Virasoro representations with central charges $\frac{1}{2}$ and 1 on the real neutral fermion Fock space $F^{\otimes \frac{1}{2}}$

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Abstract. We study a family of fermionic oscillator representations of the Virasoro algebra via 2-point-local Virasoro fields on the Fock space $F^{\otimes \frac{1}{2}}$ of a neutral (real) fermion. We obtain the decomposition of $F^{\otimes \frac{1}{2}}$ as a direct sum of irreducible highest weight Virasoro modules with central charge $c = 1$. As a corollary we obtain the decomposition of the irreducible highest weight Virasoro modules with central charge $c = \frac{1}{2}$ into irreducible highest weight Virasoro modules with central charge $c = 1$. As an application we show how positive sum (fermionic) character formulas for the Virasoro modules of charge $c = \frac{1}{2}$ follow from these decompositions.

1. Introduction

This paper is a continuation of [Ang14], and is a part of a series studying various particle correspondences in 2 dimensional conformal field theory from the point of view of chiral algebras (vertex algebras) and representation theory. The study of fermionic oscillator representations of the Virasoro algebra (Vir) is long-standing, dating back to [Fre81], [FF83], [GOS85], [GNO85], [GKO86], [KR87], [FFR91], and many others. What was new in [Ang14] is that we used 2-point-local (local at both $z = w$ and $z = -w$ as opposed to just local at $z = w$) Virasoro fields generating the fermionic oscillator representations. In particular, in [Ang14] we constructed a 2-parameter family (depending on parameters $\lambda, b \in \mathbb{C}$) of 2-point-local Virasoro fields with central charge $-2 + 12\lambda - 12\lambda^2$ on the fermionic Fock space $F^{\otimes \frac{1}{2}}$ of the real fermion. In this paper we study the nature of these representations depending on the parameters, and their decomposition into irreducible modules. We show that for particular choices of the parameters (λ, b) these two-point local Virasoro field representations can produce each one of the well known discrete series of Virasoro representations. Another important and interesting particular case of the parameters is that of $\lambda = \frac{1}{2}$, i.e., central charge 1, and $b \in \frac{1}{\sqrt{2}}\mathbb{Z}$. For this choice of (λ, b) we obtain the decomposition of $F^{\otimes \frac{1}{2}}$ into irreducible highest weight modules for Vir of central charge 1. It is well known (going back to D. Friedan and I. Frenkel) that $F^{\otimes \frac{1}{2}}$ is a natural highest weight module for Vir of central charge $\frac{1}{2}$ via a one-point local Virasoro field, and as such it decomposes into two irreducible highest weight modules for Vir of central charge $\frac{1}{2}$ (these are well known as two of the Ising minimal models). We show how each of these irreducible Vir highest weight modules of central charge $\frac{1}{2}$ decomposes into irreducible Vir highest weight modules of central charge 1. As an application, this allows us to directly write a positive sum (fermionic) character formula for these irreducible highest weight modules for Vir of central charge $\frac{1}{2}$. There has been extensive research on positive sum (fermionic) Virasoro character formulas, see for instance [KKMM93], [FQ97], [BF00], [Wel05], [FFW08]. Fermionic-type character formulas are known for all the Virasoro minimal models (the discrete Virasoro series), although the vertex algebra (field theory) foundation for such character formulas is still lacking in the general case. It is new here that we obtain these specific fermionic formulas as a direct result of the decomposition of the charge $\frac{1}{2}$ "1-point-local" modules into charge 1 "2-point-local" modules (i.e., we explicitly use multi-local fields and twisted vertex algebra techniques). We hope that such approach is possible also more generally for the discrete Virasoro series by using quasi-particle Fock spaces in the place of a single real fermionic Fock space of central charge $\frac{1}{2}$. After bosonising such quasi-particle Fock spaces, the idea is then to obtain a decomposition of the irreducible modules representing the one-point local Virasoro field of discrete-series-type central charge into irreducible modules representing multi-local Virasoro fields of charge 1.

2. Notation and background

We use the term "field" to mean a series of the form

$$a(z) = \sum_{n\in\mathbf{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V), \quad \text{such that} \;\; a_{(n_v)}v = 0 \;\; \text{for any } v \in V, \; n_v \gg 0. \tag{2.1}$$

Let

$$a(z)_- := \sum_{n\geq 0} a_n z^{-n-1}, \quad a(z)_+ := \sum_{n<0} a_n z^{-n-1}. \tag{2.2}$$

Definition 2.1 (Normal ordered product) *Let $a(z), b(z)$ be fields on a vector space V. Define*

$$:a(z)b(w):= a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a_-(z). \tag{2.3}$$

One calls this the normal ordered product of $a(z)$ and $b(w)$.

