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# Classical-quantum semigroups

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**Abstract.** An intriguing and deep analogy between classical and quantum states is revealed using the notion of positive definite function. By Bochner's theorem, (classical) positive definite functions on phase space can be obtained by taking the Fourier transform of probability measures; similarly, *quantum* positive definite functions are the image via the Fourier-Plancherel operator of Wigner quasi-probability distributions. Considering the basic properties of positive definite functions — classical and quantum — one is led to define a class of semigroups of operators, the so-called *classical-quantum semigroups*. It is then natural to wonder whether they have any physical meaning. It turns out that the classical-quantum semigroups can also be obtained by dequantizing a certain class of quantum dynamical semigroups, namely, the *classical noise semigroups*. This correspondence fits in a more general group-theoretical framework in which a larger class of quantum dynamical semigroups, the *twirling semigroups*, can be suitably dequantized. Connections with quantum information science will be briefly discussed.

## 1. Introduction and outline

The evolution of an *open* quantum system — a quantum system interacting with an environment system or reservoir — is well described by a *quantum dynamical semigroup*, provided that suitable assumptions on the nature of such interaction are satisfied (suitable regimes, e.g., weak coupling, singular coupling or low density; various approximations, e.g., the Markovian approximation) [1, 2]. Like any other semigroup of operators, a quantum dynamical semigroup is completely determined by its infinitesimal generator [3]. In the case of a finite-dimensional open system, this operator has a canonical form, independently derived by Gorini, Kossakowski and Sudarshan [4], and by Lindblad [5], around the mid-1970s.

Physicists usually prefer to express the fundamental laws and properties ruling the behaviour of a physical system in 'infinitesimal form' (rather than in 'integrated form') — e.g., Schrödinger equation (versus evolution operator), canonical commutation relations (versus commutation relations in the Weyl form [6]), etc. — and the integrated form is often just a formal expression. In the case we are considering, the usual approach amounts to fixing the infinitesimal generator of the evolution of a given open quantum system by writing a suitable *master equation*. In this regard, it turns out that the Gorini-Kossakowski-Lindblad-Sudarshan canonical form of the master equation appears not only when studying the dynamics of a finite-dimensional system, but in a more general setting — e.g., it is associated with every norm-continuous quantum dynamical semigroup [5] — and deriving the canonical master equation describing a given open quantum system is a fundamental task [2]. However, in some noteworthy cases this 'master equation approach' can be reversed by considering directly the integrated form of the evolution: the semigroup of operators itself is given — in some more or less explicit form — whereas the infinitesimal generator (thus, the master equation) can be derived. We will call this point of view the *semigroup approach.*

Undertaking the semigroup approach, we will consider a class of semigroups of operators whose definition stems in a natural and simple way from the notion of *positive definite function.* As briefly recalled in sect. 2, here one should actually distinguish two different cases. On one hand, we have the standard positive definite functions on $\mathbb{R}^n$, $n \geq 1$ — with $\mathbb{R}^n$ regarded as an (additive) abelian group — arising in harmonic analysis [7] and probability theory [8, 9], and directly related to the notion of *classical* state. On the other hand, associated with quantum states we have the *quantum* positive definite functions on $\mathbb{R}^{2n}$, a notion arising in the context of the phase-space formulation of quantum mechanics *à la* Weyl-Wigner-Groenewold-Moyal [10–14] (which is almost as old as the standard formulation itself), but seems to have been introduced only in the mid-1960s by Kastler [15], and by Loupias and Miracle-Sole [16,17]. It turns out that there is a nice interplay between classical and quantum positive definite functions [18], and this interplay is at the root of the definition of the aforementioned class of semigroups of operators, that will be therefore called *classical-quantum* semigroups.

The classical-quantum semigroups, whose mere definition — see sect. 3 — may be regarded as sort of mathematical curiosity, can actually be considered as quantum dynamical semigroups 'in disguise'. Otherwise stated, they can be regarded as a nonconventional representation of the dynamics of certain open quantum systems. The main aim of the present contribution is to give a precise sense to this claim, and in the rest of this section we will briefly outline the basic ideas and try to give a sketch of the general picture.

Indeed, a standard quantum dynamical semigroup acts in a Banach space of trace class operators, where the quantum states are realized by *density operators* (normalized, positive trace class operators). However, as already observed, quantum mechanics admits a phase-space formulation, which ultimately relies on group-theoretical methods; see [19–21] and references therein. In this formulation, a density operator is replaced with an ordinary function, living on a (symmetry) group or, more generally, on a homogeneous space of that group. These functions are often called *generalized Wigner functions* or *quantum tomograms* in the literature [19–24]. Of course, the archetypical approach is based on the standard Wigner functions, which amounts to considering phase-space translations as the relevant symmetry group. It is then natural to wonder what is the expression of the master equation or of the semigroup of operators, associated with an open quantum system, in terms of tomograms. Clearly, how unwieldy these expressions are strongly depends on the class of quantum dynamical semigroups one is considering.

There is a class of semigroups of operators whose definition involves two basic ingredients: a representation of a locally compact group in a Banach space and a convolution semigroup of probability measures on that group; see sect. 4 and references therein. We call the semigroups in this class *randomly generated semigroups* (RGSs). The class of RGSs contains, in particular, both 'classical' and 'quantum' objects. E.g., it contains the semigroups of operators describing the statistical properties of classical Brownian motion on Lie groups, i.e., the *probability semigroups* [25–27].

