrix} p_1 & 1 & 0 \\ \gamma_0 & -p_1 & 0 \\ \delta_0 & p_2 & 0 \end{pmatrix}.$$

with $\gamma_0(x)$ and $\delta_0(x)$ arbitrary functions. Again we have

$$(V_0)_x + [Q(x) + B, V_0] = \begin{pmatrix} p_{1x} - q_1 + \gamma_0 & 0 & 0 \\ \gamma_{0x} + 2q_1p_1 - 2p_1\gamma_0 & -p_{1x} + q_1 - \gamma_0 & 0 \\ \delta_{0x} + q_2p_1 + p_2q_1 + p_1\delta_0 + p_2\gamma_0 & p_{2x} + q_2 - \delta_0 & 0 \end{pmatrix}.$$

Therefore in order to have this latter expression in $\text{Im}(\text{ad}A)$ (i.e., of the form (16)) and solve equation (14) at the next step we must impose $\delta_0 = p_{2x} + p_{1x} + q_2 - q_1 + \gamma_0$, while γ_0 remains arbitrary.

It should now be clear how to proceed by induction. Observe indeed that this still arbitrary function γ_0 will appears as linear factor in the entries of position (1, 1) and (2, 2) of V_1 . Suppose by induction that we have already determine the elements V_k until the index $k = n$ with

$$V_n = \begin{pmatrix} a + \gamma_{n-1} & b & 0 \\ \gamma_n & -a - \gamma_{n-1} & 0 \\ \delta_n & c & 0 \end{pmatrix}$$

where γ_{n-1} , γ_n and δ_n still arbitrary. Since we have

$$(V_n)_x + [Q(x) + B, V_n] = \begin{pmatrix} a_x + \gamma_{(n-1)x} - bq_1 + \gamma_n & b_x + 2p_1b - 2a - 2\gamma_{n1} & 0 \\ \gamma_{nx} + 2q_1a + 2q - 1\gamma_{n-1} - 2p_1\gamma_n & -a_x - \gamma_{(n-1)x} + bq_1 - \gamma_n & 0 \\ \delta_{nx} + q_2a + q_2\gamma_{n-1} - p_1\delta_n + p_2\gamma_n - cq_1 & c_x + q_2b - p_1a - p_2\gamma_{n-1} + cp_1 - \delta_n & 0 \end{pmatrix}$$

to solve now equation (14) for V_{n+1} we have to impose

$$\begin{aligned}\gamma_{n-1} &= a - p_1 b - \frac{1}{2} b_x \\ \delta_n &= -\frac{1}{2} b_{xx} + c_x + 2a - x + p_{1x} b + p_1 b_x + \frac{1}{2} p_2 b_x + b(q_2 - q_1) \\ -p_1 a_1 - p_2 a + p_2 p_1 b &= c p_2 + \gamma_n\end{aligned}$$

while γ_n still remains arbitrary and will be determined at the next step. Hence V_{n+1} has the form required by the induction Hypothesis and this closes the proof. \square

From the last theorem follows almost immediately

Theorem 3.5 *The equations of the hierarchy (3.2) are linear combination of*

$$\frac{\partial}{\partial t_k} L = [(\lambda^k V_k)_+, L]. \quad (17)$$

Proof The above equations (17) follow directly from equation (14). Now since V commutes with the Lax operator L from the dressing Lemma 10 and the property of Λ follows that $e^{\text{ad}(-F)} V$ must belong to $\mathcal{L}(\mathfrak{s})_\Lambda$. \square

The first non trivial equations written in components are

$$\begin{aligned}\frac{\partial p_1}{\partial t_3} &= -\frac{\partial p_2}{\partial t_3} = \frac{1}{2} (p_{1x} + q_1 + p_1^2) \\ \frac{\partial q_1}{\partial t_3} &= \frac{1}{2} p_{1xx} + \frac{1}{2} q_{1x} - q_1 p_1 - p_1^3 \\ \frac{\partial q_2}{\partial t_3} &= p_{2xx} + \frac{1}{2} p_{1xx} + q_{2x} + q_{1x} + 2p_1 p_{1x} + p_{2x} p_2 - \frac{1}{2} p_{2x} p_1 + 2q_2 p_1 + 2q_1 p_1 \\ &\quad + 2q_2 p_2 + p_1^2 p_2 + p_1^3.\end{aligned}$$

Let us here remark that usually along with the Lax theory of integrable systems one can develop a bi-Hamiltonian theory. See for example for detailed descriptions of these theory the papers [11] [12] [10] [17] [34] or the books [16] [1].