Remark 2.2 Let $a(z), b(z)$ be fields on a vector space V. Then $:a(z)b(\lambda z):$ and $:a(\lambda z)b(z):$ are well defined fields on V for any $\lambda \in \mathbb{C}^*$.

Definition 2.3 *([ACJ14])* **(2-point-local fields)** *We say that a field $a(z)$ on a vector space V is* **even** *and 2-point self-local at $(1;-1)$, if there exist $n_0, n_1 \in \mathbb{N}$ such that*

$$(z-w)^{n_0}(z+w)^{n_1}[a(z), a(w)] = 0. \tag{2.4}$$

In this case we set the **parity** *$p(a(z))$ of $a(z)$ to be 0.*
We set $\{a, b\} = ab + ba$. We say that a field $a(z)$ on V is 2-point self-local at $(1;-1)$ and **odd** *if there exist $n_0, n_1 \in \mathbb{N}$ such that*

$$(z-w)^{n_0}(z+w)^{n_1}\{a(z), a(w)\} = 0. \tag{2.5}$$

In this case we set the **parity** *$p(a(z))$ to be 1. For brevity we will just write $p(a)$ instead of $p(a(z))$.*
Finally, if $a(z), b(z)$ are fields on V, we say that $a(z)$ and $b(z)$ are 2-point mutually local at $(1;-1)$ if there exist $n_0, n_1 \in \mathbb{N}$ such that

$$(z-w)^{n_0}(z+w)^{n_1}\Big(a(z)b(w) - (-1)^{p(a)p(b)} b(w)a(z)\Big) = 0. \tag{2.6}$$

For a rational function $f(z, w)$, with poles only at $z = 0$, $z = \pm w$, we denote by $i_{z,w} f(z, w)$ the expansion of $f(z, w)$ in the region $|z| \gg |w|$ (the region in the complex z plane outside the points $z = \pm w$), and correspondingly for $i_{w,z} f(z, w)$. The mathematical background of the well-known and often used (both in physics and in mathematics) notion of Operator Product Expansion (OPE) of product of two fields for case of usual locality ($N = 1$) has been established for example in [Kac98], [LL04]. The following lemma extended the mathematical background to the case of 2-point locality (in fact to N-point locality, for $N \in \mathbb{N}$):

Lemma 2.4 *([ACJ14])* **(Operator Product Expansion (OPE))**
Let $a(z)$, $b(w)$ be 2-point mutually local. Then exists fields $c_{jk}(w)$, $j = 0, 1; k = 0, \dots, n_j - 1$, such that we have

$$a(z)b(w) = i_{z,w} \sum_{k=0}^{n_0-1} \frac{c_{0k}(w)}{(z-w)^{k+1}} + i_{z,w} \sum_{k=0}^{n_1-1} \frac{c_{1k}(w)}{(z+w)^{k+1}} + :a(z)b(w): . \tag{2.7}$$

We call the fields $c_{jk}(w)$, $j = 0, 1; k = 0, \dots, n_j - 1$ OPE coefficients. We will write the above OPE as

$$a(z)b(w) \sim \sum_{k=0}^{n_0-1} \frac{c_{0k}(w)}{(z-w)^{k+1}} + \sum_{k=0}^{n_1-1} \frac{c_{1k}(w)}{(z+w)^{k+1}}. \tag{2.8}$$

The $\sim$ signifies that we have only written the singular part, and also we have omitted writing explicitly the expansion $i_{z,w}$, which we do acknowledge tacitly. Often also the following notation is used for short:

$$\lfloor ab \rfloor = a(z)b(w) - :a(z)b(w): := [a(z)_-, b(w)], \tag{2.9}$$

i.e., the *contraction* of any two fields $a(z)$ and $b(w)$ is in fact also the $i_{z,w}$ expansion of the singular part of the OPE of the two fields $a(z)$ and $b(w)$.

The OPE expansion in the multi-local case allowed us to extend the Wick's Theorem (see e.g., [BS83], [Hua98]) to the case of multi-locality (see [ACJ14]), and we will use it in what follows, together with the Taylor Expansion Lemma (see [ACJ14]).