The mentioned group-theoretical framework of RGSs turns out to be a natural bridge between quantum mechanics on phase space — formulation that is based on group representation theory, as recalled above — and the theory of open quantum systems. In fact, for a suitable choice of the relevant (group, i.e. phase-space translations, and) group representation, by varying the convolution semigroup of measures involved in the construction one spans the (sub-)class of the *Wigner (quasi-probability) semigroups.* The Wigner semigroups act on phase-space functions, the standard Wigner functions, so by applying the symplectic Fourier-Plancherel transform one obtains a further class of semigroups of operators. The semigroups in this class turn out to coincide with the classical-quantum semigroups that are the main topic of the paper.

This way of re-deriving the classical-quantum semigroups simultaneously confirms a profound 'group-theoretical nature' of these semigroups of operators and shows how to generalize their definition; see sect. 4. This generalization leads to the notion of *tomographic semigroup*. At this point, one is ready to recognize that the tomographic semigroups (more precisely, the *proper* tomographic semigroups) — hence, in particular, the classical-quantum semigroups (or the strictly related Wigner semigroups) — can be obtained by transforming via a linear isometry quantum dynamical semigroups of a certain type, the so-called *twirling semigroups*, introduced by Kossakowski in a seminal paper [28]; see sect. 5 and further references therein.

This last step closes our circle of ideas: the classical-quantum semigroups, despite their extremely simple definition, are quantum dynamical semigroups in disguise, where the disguise is obtained by a linear transformation mapping operators into phase-space functions. Precisely, the classical-quantum semigroups are the disguised counterpart of certain twirling semigroups, the so-called *classical-noise* semigroups. The origin of such a peculiar denomination for this class of quantum dynamical semigroups will become clear in sect. 6, where connections with quantum information science will be established.

Interestingly, not only the probability and the Wigner quasi-probability semigroups, but also all other semigroups of operators considered here — the tomographic semigroups (in particular, the classical-quantum semigroups) and the twirling semigroups (in particular, the classical-noise semigroups) — belong to the class of RGSs.

## 2. Positive definite functions: classical and quantum

In classical (statistical) mechanics, *states* are usually realized as probability measures on phase space — for the sake of notational simplicity, here and in the following we will consider the (1+1)-dimensional phase space $\mathbb{R}\times\mathbb{R}$ — and the expectation value of an *observable* $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ in the state $\mu$ is provided by the expression

$$\langle f\rangle_\mu = \int_{\mathbb{R}\times\mathbb{R}} f(q,p)\,\mathrm{d}\mu(q,p), \qquad f\in \mathsf{C}_0(\mathbb{R}\times\mathbb{R}), \tag{1}$$

where $\mathsf{C}_0(\mathbb{R}\times\mathbb{R})$ is the space of continuous *real* functions on $\mathbb{R}\times\mathbb{R}$ vanishing at infinity. Here we are thinking of $\mu$ as a normalized, positive bounded functional on the C*-algebra of classical observables (whose selfadjoint part is $\mathsf{C}_0(\mathbb{R}\times\mathbb{R})$) but, clearly, formula (1) makes sense for every $\mu$-integrable function $f$.

If one wants to deal with ordinary functions, rather than with probability measures, the state (associated with) $\mu$ can be replaced with its symplectic Fourier transform

$$\widetilde{\mu}(q,p) = \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{\mathrm{i}(qp'-pq')}\,\mathrm{d}\mu(q',p') = \big\langle (q',p')\mapsto \mathrm{e}^{\mathrm{i}(qp'-pq')}\big\rangle_\mu, \tag{2}$$

which is a bounded continuous function usually called the *characteristic function* associated with $\mu$, in the context of probability theory [8, 9]. By *Bochner's theorem* [7–9], characteristic functions admit an *intrinsic* characterization: the convex set that they form (in the vector space of bounded, continuous $\mathbb{C}$-valued functions on $\mathbb{R}\times\mathbb{R}$) *coincides* with the convex set of normalized, continuous *positive definite functions*; i.e., with the convex set containing every continuous complex function $\widetilde{\mu}$ on $\mathbb{R}\times\mathbb{R}$ satisfying

$$\sum_{j,k} \widetilde{\mu}(z_j - z_k)\, c_j\, c_k^* \ge 0, \quad \text{(positivity)} \tag{3}$$

— for any finite set

$$\{z_1\equiv(q_1,p_1),\ldots,z_n\equiv(q_n,p_n)\}\subset\mathbb{R}\times\mathbb{R} \tag{4}$$

and arbitrary complex numbers $c_1, \ldots, c_n$ — and normalized in such a way that

$$\widetilde{\mu}(0) = 1, \tag{5}$$

with $0 \equiv (0,0)$ denoting the origin in $\mathbb{R}\times\mathbb{R}$. It is worth stressing that in the first of conditions (3) the difference $z_j - z_k = (q_j - q_k, p_j - p_k)$ should be regarded as a group operation, where obviously the (additive) group $\mathbb{R}\times\mathbb{R}$ is involved. Indeed, the notion of positive definite function on a vector group extends in a natural way to abelian groups [7] and, more generally, to locally compact groups [29]. In the following, we will denote by $\mathsf{C}$ (alternatively, by $\check{\mathsf{C}}$) the convex cone of positive definite functions (respectively, the convex set of *normalized* positive definite functions) on $\mathbb{R} \times \mathbb{R}$.