A bi-Hamiltonian manifold \mathcal{M} is a manifold equipped with two compatible Poisson structures, i.e., two Poisson tensors P_0 and P_1 such that the pencil $P_\lambda = P_1 - \lambda P_0$ is a Poisson tensor for any $\lambda \in \mathbb{C}$. A bihamiltonian vector field X on \mathcal{M} is a vector field which is Hamiltonian with respect to both Poisson tensors (and therefore with respect to any Poisson tensors in the pencil).

The central idea of the bihamiltonian theory of the hierarchies of PDE's is to view them as collections of bihamiltonian vector fields on a (usually infinite dimensional) bihamiltonian manifold \mathcal{M} .

In our case where $\mathcal{M} = C^\infty(S^1, \mathcal{L}(\mathfrak{s}))$ it is still possible to define the canonical Lie Poisson tensor as

$$P_S(V) = -\partial_x V + [S, V] \quad S, V \in C^\infty(S^1, \mathcal{L}(\mathfrak{s})), \quad (18)$$

and it can be easily shown [38] that this Poisson tensor is compatible with the constant Poisson tensors obtained by freezing it in any point, giving to \mathcal{M} the structure of a bi-Hamiltonian manifold. But, since we do not have a non degenerate ad-invariant bilinear form on \mathfrak{s} , we are

left with to choices either we take an ad-invariant degenerated linear form but the corresponding Poisson bracket $\{\cdot, \cdot\}$:

$$\{F, G\}(m) = \langle dF, P_m dG \rangle_m \quad \forall F, G \in C^\infty(\mathcal{M}, \mathbb{C}) \quad m \in \mathcal{M}$$

would be degenerate, or we use a non degenerate but not ad-invariant form and then the Poisson tensor would be not anymore skew symmetric.

4. The reduced Hierachy

Viewed on \mathcal{M} the equations of the hierarchy seem to be rather complicated. However we can show that there exist a gauge transformation which brings them to a hierarchy of a system of two coupled equations, where the first ones are those of the KdV hierarchy.

Let T be the abelian group

Theorem 4.1 *The gauge group*

$$T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix} \mid t_i \in C^\infty(S^1, \mathbb{C}) \right\}.$$

bring the Lax operator in the form:

$$L' = TLT^{-1} = \partial_x + TT_x^{-1} + T(Q + B)T^{-1} = \partial_x + \begin{pmatrix} 0 & 1 & 0 \\ u_1 & 0 & 0 \\ u_2 & 0 & 0 \end{pmatrix}.$$

with

$$\begin{aligned} u_1 &= p_{1x} + 2p_1^2 + q_1^2 \\ u_2 &= -p_{2x} - p_1p_2 + q_2 \end{aligned} \tag{19}$$

Proof by direct computation. □

We can compute using the transformation (19) the first non trivial gauge fixed equations in term of the variable u_1 u_2 obtaining

$$\begin{aligned} \frac{\partial u_1}{\partial t_3} &= \frac{1}{4}u_{1xxx} + \frac{3}{2}u_1u_{1x} \\ \frac{\partial u_2}{\partial t_3} &= u_{2xx} - \frac{3}{4}u_{1xx} + \frac{3}{4}u_2u_{1x} + \frac{1}{2}u_{2x}u_2 - \frac{3}{4}u_1^2 + u_1u_2. \end{aligned} \tag{20}$$

which is a sistem of coupled equation which the KdV as first one. As far as we know this hierarchy of coupled KdV equation has been not yet considered in the literature.

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