3. The Fock space $F^{\otimes \frac{1}{2}}$ and 2-point-local field representations of Vir

We recall the definitions and notations for the Fock space $F^{\otimes \frac{1}{2}}$ as in [Fre81], [DJKM81a], [Kac90], [Wan99b]; in particular we follow the notation of [Wan99b], [Wan99a].

Consider a single odd self-local field $\phi^D(z)$, which we index in the form $\phi^D(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \phi^D_n z^{-n-\frac{1}{2}}$. The OPE of $\phi^D(z)$ is given by

$$\phi^D(z)\phi^D(w) \sim \frac{1}{z-w}. \tag{3.1}$$

This OPE completely determines the commutation relations between the modes ϕ^D_n, $n \in \mathbb{Z} + \frac{1}{2}$:

$$\{\phi^D_m, \phi^D_n\} := \phi^D_m \phi^D_n + \phi^D_n \phi^D_m = \delta_{m,-n}1. \tag{3.2}$$

and so the modes generate a Clifford algebra Cl_D. The field $\phi^D(z)$ is usually called a "real neutral fermion field". The Fock space, denoted by $F^{\otimes \frac{1}{2}}$, of the field $\phi^D(z)$ is the highest weight module of Cl_D with vacuum vector $|0\rangle$, so that $\phi^D_n|0\rangle = 0$ for $n > 0$. This well known Fock space is often called the Fock space of the free real neutral fermion (see e.g. [DJKM81b], [FFR91], [KW94], [KWY98], [Wan99a], [Wan99b], [RT12]). $F^{\otimes \frac{1}{2}}$ has basis

$$\{\phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_2-\frac{1}{2}} \phi^D_{-n_1-\frac{1}{2}}|0\rangle, |0\rangle \mid \text{where } n_k > \dots > n_2 > n_1 \geq 0, n_i \in \mathbb{Z}_{\geq 0}, i = 1, 2, \dots, k\} \tag{3.3}$$

We recall the various gradings on $F^{\otimes \frac{1}{2}}$. The space $F^{\otimes \frac{1}{2}}$ has a well known $\mathbb{Z}_2$ grading given by $k \ mod\ 2$,

$$F^{\otimes \frac{1}{2}} = F^{\otimes \frac{1}{2}}_{\bar{0}} \oplus F^{\otimes \frac{1}{2}}_{\bar{1}},$$

where $F^{\otimes \frac{1}{2}}_{\bar{0}}$ (resp. $F^{\otimes \frac{1}{2}}_{\bar{1}}$) denote the even (resp. odd) components of $F^{\otimes \frac{1}{2}}$. This $\mathbb{Z}_2$ grading can be extended to a $\mathbb{Z}_{\geq 0}$ grading $\tilde{L}$, called "length", by setting

$$\tilde{L}(\phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_2-\frac{1}{2}} \phi^D_{-n_1-\frac{1}{2}}|0\rangle) = k. \tag{3.4}$$

Using the $\mathbb{Z}_2$ grading the space $F^{\otimes \frac{1}{2}}$ can be given a super-vertex algebra structure, as is known from e.g. [FFR91], [KW94], [Kac98].

In [ACJ13] and [Ang14] we introduced a $\mathbb{Z}$ grading dg on $F^{\otimes \frac{1}{2}}$ by assigning $dg(|0\rangle) = 0$ and

$$dg(\phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_2-\frac{1}{2}}\phi^D_{-n_1-\frac{1}{2}}|0\rangle) = \#\{i = 1, 2, \dots, k|n_i = \text{odd}\} \ - \#\{i = 1, 2, \dots, k|n_i = \text{even}\}.$$

Denote the homogeneous component of degree $dg = n \in \mathbb{Z}$ by $F^{\otimes \frac{1}{2}}_{(n)}$, hence as vector spaces we have

$$F^{\otimes \frac{1}{2}} = \oplus_{n\in\mathbb{Z}} F^{\otimes \frac{1}{2}}_{(n)}. \tag{3.5}$$

We define the special vectors $v_n \in F^{\otimes \frac{1}{2}}_{(n)}$ by

$$v_0 = |0\rangle \in F^{\otimes \frac{1}{2}}_{(0)}; \tag{3.6}$$

$$v_n = \phi^D_{-2n+1-\frac{1}{2}} \dots \phi^D_{-3-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle \in F^{\otimes \frac{1}{2}}_{(n)}, \quad \text{for } n > 0; \tag{3.7}$$

$$v_{-n} = \phi^D_{-2n+2-\frac{1}{2}} \dots \phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle \in F^{\otimes \frac{1}{2}}_{(-n)}, \qquad n > 0. \tag{3.8}$$