On the other hand, in quantum mechanics (normal) states are usually realized as *density operators*, i.e., as normalized, positive trace class operators in a given Hilbert space. In this formalism, there is no direct analogue of the notion positive definite function. However, if one wishes to deal with ordinary functions — as in the classical case, where a state can be represented by a characteristic function, see (2) — 'phase-space formulations' of quantum theory are possible. The archetype — and, under certain respects, the most remarkable — of these approaches is mainly due to the pioneering work of Weyl, Wigner, Groenewold and Moyal [10–13], so that we will call it the *WWGM formulation*.

Setting $\hbar = 1$, in the WWGM formulation of quantum mechanics a *pure* state $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$, $\psi \in \mathrm{L}^2(\mathbb{R})$ ($\|\psi\| = 1$), is (injectively) replaced with a function $\varrho_\psi$ on phase space according to Wigner's prescription

$$\varrho_\psi(q,p) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}px}\, \psi\Big(q - \frac{x}{2}\Big)^* \psi\Big(q + \frac{x}{2}\Big)\ \mathrm{d}x\,. \tag{6}$$

This recipe, exploiting the spectral decomposition of a positive trace class operator, immediately extends to any mixed state in $\mathrm{L}^2(\mathbb{R})$ and, hence, to every trace class operator (by taking linear superpositions). The (complex Banach) space of functions that one obtains with such construction will be denoted by $\mathsf{LW}$. The linear space $\mathsf{LW}$ contains a convex cone $\mathsf{W}$, formed by those functions that correspond to *positive* trace class operators, and $\mathsf{W}$ contains the convex set $\check{\mathsf{W}}$ formed by the *Wigner functions*, i.e., the functions associated with density operators. Within the convex cone $\mathsf{W}$, the Wigner functions are characterized by the normalization condition

$$\lim_{r\to+\infty}\int_{q^2+p^2<r} \varrho(q,p)\ \mathrm{d}q\,\mathrm{d}p = \mathrm{tr}(\hat{\rho}) = 1, \tag{7}$$

see e.g. [30], where $\varrho \in \check{\mathsf{W}}$ is the phase-space function associated with a certain state $\hat{\rho}$.

The function $\varrho$ is real and, although not (in general) a genuine probability distribution — it may also assume negative values — it allows us to express the expectation value of an observable $\hat{A}$ in the state $\hat{\rho}$ (i.e., $\langle\hat{A}\rangle_{\hat{\rho}} = \mathrm{tr}(\hat{A}\,\hat{\rho})$) as a phase-space integral,

$$\langle\hat{A}\rangle_{\hat{\rho}} = \int_{\mathbb{R}\times\mathbb{R}} \mathcal{A}(q,p)\,\varrho(q,p)\ \mathrm{d}q\,\mathrm{d}p, \tag{8}$$

where $\mathcal{A}$ is another real function suitably associated with the selfadjoint operator $\hat{A}$. Due to the integral formulae (7) and (8), the Wigner function $\varrho$ is often called a *quasi-probability distribution.*

**Remark 1** To be precise, not all the observables can be realized as ordinary functions in the WWGM formulation. More generally, they will be suitable distributions (generalized functions), see [31]. However, such technicalities are not relevant for our purposes.

It is natural to wonder whether there is any *intrinsic* characterization of Wigner quasi-probability distributions — apart from the mere normalization condition (7), which makes sense within $\mathsf{W}$. Only if such a characterization is possible, the WWGM approach can be regarded as a fully self-consistent formulation of quantum theory. A simple answer to this question is provided by a quantum version of Bochner's theorem (Kastler 1965 [15], Loupias and Miracle-Sole 1966 [16, 17]; also see [32, 33]). As in the case of classical states, a Wigner quasi-probability distribution $\varrho$ admits a remarkable characterization in terms of its symplectic Fourier transform $\widetilde{\varrho}$. Precisely, a function $\widetilde{\varrho}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is the symplectic Fourier transform of a Wigner function,

$$\widetilde{\varrho} = 2\pi\, \mathcal{F}_{\mathrm{sp}}\, \varrho, \tag{9}$$

if and only if it is continuous, it satisfies the condition

$$\sum_{j,k} \widetilde{\varrho}(z_j - z_k)\, \mathrm{e}^{\mathrm{i}\omega(z_k,z_j)/2}\, c_j\, c_k^* \ge 0, \quad (\omega\text{-positivity}) \tag{10}$$

for every finite set $\{z_1 \equiv (q_1,p_1),\ldots,z_n \equiv (q_n,p_n)\} \subset \mathbb{R}\times\mathbb{R}$ and complex numbers $c_1,\ldots,c_n$ — where $\omega\colon (\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R}) \to \mathbb{R}$ is the standard *symplectic form* — and it is suitably normalized:

$$\widetilde{\varrho}(0) = 1. \tag{11}$$

**Remark 2** The symplectic Fourier transform denoted above by $\mathcal{F}_{\mathrm{sp}}$ should be regarded as a unitary operator in the Hilbert space $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ — a symplectic Fourier-Plancherel operator — rather than an integral expression like formula (2). Indeed, as shown in [30], a Wigner function is square integrable but, in general, it is *not* integrable (fact that, as the reader may have noted, is taken into account in the lhs of formula (7)). In order to obtain a unitary operator, we fix the normalization of $\mathcal{F}_{\mathrm{sp}}$ as follows:

$$\big(\mathcal{F}_{\mathrm{sp}} f\big)(q,p) = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} f(q',p')\, \mathrm{e}^{\mathrm{i}(qp'-pq')}\, \mathrm{d}q'\mathrm{d}p', \quad f \in \mathrm{L}^1(\mathbb{R}\times\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R}\times\mathbb{R}). \tag{12}$$

This explains the factor $2\pi$ appearing on the rhs of relation (9).