Note that the vectors $v_n \in F^{\otimes \frac{1}{2}}_{(n)}$ have minimal length $\tilde{L} = |n|$ among the vectors within $F^{\otimes \frac{1}{2}}_{(n)}$, and they are in fact the unique (up-to a scalar) vectors minimizing the length $\tilde{L}$, such that the index n_k is minimal too (n_k is identified from the smallest index appearing). One can view each of the vectors v_n as a vacuum-like vector in $F^{\otimes \frac{1}{2}}_{(n)}$, see below, and the dg grading as the analogue of the "charge" grading.

Note also that the super-grading derived from the second grading dg is compatible with the super-grading derived from the first grading $\tilde{L}$, as $dg \ mod\ 2 = \tilde{L}\ mod\ 2$.

Lastly, we recall the grading deg_h (from [Ang14])) on each of the components $F^{\otimes \frac{1}{2}}_{(n)}$ for each $n \in \mathbb{Z}$. Consider a monomial vector $v = \phi^D_{-n_k-\frac{1}{2}} \dots \phi^D_{-n_2-\frac{1}{2}}\phi^D_{-n_1-\frac{1}{2}}|0\rangle$ from $F^{\otimes \frac{1}{2}}_{(n)}$. One can view this vector as an "excitation" from the vacuum-like vector v_n, and count the n_i that should have been in v as compared to v_n, and also the n_i that should not have been in v as compared to v_n. Thus the grading deg_h (one can think of it as "energy") is defined as

$$\begin{aligned} deg_h(v) = &\sum\{\lfloor\frac{n_l+1}{2}\rfloor \mid n_l \text{ that should have been there but are not}\} \\ &+ \sum\{\lfloor\frac{n_l+1}{2}\rfloor \mid n_l \text{ that should not have been there but are}\}; \end{aligned}$$

here $\lfloor x\rfloor$ denotes the floor function, i.e., $\lfloor x\rfloor$ denotes the largest integer smaller or equal to x. Denote by $F^{\otimes \frac{1}{2}}_{(n,k)}$ the linear span of all vectors of grade $deg_h = k$ in $F^{\otimes \frac{1}{2}}_{(n)}$.

Recall the Heisenberg algebra $\mathcal{H}_{\mathbb{Z}}$ with relations $[h_m, h_n] = m\delta_{m+n,0}1$, m, n integers. We proved in [Ang14] (and by the less traditional bicharacter construction in [Ang13]) that each $F^{\otimes \frac{1}{2}}_{(n)}$ is an irreducible highest weight module for the Heisenberg algebra via a 2-point-local field:

Proposition 3.1 *([Ang13], [Ang14]) The 2-point-local field $h^D(z)$ given by:*

$$h^D(z) = \frac{1}{2} : \phi^D(z)\phi^D(-z) : \tag{3.9}$$

has only odd-indexed modes ($h^D(z) = -h^D(-z)$), $h^D(z) = \sum_{n\in\mathbb{Z}} h_n z^{-2n-1}$, and has OPE with itself given by:

$$h^D(z)h^D(w) \sim \frac{zw}{(z^2-w^2)^2} \sim \frac{1}{4}\frac{1}{(z-w)^2} - \frac{1}{4}\frac{1}{(z+w)^2}. \tag{3.10}$$

Hence its modes, h_n, $n \in \mathbb{Z}$, generate the Heisenberg algebra $\mathcal{H}_{\mathbb{Z}}$. The neutral real fermion Fock space $F^{\otimes\frac{1}{2}}$ is thus a module for $\mathcal{H}_{\mathbb{Z}}$ via this 2-point-local field representation and decomposes into irreducible highest weight modules for $\mathcal{H}_{\mathbb{Z}}$ under this action as follows:

$$F^{\otimes\frac{1}{2}} = \oplus_{m\in\mathbb{Z}} F^{\otimes\frac{1}{2}}_{(m)} \cong \oplus_{m\in\mathbb{Z}} B_m, \quad where \quad F^{\otimes\frac{1}{2}}_{(m)} \cong B_m, \quad B_m \cong \mathbb{C}[x_1, x_2, \dots, x_n, \dots], \quad \forall\, m \in \mathbb{Z}. \tag{3.11}$$