**Remark 3** By the previous facts, a real function $\varrho$ on phase space is a Wigner distribution if and only if it belongs to $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ and $\widetilde{\varrho} := 2\pi\,\mathcal{F}_{\mathrm{sp}}\,\varrho$ is a normalized *quantum positive definite function*, i.e., it is a continuous function satisfying the $\omega$-positivity condition (10) and normalized according to (11) (the requirement that $\widetilde{\varrho}$ be continuous can be slightly relaxed — see [33] — but, again, this technical aspect is not relevant here). By contrast, the characteristic function of a probability measure will be sometimes referred to as a *classical positive definite function*.

In the following, we will denote by $\mathsf{Q}$ (alternatively, by $\breve{\mathsf{Q}}$) the convex cone of quantum positive definite functions (respectively, the convex set of *normalized* quantum positive definite functions) on $\mathbb{R}\times\mathbb{R}$; namely, $\mathsf{Q} = \mathcal{F}_{\mathrm{sp}}\mathsf{W}$ and $\breve{\mathsf{Q}} = 2\pi\,\mathcal{F}_{\mathrm{sp}}\breve{\mathsf{W}}$.

Comparing conditions (3) and (10), the reader should note something more than a formal similarity. In the latter, the function

$$(\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R}) \ni (z_1,z_2) \mapsto \mathrm{e}^{\mathrm{i}\omega(z_1,z_2)/2} \tag{13}$$

should be regarded as a (nontrivial) multiplier for the group $\mathbb{R}\times\mathbb{R}$ [34], the multiplier associated with the Weyl system and with the integrated form of canonical commutation relations [6], whereas in the former the trivial multiplier is implicitly involved. Otherwise stated, relation (3) is a clue of classical commutativity as opposed to quantum non-commutativity which is implicit in relation (10).

## 3. Playing with positive definite functions: classical-quantum semigroups

The *convolution* $\mu_1 \circledcirc \mu_2$ of two probability measures $\mu_1$ and $\mu_2$ (say, on a locally compact abelian group [29]) is a probability measure too; hence, taking the Fourier transform, by Bochner's theorem the point-wise product $\mathcal{C}_1\,\mathcal{C}_2$ of two positive definite functions $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathsf{C}$ is again a positive definite function. To the same conclusion one is led exploiting relation (3) and Schur's product theorem [35], according to which the *Hadamard product* (i.e., the entrywise product) of two positive (semi-definite) matrices is positive too. What happens if we now take the point-wise product of a (classical) positive definite function $\mathcal{C} \in \mathsf{C}$ by a quantum positive definite function $\mathcal{Q} \in \mathsf{Q}$? It is immediately clear, by Schur's product theorem, that $\mathcal{C}\,\mathcal{Q}$ is once again a *quantum* positive definite function.

**Remark 4** Clearly, the point-wise product of two classical positive definite functions, or of a classical positive definite function by a quantum one, preserves normalization: $\mathcal{C}\,, \mathcal{C}' \in \check{\mathsf{C}}$, $\mathcal{Q} \in \check{\mathsf{Q}} \;\Rightarrow\; \mathcal{C}\,\mathcal{C}' \in \check{\mathsf{C}}$, $\mathcal{C}\,\mathcal{Q} \in \check{\mathsf{Q}}$.

Comforted by the previous results, we can now feel free to play with the point-wise product of positive definite functions (classical or quantum), without paying attention, for the moment, to the physical meaning of our mathematical manipulations.

Consider then a *semigroup of positive definite functions*, i.e., a set $\{\mathcal{C}_t\}_{t\in\mathbb{R}^+}$ of normalized (classical) positive definite functions on $\mathbb{R}\times\mathbb{R}$ such that

$$\mathcal{C}_t\;\mathcal{C}_s = \mathcal{C}_{t+s}\,,\; t,s \geq 0, \quad \mathcal{C}_0 \equiv 1, \tag{14}$$

where $\mathcal{C}_t\;\mathcal{C}_s$ is a point-wise product. Such semigroups can be completely classified. Indeed, the (symplectic) Fourier transform of a semigroup of positive definite functions is a convolution semigroup of probability measures, and convolution semigroups admit a well known classification related to the Lévy-Kintchine formula [25–27]. Since, for every $t \geq 0$, $\mathcal{C}_t$ is a bounded continuous function, we can define in $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ a bounded operator $\hat{\mathfrak{C}}_t$ by setting

$$\big(\hat{\mathfrak{C}}_t\, f\big)(q,p) := \mathcal{C}_t(q,p)\, f(q,p), \quad f \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}). \tag{15}$$

The set $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ is a semigroup of operators:

(i) $\hat{\mathfrak{C}}_t\,\hat{\mathfrak{C}}_s = \hat{\mathfrak{C}}_{t+s}$, $t,s \geq 0$ (one-parameter semigroup property);

(ii) $\hat{\mathfrak{C}}_0 = I$ ($I$ denoting the identity operator).

It is now natural to consider the restriction of the semigroup of operators $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ to a linear subspace of $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$. Indeed, by linear superpositions, one can extend in a natural way the convex cone $\mathsf{Q}$ of *quantum* positive definite functions on $\mathbb{R}\times\mathbb{R}$ to a *complex* vector space $\mathsf{LQ}$ which turns out to be a *dense* linear subspace of $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$. A semigroup of operators $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ in $\mathsf{LQ}$ is defined as follows. Since the point-wise product of a classical positive definite function by a quantum positive definite function is again quantum positive definite, we can set

$$\big(\mathfrak{C}_t\;\mathcal{Q}\big)(q,p) := \mathcal{C}_t(q,p)\;\mathcal{Q}(q,p), \quad \mathcal{Q} \in \mathsf{LQ}, \tag{16}$$

where, with a slight abuse of notation, $\mathcal{Q}$ here is a linear superposition of quantum positive definite functions:

$$\mathcal{Q} = \mathcal{Q}_1 - \mathcal{Q}_2 + \mathrm{i}\,(\mathcal{Q}_3 - \mathcal{Q}_4), \quad \mathcal{Q}_1, \ldots, \mathcal{Q}_4 \in \mathsf{Q}\,. \tag{17}$$

It is clear that we have:

$$\mathfrak{C}_t\,\mathsf{Q} \subset \mathsf{Q}, \quad \mathfrak{C}_t\,\check{\mathsf{Q}} \subset \check{\mathsf{Q}}. \tag{18}$$

**Remark 5** It can be shown that the semigroup of operators $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ — regarded as the map $t \mapsto \hat{\mathfrak{C}}_t$ — is continuous; moreover, since $|\mathcal{C}_t(q,p)| \le \mathcal{C}_t(0) = 1$, it is a *contraction semigroup*, i.e., $\|\hat{\mathfrak{C}}_t\| \le 1$. Similarly, endowing the sets $\mathsf{C}$ and $\mathsf{LQ}$ with suitable topologies, the semigroups $\{\mathcal{C}_t\}_{t\in\mathbb{R}^+}$ and $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ turn out to be continuous too. In particular, the linear space $\mathsf{LQ}$ can be regarded as the image, via the Weyl (dequantization) map, of the Banach space of trace class operators, as it will be briefly recalled in sect. 4. However, as already stressed, we will focus on the main ideas rather than on technical details.

We will call the semigroups of operators $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ and $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ a *classical-quantum semigroup* and a *proper classical-quantum semigroup*, respectively.

## 4. Classical-quantum semigroups as tomographic semigroups

The mere definition of a classical-quantum semigroup in terms of positive definite functions is so simple and abstract that the reader may hardly suspect, at first sight, that it has anything to do with the description of an open quantum system.

To clarify this point, consider the unitary operator $\mathcal{T}(q,p)$ in $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ defined by

$$\big(\mathcal{T}(q,p)\,f\big)(\tilde{q},\tilde{p}) = \mathrm{e}^{-\mathrm{i}(q\tilde{p}-p\tilde{q})}\, f(\tilde{q},\tilde{p}). \tag{19}$$

The map $\mathbb{R}\times\mathbb{R} \ni (q,p) \mapsto \mathcal{T}(q,p)$ is a unitary representation. Assume that $\{\mu_t\}_{t\in\mathbb{R}^+}$ is the convolution semigroup of probability measures associated, via the symplectic Fourier transform, with the semigroup $\{\mathcal{C}_t\}_{t\in\mathbb{R}^+}$ of (normalized) positive definite functions which generates the classical-quantum semigroup (15). Observe now that

$$\hat{\mathfrak{C}}_t\, f = \int_{\mathbb{R}\times\mathbb{R}} \mathcal{T}(q,p) f \; \mathrm{d}\mu_t(q,p), \quad \forall f \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}), \tag{20}$$

where the rhs of (20) should be regarded as an integral of vector-valued functions.

Formula (20) shows that a classical-quantum semigroup can be regarded as a particular case of a class of semigroups of operators constructed in the following way. Given a *locally compact group* $G$, let $\mathfrak{V}$ be a (weakly continuous) representation of $G$ in a real or complex Banach space $\mathcal{J}$, and let $\{\mu_t\}_{t\in\mathbb{R}^+}$ a convolution semigroup on $G$. The set $\{\mu_t[\mathfrak{V}]\}_{t\in\mathbb{R}^+}$, with $\mu_t[\mathfrak{V}]\colon \mathcal{J}\to\mathcal{J}$ denoting the bounded linear operator defined by

$$\mu_t[\mathfrak{V}]\,\Psi := \int_G \mathfrak{V}(g)\,\Psi\; \mathrm{d}\mu_t(g), \quad \forall \Psi \in \mathcal{J}, \tag{21}$$

is a semigroup of operators, a so-called *randomly generated semigroup* [36, 37].

**Remark 6** The representation $\mathfrak{V}$ is also assumed to be uniformly bounded and, if the Banach space $\mathcal{J}$ is separable, the integral on the rhs of (21) can be regarded as a Bochner integral (in particular, this is the case of the integral on the rhs of (20)); see [36].

The (selfadjoint) unitary operator $\mathcal{F}_{\mathrm{sp}}$ determined by (12) intertwines the representation $\mathcal{T}$ with a unitary representation $\mathcal{S}$ of the group $\mathbb{R}\times\mathbb{R}$ in $\mathrm{L}^2(\mathbb{R}\times\mathbb{R})$,

$$\mathcal{S}(q,p) := \mathcal{F}_{\mathrm{sp}}\, \mathcal{T}(q,p)\, \mathcal{F}_{\mathrm{sp}}, \quad \forall (q,p) \in \mathbb{R}\times\mathbb{R}, \tag{22}$$

and obviously the same construction as above can be applied to this new representation. Taking into account the fact that

$$\big(\mathcal{S}(q,p)\,f\big)(\tilde{q},\tilde{p}) = f(\tilde{q}-q, \tilde{p}-p), \quad f \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}) \tag{23}$$