Recall the well-known Virasoro algebra Vir, the central extension of the complex polynomial vector fields on the circle. The Virasoro algebra Vir is the Lie algebra with generators L_n, $n \in \mathbb{Z}$, and central element C, with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{(m^3-m)}{12}C; \quad [C, L_m] = 0,\ m,n \in \mathbb{Z}. \tag{3.12}$$

Equivalently, the 1-point-local Virasoro field $L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ has OPE with itself given by:

$$L(z)L(w) \sim \frac{C/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}. \tag{3.13}$$

We denote by $M(c,h)$ the irreducible highest weight Vir module with central charge $c \in \mathbb{C}$, where $h \in \mathbb{C}$ is the weight of the operator L_0 acting on the highest weight vector v_h, i.e., $M(c,h)$ is an irreducible Vir module generated by a vector $v_h \in M(c,h)$ with $Cv_h = cv_h$, $L_n v_h = 0$ for any $n > 0$, and $L_0 v_h = hv_h$. (Note that in some of the literature such Vir modules are called "lowest weight", instead of "highest weight", here we follow the convention of Kac, as in [Kac80], [KR87], [Kac90]). The seminal paper [FF83] (using the Kac determinant formula, [Kac80]) delineates the cases where $M(c,h)$ is a quotient by a nontrivial factor of the corresponding Verma module $VM(c,h)$, as opposed to the "generic" cases where $M(c,h) \cong VM(c,h)$.

Denote the formal character of a highest weight module V with highest weight (c,h) to be

$$ch^{Vir}_q V := tr_V q^{L_0} := \sum_{j\in\mathbb{Z}_+} (dim V_{h+j}) q^{h+j},$$

where V_{h+j} is the eigenspace of L_0 of weight $h+j$.

It is well known that $F^{\otimes\frac{1}{2}}$ is a module for the Virasoro algebra with central charge $c = \frac{1}{2}$ (see for example [FFR91], [KW94], [Wan99a]) via the 1-point local Virasoro field $L^{1/2}(z)$ given by

$$L^{1/2}(z) = \frac{1}{2} : \partial_z \phi^D(z)\phi^D(z) : . \tag{3.14}$$

Furthermore, it is well known that as Vir modules

$$F^{\otimes\frac{1}{2}}_{\bar{0}} \cong M\left(\frac{1}{2}, 0\right); \quad F^{\otimes\frac{1}{2}}_{\bar{1}} \cong M\left(\frac{1}{2}, \frac{1}{2}\right); \quad F^{\otimes\frac{1}{2}} \cong M\left(\frac{1}{2}, 0\right) \oplus M\left(\frac{1}{2}, \frac{1}{2}\right).$$

In [Ang14] we proved the following Jacobi identity holds:

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Proposition 3.2 *(Corollary to Proposition 3.1) Define the graded dimension (character) of the Fock space $F = F^{\otimes\frac{1}{2}}$ as*

$$chF := tr_F q^{L_0^{1/2}} z^{h_0}.$$

We have

$$chF = \prod_{i=1}^{\infty}(1 + zq^{2i-1+\frac{1}{2}})(1 + z^{-1}q^{2i-2+\frac{1}{2}}) \tag{3.15}$$

$$= \frac{1}{\prod_{i=1}^{\infty}(1-q^{2i})} \sum_{n\in\mathbb{Z}} z^n q^{\frac{n}{2}} q^{n^2}. \tag{3.16}$$

By comparing the two identities we get the Jacobi identity

$$\prod_{i=1}^{\infty}(1-q^{2i})(1 + zq^{2i-\frac{1}{2}})(1 + z^{-1}q^{2i-\frac{3}{2}}) = \sum_{m\in\mathbb{Z}} z^m q^{\frac{m(2m+1)}{2}}. \tag{3.17}$$

Observe that we can specialize the graded dimension chF to the Virasoro character by evaluating at $z = 1$:

Corollary 3.3

$$ch_q^{Vir} F = ch_q M\left(\frac{1}{2}, 0\right) + ch_q M\left(\frac{1}{2}, \frac{1}{2}\right) = \prod_{i=0}^{\infty}(1 + q^{\frac{1}{2}}q^i) \tag{3.18}$$

$$= \frac{1}{\prod_{i=1}^{\infty}(1-q^{2i})} \sum_{n\in\mathbb{Z}} q^{n^2+\frac{n}{2}}. \tag{3.19}$$