— i.e., $\mathcal{S}$ is the regular representation of the group $\mathbb{R}\times\mathbb{R}$ — we obtain the semigroup of operators

$$\{\hat{\mathfrak{W}}_t \equiv \mu_t[\mathcal{S}] \colon \mathrm{L}^2(\mathbb{R}\times\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}\times\mathbb{R})\}_{t\in\mathbb{R}^+} \tag{24}$$

defined by

$$\big(\hat{\mathfrak{W}}_t f\big)(\tilde{q},\tilde{p}) = \int_{\mathbb{R}\times\mathbb{R}} \big(\mathcal{S}(q,p) f\big)(\tilde{q},\tilde{p})\; \mathrm{d}\mu_t(q,p) = \int_{\mathbb{R}\times\mathbb{R}} f(\tilde{q}-q,\tilde{p}-p)\; \mathrm{d}\mu_t(q,p). \tag{25}$$

One can show, moreover, that the representation $\mathcal{T}$ satisfies

$$\mathcal{T}(q,p)\,\breve{\mathsf{Q}} = \breve{\mathsf{Q}}\,. \tag{26}$$

From this relation it follows that

$$\mathcal{T}(q,p)\,\mathsf{LQ} = \mathsf{LQ} \quad \text{and} \quad \mathcal{T}(q,p)\,\mathsf{Q} = \mathsf{Q}\,. \tag{27}$$

The first of relations (27) is coherent with the fact that, as already observed, $\hat{\mathfrak{C}}_t\,\mathsf{LQ} \equiv \mu_t[\mathcal{T}]\,\mathsf{LQ} \subset \mathsf{LQ}$, and hence one can define a further semigroup of operators — the proper classical-quantum semigroup $\{\mathfrak{C}_t\}_{t\in\mathbb{R}^+}$ — just by restriction to the Banach space $\mathsf{LQ}$. The second of relations (27) and relation (26) are then coherent with relations (18).

Relation (26) also implies that

$$\mathcal{S}(q,p)\,\mathsf{LW} = \mathsf{LW}\,, \quad \mathcal{S}(q,p)\,\mathsf{W} = \mathsf{W} \quad \text{and} \quad \mathcal{S}(q,p)\,\breve{\mathsf{W}} = \breve{\mathsf{W}}\,. \tag{28}$$

Accordingly, we can restrict the semigroup of operators $\{\hat{\mathfrak{W}}_t\}_{t\in\mathbb{R}^+}$ to the Banach space $\mathsf{LW}$, so obtaining a new semigroup of operators $\{\mathfrak{W}_t \colon \mathsf{LW} \to \mathsf{LW}\}_{t\in\mathbb{R}^+}$, which is *positive* — namely, $\mathfrak{W}_t\,\mathsf{W} \subset \mathsf{W}$ — and preserves the normalization condition (7), i.e., $\mathfrak{W}_t\,\breve{\mathsf{W}} \subset \breve{\mathsf{W}}$ (thus, we may say that it is *trace-preserving*). The semigroups of operators $\{\hat{\mathfrak{W}}_t\}_{t\in\mathbb{R}^+}$ and $\{\mathfrak{W}_t\}_{t\in\mathbb{R}^+}$ will be called a *Wigner semigroup* and a *proper Wigner semigroup*, respectively.

Apart from *complete* positivity [1], whose role in the theory of open quantum systems was first recognized by Lindblad [5] (for the sake of simplicity, we will not consider this property here), positivity and preservation of the trace are the basic features of a quantum dynamical semigroup. It is then clear that the (proper) classical-quantum semigroups, and the strictly related (proper) Wigner semigroups, describe the evolution of open quantum systems in terms of phase-space functions.

It is now worth observing that, in their integral formulation (20), the classical-quantum semigroups can be suitably generalized. This generalization, which gives rise to a special class of randomly generated semigroups, the tomographic semigroups [18,36,37], is based on considering a certain representation of a locally compact group $G$.

Precisely, let $\Delta_G$ be the modular function on $G$, and let $\mathsf{m} : G\times G \to \mathbb{T}$ — with $\mathbb{T}$ denoting the circle group — be a multiplier for $G$ [34], i.e.,

$$\mathsf{m}(g,e) = \mathsf{m}(e,g) = 1, \quad \forall g\in G, \tag{29}$$

and

$$\mathsf{m}(g_1,g_2g_3)\,\mathsf{m}(g_2,g_3) = \mathsf{m}(g_1g_2,g_3)\,\mathsf{m}(g_1,g_2), \quad \forall g_1,g_2,g_3\in G. \tag{30}$$

Associated with the multiplier $\mathsf{m}$, there is a function $\breve{\mathsf{m}} : G\times G \to \mathbb{T}$ defined by

$$\breve{\mathsf{m}}(g,h) := \mathsf{m}(g,g^{-1}h)^*\,\mathsf{m}(g^{-1}h,g), \quad \forall g,h\in G. \tag{31}$$

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Consider then the map

$$\mathcal{T}_{\mathsf{m}} \colon G \to \mathcal{U}(\mathrm{L}^2(G)) \tag{32}$$

— with $\mathcal{U}(\mathrm{L}^2(G))$ denoting the unitary group of the Hilbert space $\mathrm{L}^2(G)$, of square integrable functions on $G$ with respect to the left Haar measure — defined by

$$\big(\mathcal{T}_{\mathsf{m}}(g)\, f\big)(h) := \Delta_G(g)^{\frac{1}{2}}\ \check{\mathsf{m}}(g,h)\, f(g^{-1}hg), \quad f \in \mathrm{L}^2(G), \tag{33}$$

It turns out that $\mathcal{T}_{\mathsf{m}}$ is a strongly continuous unitary representation [21].