If we use the q-Pochhammer symbol notation,

$$(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i); \quad (a;q)_m := \prod_{i=0}^{m-1}(1-aq^i), \quad \text{with } (a;q)_0 := 1;$$

we can rewrite the first formula (3.18)

$$ch_q^{Vir} F = ch_q M\left(\frac{1}{2}, 0\right) + ch_q M\left(\frac{1}{2}, \frac{1}{2}\right) = (-q^{\frac{1}{2}}; q)_\infty.$$

The formula (3.18) is well known (going back to I. Frenkel and D. Friedan, as well as [FF83]), but here we obtain it as a direct evaluation at $z = 1$ of the formula (3.15). We can refine the proof of (3.16) from [Ang14] to obtain separate formulas for $ch_q M\left(\frac{1}{2}, 0\right)$ and $ch_q M\left(\frac{1}{2}, \frac{1}{2}\right)$:

Proposition 3.4

$$ch_q M\left(\frac{1}{2}, 0\right) = \frac{1}{\prod_{i=1}^{\infty}(1-q^{2i})} \sum_{\substack{n\in\mathbb{Z} \\ n \text{ even}}} q^{n^2+\frac{n}{2}}; \tag{3.20}$$

$$ch_q M\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\prod_{i=1}^{\infty}(1-q^{2i})} \sum_{\substack{n\in\mathbb{Z} \\ n \text{ odd}}} q^{n^2+\frac{n}{2}} \tag{3.21}$$

These formulas have all positive coefficients in their sum, as

$$\frac{1}{\prod_{i=1}^{\infty}(1-q^i)} = \sum_{k\geq 0} p(k)q^k,$$

where $p(k)$ is the number of partitions $0 < k_1 \leq \cdots \leq k_l$ of $k = k_1 + \cdots + k_l$. Such formulas as (3.20) and (3.21) are called fermionic (see e.g. [KKMM93]), to distinguish them from the character sums with alternating sign coefficients (as in [RC85]), which are called bosonic.
Proof: We have, as vector spaces, that

$$M\left(\frac{1}{2},0\right) \cong F_{\bar{0}}^{\otimes\frac{1}{2}} = \oplus_{\substack{n\in\mathbb{Z}\\ n \text{ even}}} F_{(n)}^{\otimes\frac{1}{2}}; \qquad M\left(\frac{1}{2},\frac{1}{2}\right) \cong F_{\bar{1}}^{\otimes\frac{1}{2}} = \oplus_{\substack{n\in\mathbb{Z}\\ n \text{ odd}}} F_{(n)}^{\otimes\frac{1}{2}}.$$

We can now use the decomposition of each $F_{(n)}^{\otimes\frac{1}{2}}$ into $F_{(n,k)}^{\otimes\frac{1}{2}}$ by deg_h. From Proposition 3.1 it follows that a basis for $F_{(n,k)}^{\otimes\frac{1}{2}}$ is given by the elements $v_{h,n,\vec{k}} = h_{-k_l}\dots h_{-k_1}v_n$ with indecies varying with partitions $0 < k_1 \leq \cdots \leq k_l$ of $k = k_1 + \cdots + k_l$. First, we have

$$\begin{aligned}
L_0^{1/2}v_0 &= L_0^{1/2}|0\rangle = 0\\
L_0^{1/2}v_n &= L_0^{1/2}\phi^D_{-2n+1-\frac{1}{2}}\cdots\phi^D_{-3-\frac{1}{2}}\phi^D_{-1-\frac{1}{2}}|0\rangle\\
&= \left((1+\frac{1}{2})+(3+\frac{1}{2})+\cdots+(2n-1+\frac{1}{2})\right)v_n = (n^2+\frac{n}{2})v_n, \quad \text{for } n>0;\\
L_0^{1/2}v_{-n} &= L_0^{1/2}\phi^D_{-2n+2-\frac{1}{2}}\cdots\phi^D_{-2-\frac{1}{2}}\phi^D_{-\frac{1}{2}}|0\rangle\\
&= \left((0+\frac{1}{2})+(2+\frac{1}{2})+(4+\frac{1}{2})+\cdots+(2n-2+\frac{1}{2})\right)v_{-n} = (n^2-\frac{n}{2})v_{-n}, \quad \text{for } n>0.
\end{aligned}$$

Now a direct calculation shows that ([Ang14]):

$$:h^D(w)^2: =: h^D(w)h^D(w) := \frac{1}{4}:(\partial_{-w}\phi^D(-w))\phi^D(-w): + \frac{1}{4}:(\partial_w\phi^D(w))\phi^D(w): - \frac{1}{2w}h^D(w). \tag{3.22}$$