Therefore, for every convolution semigroup $\{\mu_t\}_{t\in\mathbb{R}^+}$ of probability measures on $G$ we have a randomly generated semigroup associated with the pair $(\mathcal{T}_{\mathsf{m}}, \{\mu_t\}_{t\in\mathbb{R}^+})$, namely,

$$\mathfrak{T}_t^{\mathsf{m}} f := \mu_t[\mathcal{T}_{\mathsf{m}}]\, f = \int_G \mathcal{T}_{\mathsf{m}}(g) f \, \mathrm{d}\mu_t(g), \quad \forall\, f \in \mathrm{L}^2(G). \tag{34}$$

As anticipated, we call the semigroup of operators $\{\mathfrak{T}_t^{\mathsf{m}}\}_{t\in\mathbb{R}^+}$ a *tomographic semigroup*. The origin of this term becomes clear when one considers the case where there exists a *square integrable* projective representation $U$ of the group $G$, with multiplier $\mathsf{m}$ [38,39]. In this case, one can define a map $\mathscr{D}$ that associates injectively a function in $\mathrm{L}^2(G)$ with every Hilbert-Schmidt operator — in particular, with every trace class operator — in the carrier Hilbert space of the representation $U$ [21]. This map should be thought of as a *dequantization map*. If the group $G$ is unimodular (e.g., a compact group) — i.e., $\Delta_G \equiv 1$ — the function $\varrho$ associated with a density operator $\hat{\rho}$ is of the form

$$\varrho(g) \equiv (\mathscr{D}\hat{\rho})(g) = d_U^{-1}\ \mathrm{tr}(U(g)^*\hat{\rho}), \tag{35}$$

where $d_U > 0$ is a normalization constant depending on the representation $U$ and on the normalization of the Haar measure on $G$. As mentioned in sect. 1, $\varrho$ is sometimes called the quantum *tomogram* associated with $\hat{\rho}$. The range $\mathrm{Ran}(\mathscr{D})$ of the dequantization map is a closed subspace of $\mathrm{L}^2(G)$, invariant with respect to the representation $\mathcal{T}_{\mathsf{m}}$ [21]. Also invariant for $\mathcal{T}_{\mathsf{m}}$ is a dense linear subspace of $\mathrm{Ran}(\mathscr{D})$, i.e., the image via $\mathscr{D}$ of the trace class operators, and we will denote this space with $\mathscr{T\!C}(U)$. As a consequence, we obtain two further semigroups of operators by restricting $\{\mathfrak{T}_t^{\mathsf{m}}\}_{t\in\mathbb{R}^+}$ to $\mathrm{Ran}(\mathscr{D})$ and to $\mathscr{T\!C}(U)$. We will call the latter semigroup a *proper tomographic semigroup* since it acts on quantum tomograms.

The classical-quantum semigroups fit in this scheme with the following identifications. The group $G$ has obviously to be identified with the vector group $\mathbb{R}\times\mathbb{R}$. The representation $U$ is the *Weyl system* [6,21], i.e.,

$$U(q,p) = \exp(\mathrm{i}(p\,\hat{q} - q\hat{p}))\,, \tag{36}$$

with $\hat{q}$, $\hat{p}$ denoting the position and momentum operators in $\mathrm{L}^2(\mathbb{R})$. This is a (square integrable) projective representation,

$$U(q+\tilde{q}, p+\tilde{p}) = \mathsf{m}(q,p\,;\tilde{q},\tilde{p})\, U(q,p)\, U(\tilde{q},\tilde{p}), \tag{37}$$

and the multiplier $\mathsf{m}$ is given by

$$\mathsf{m}(q,p\,;\tilde{q},\tilde{p}) := \exp\Big(\frac{\mathrm{i}}{2}(q\tilde{p} - p\tilde{q})\Big). \tag{38}$$

Hence, the associated representation $\mathcal{T}_{\mathsf{m}}$ coincides precisely with $\mathcal{T}$, see (19). Although not immediately clear, it turns out that applying the symplectic Fourier-Plancherel transform to the expression on the rhs of (35) on gets precisely the Wigner function associated with $\hat{\rho}$ (up to a normalization factor) [20,21]. Moreover, in this case $\mathrm{Ran}(\mathscr{D}) = \mathrm{L}^2(\mathbb{R}\times\mathbb{R})$ [21,40], so that the proper tomographic semigroup is obtained by restricting the tomographic semigroup $\{\hat{\mathfrak{C}}_t\}_{t\in\mathbb{R}^+}$ to the (Banach) space $\mathscr{T\!C}(U) = \mathsf{LQ}$.

## 5. Classical-quantum semigroups as quantum dynamical (twirling) semigroups

In the picture emerging in the previous section, there are two points that still need to be clarified:

- the meaning of the representation $\mathcal{T}_{\mathsf{m}}$ characterizing the tomographic semigroups within the full class of randomly generated semigroups;
- the form that a proper tomographic semigroup assumes once expressed in terms of standard Hilbert space operators.

These two issues are actually non unrelated as it will become soon evident.