We can calculate by direct use of Wick's Theorem the OPE between $:h^D(z)^2:$ and $h^D(w)$, and thus by use of the equation above the commutator of $L_0^{1/2}$ and $h^D(w)$; and we obtain that

$$[L_0^{1/2}, h_k^D] = -2kh_k^D. \tag{3.23}$$

Hence

$$\begin{aligned}
L_0^{1/2}v_{h,0,\vec{k}} &= L_0^{1/2}h_{-k_l}\dots h_{-k_1}v_n = (2k_1+\cdots+2k_l)\,v_{h,0,\vec{k}};\\
L_0^{1/2}v_{h,n,\vec{k}} &= L_0^{1/2}h_{-k_l}\dots h_{-k_1}v_n = \left(2k_1+\cdots+2k_l+n^2+\frac{n}{2}\right)v_{h,n,\vec{k}} \quad \text{for } n>0;\\
L_0^{1/2}v_{h,-n,\vec{k}} &= L_0^{1/2}h_{-k_l}\dots h_{-k_1}v_{-n} = \left(2k_1+\cdots+2k_l+n^2-\frac{n}{2}\right)v_{h,-n,\vec{k}} \quad \text{for } n>0.
\end{aligned}$$

Now since there are $p(k)$ such elements for partitions $0 < k_1 \leq \cdots \leq k_l$ of $k = k_1 + \cdots + k_l$, we

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have for $F_{\bar{0}}^{\otimes\frac{1}{2}} \cong M\big(\frac{1}{2},0\big)$

$$ch_q M\big(\frac{1}{2},0\big) = tr_{F_{\bar{0}}^{\otimes\frac{1}{2}}} q^{L_0^{1/2}} = \sum_{k\geq 0} p(k)q^{2k} + \sum_{\substack{n,k\in\mathbb{Z}_+\\ n\ even}} p(k)\left(q^{2k+n^2+\frac{n}{2}} + q^{2k+n^2-\frac{n}{2}}\right)$$

$$= \sum_{k\in\mathbb{Z},k\geq 0} p(k)q^{2k}\cdot\left(1 + \sum_{\substack{n\in\mathbb{Z}_+\\ n\ even}}\left(q^{n^2+\frac{n}{2}} + q^{n^2-\frac{n}{2}}\right)\right)$$

$$= \frac{1}{\prod_{i=1}^{\infty}(1-q^{2i})}\sum_{\substack{n\in\mathbb{Z}\\ n\ even}} q^{n^2+\frac{n}{2}}.$$

The calculation for $F_{\bar{1}}^{\otimes\frac{1}{2}} \cong M\big(\frac{1}{2},\frac{1}{2}\big)$ is even simpler. □

We now turn to the new representations in this paper, the 2-point local Virasoro field representations on the Fock space $F^{\otimes\frac{1}{2}}$. In [Ang14] we proved that besides $L^{1/2}(z)$ there is a 2-parameter family of 2-point-local Virasoro fields on $F^{\otimes\frac{1}{2}}$:

Proposition 3.5 *([Ang14]) The 2-point-local field*

$$L^1(z^2) := \frac{1}{2z^2} : h^D(z)h^D(z) : \tag{3.24}$$

has only even-indexed modes, $L^1(z^2) := \sum_{n\in\mathbb{Z}} L_n^1(z^2)^{-n-2}$ and its modes L_n satisfy the Virasoro algebra commutation relations with central charge $c = 1$:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{(m^3-m)}{12}.$$

Equivalently, the 2-point-local field $L^1(z^2)$ has OPE with itself given by:

$$L^1(z^2)L^1(w^2) \sim \frac{1/2}{(z^2-w^2)^4} + \frac{2L^1(w^2)}{(z^2-w^2)^2} + \frac{\partial_{w^2}L^1(w^2)}{(z^2-w^2)}. \tag{3.25}$$

Furthermore, the 2-point-local field

$$L^{\lambda,b}(z^2) := L^1(z^2) + \frac{1-2\lambda}{4z^2}\partial_z h^D(z) - \frac{b}{z^3}h^D(z) + \frac{\left(b+\frac{1-2\lambda}{4}\right)^2 - 4\left(\frac{1-2\lambda}{4}\right)^2}{2z^4} = \sum_{n\in\mathbb{Z}} L_n^{\lambda,b}(z^2)^{-n-2} \tag{3.26}$$

is a Virasoro field for every $\lambda, b\in\mathbb{C}$ with central charge $-12\lambda^2+12\lambda-2$. If $\lambda = \frac{1}{2}$, $b = 0$, $L^{\frac{1}{2},0}(z^2) = L^1(z^2)$.