Let $\mu$ be a probability measure on $G$. Denoting by $\mathcal{H}$ the carrier Hilbert space of the projective representation $U$ and by $\mathcal{B}_1(\mathcal{H})$ the Banach space of trace class operators in $\mathcal{H}$, we can define a bounded linear operator

$$\mu[U\vee U]\colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H}), \tag{39}$$

where

$$\mu[U\vee U]\,\hat{\rho} := \int_G U(g)\,\hat{\rho}\,U(g)^*\,\mathrm{d}\mu(g). \tag{40}$$

We call $\mu[U\vee U]$ a *twirling operator* [27]. This bounded operator is a (completely) positive and trace-preserving map, namely, a *quantum channel.* The notation $U\vee U$ has to do with the fact that the map $G \ni g \mapsto U\vee U(g)$, with

$$U\vee U(g)\colon \mathcal{B}_1(\mathcal{H}) \ni \hat{\rho} \mapsto U(g)\,\hat{\rho}\,U(g)^* \in \mathcal{B}_1(\mathcal{H}), \tag{41}$$

is an isometric representation, i.e., the standard symmetry action of $G$, on trace class operators, associated with $U$. Having once again a group representation at our disposal, we can consider the corresponding randomly generated semigroups; i.e., for every convolution semigroup $\{\mu_t\}_{t\in\mathbb{R}^+}$ on $G$ we obtain a semigroup of operators $\{\mu_t[U\vee U]\}_{t\in\mathbb{R}^+}$ by setting

$$\mu_t[U\vee U]\,\hat{\rho} = \int_G U\vee U(g)\,\hat{\rho}\,\mathrm{d}\mu_t(g). \tag{42}$$

Since every twirling operator is a quantum channel, the *twirling semigroup* $\{\mu_t[U\vee U]\}_{t\in\mathbb{R}^+}$ is a quantum dynamical semigroup [27, 36, 37]. Clearly, if a probability measure on $G$ is a member of a convolution semigroup, then a twirling operator associated with this measure is member of a twirling semigroup.

In order to illustrate the role that the twirling semigroups play in our framework — assuming that the representation $U$ is square integrable — let us *dequantize* them. To this aim, it is fundamental to observe that the dequantization map $\mathscr{D}$ associated with $U$, satisfies the intertwining relation

$$\mathscr{D}\,U\vee U(g)\,\hat{\rho} = \mathcal{T}_{\mathsf{m}}(g)\,\mathscr{D}\,\hat{\rho}, \quad \forall\,\hat{\rho}\in\mathcal{B}_1(\mathcal{H}). \tag{43}$$

It follows that, for every $t \geq 0$, we have:

$$\mathscr{D}\,\mu_t[U\vee U]\,\hat{\rho} = \mathfrak{T}_t^{\mathsf{m}}\,\mathscr{D}\,\hat{\rho}. \tag{44}$$

Therefore, the representation $\mathcal{T}_{\mathsf{m}}$ is (a natural extension to $\mathrm{L}^2(G)$ of) the quantum symmetry action of $G$ on the tomograms living in $\mathscr{T}\mathscr{C}(U) \subset \mathrm{Ran}(\mathscr{D})$, and the proper tomographic semigroup is nothing but a twirling semigroup expressed in terms of such tomograms.

## 6. Conclusions and connections with quantum information science

In classical statistical mechanics, there is a natural one-to-one correspondence between *states*, realized as probability measures on phase space, and positive definite functions. The latter are usually more convenient to deal with, since they are ordinary $\mathbb{C}$-valued functions, rather than measures that are in general quite abstract objects. A similar correspondence arises in the context of the WWGM formulation of quantum mechanics, where the quantum positive definite functions can be obtained as symplectic Fourier-Plancherel transforms of Wigner quasi-probability distributions on phase space. We have shown that there is a natural notion of semigroup of (classical) positive definite functions, and by this notion one is led to define a class of semigroups of operators, the classical-quantum semigroups. The physical content of these abstract mathematical objects becomes clear as soon as one realizes that they can be regarded as the dequantized counterpart of a certain class of quantum dynamical semigroups, the *classical-noise* semigroups.

This term is borrowed from quantum information science, where a *classical-noise channel* is a completely positive, trace-preserving map $\mathscr{C}$ of the form

$$\mathscr{C}\hat{\rho} = \int_{\mathbb{C}} D(z)\,\hat{\rho}\,D(z)^*\,\gamma(z)\,\mathrm{d}^2 z, \tag{45}$$

where $\gamma$ is a Gaussian probability distribution and $D(z) := \exp(z a^\dagger - \bar{z} a)$ — with $a^\dagger$, $a$ denoting the creation and annihilation operators — is known in the context of quantum optics as the *displacement operator* [41, 42]; i.e., with the standard identifications the map $\mathbb{C} \ni z \mapsto D(z)$ is nothing but the Weyl system (36). This class of Gaussian quantum channels has been studied extensively; in particular, estimates of the channel capacity and of the minimum Rényi and Wehrl output entropies have been investigated, see [43, 44] and references therein.

Therefore, assuming that the twirling semigroup $\{\mu_t[U \vee U]\}_{t\in\mathbb{R}^+}$ is defined identifying the representation $U$ with the Weyl system and $\{\mu_t\}_{t\in\mathbb{R}^+}$ with a *Gaussian* convolution semigroup [27, 37] on $\mathbb{R}\times\mathbb{R}$, we obtain a semigroup of operators which consists of classical noise channels.

The proper classical-quantum semigroups are contained in a natural way in the class of proper tomographic semigroups. Also this larger class of semigroups of operators may be thought of as the dequantized version of certain quantum dynamical semigroups, namely, the twirling semigroups associated with square integrable representations. It is worth mentioning that the twirling semigroups acting in finite-dimensional spaces are of particular interest in quantum information science because these semigroups of operators are formed by random unitary maps [27, 37], i.e., by quantum channels that are *perfectly corrigible*, according to the definition given by Gregoratti and Werner [45].

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