We now turn to the types of Vir representations these 2-point-local Virasoro fields generate on $F^{\otimes\frac{1}{2}}$, depending on the choices of the parameters (λ, b). First, observe that these representations of Vir, although of course expressible as 2-point local fermionic oscillator representations through the generating field $\phi^D(z)$ (see (3.22)), are in fact only dependent on the descendent Heisenberg field $h^D(z)$ from Proposition 3.1. It is then immediate that each $F_{(n)}^{\otimes\frac{1}{2}}$ for $n\in\mathbb{Z}$ is also a submodule for these 2-point-local Vir representations, and is in fact a highest weight module:

$$L_m^{\lambda,b}v_n = 0, \quad \text{for any } m,n\in\mathbb{Z}, \quad \text{where } m>0, \tag{3.27}$$

and we have for the highest weight vectors v_n $(n \in \mathbb{Z})$

$$\begin{aligned} L_0^{\lambda,b} v_n &= \left(\frac{n^2}{2} - bn - (\frac{1-2\lambda}{4})n + \frac{\left(b + \frac{1-2\lambda}{4}\right)^2 - 4(\frac{1-2\lambda}{4})^2}{2} \right) v_n \\ &= \left(\frac{\left(b + \frac{1-2\lambda}{4} - n\right)^2}{2} - 2\left(\frac{1-2\lambda}{4}\right)^2 \right) v_n. \end{aligned}$$

Observe that

$$c = 1 \quad \text{only for} \quad \lambda = \frac{1}{2}; \quad c = -12\lambda^2 + 12\lambda - 2 < 1 \quad \text{for} \quad \lambda \in \mathbb{R} \setminus \{\frac{1}{2}\}.$$

For "generic" cases of real $0 \leq c \leq 1$ and h the Verma modules $VM(c, h)$ are irreducible representations ([Kac80], [FF83]), and for those "generic" (c, h) we have $F_{(n)}^{\otimes \frac{1}{2}} \cong VM(c, h) = M(c, h)$. Furthermore, one is of course interested in the cases where the representation is unitary, and that leaves only the discrete series (see [FQS84], and e.g. [KR87]), or the case of $c = 1$ (see e.g. [Kac80], [FF83], [FQS84], [KR87]). For $F_{(n)}^{\otimes \frac{1}{2}}$ to be of discrete series type we need to have ([Kac80], [FF83], [FQS84], [KR87]):

$$\begin{aligned} c &= -12\lambda^2 + 12\lambda - 2 = 1 - \frac{6}{(m+2)(m+3)}, \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \\ h &= h_{r,s} = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)} \quad \text{for } r, s \in \mathbb{Z}_+,\ 1 \leq s \leq r \leq m+1; \end{aligned}$$

which here gives us

$$(1 - 2\lambda)^2 = \frac{2}{(m+2)(m+3)}.$$

Now we have to consider for such λ also the highest weight h, which gives us

$$\frac{\left(b + \frac{1-2\lambda}{4} - n\right)^2}{2} = \frac{((m+3)r - (m+2)s)^2}{4(m+2)(m+3)} \quad \text{for some } r, s \in \mathbb{Z}_+,\ 1 \leq s \leq r \leq m+1;$$

and thus

$$\left(\frac{b + \frac{1-2\lambda}{4} - n}{\frac{1-2\lambda}{4}} \right)^2 = 4\left((m+3)r - (m+2)s\right)^2, \quad \text{for } r, s \in \mathbb{Z}_+,\ 1 \leq s \leq r \leq m+1.$$

Hence we can find $b \in \mathbb{R}$ which will produce a discrete-series-type representation on $F_{(n)}^{\otimes \frac{1}{2}}$. To summarize, if the parameters (λ, b) satisfy

$$\begin{aligned} (2\lambda - 1) &= \pm\sqrt{\frac{2}{(m+2)(m+3)}}, \quad \text{for } m \in \mathbb{Z}_+; \\ (b - n) &= \pm\frac{2\left((m+3)r - (m+2)s\right) \pm 1}{2\sqrt{2(m+2)(m+3)}}, \quad \text{for } r, s \in \mathbb{Z}_+,\ 1 \leq s \leq r \leq m+1; \end{aligned}$$

then the submodule $F_{(n)}^{\otimes \frac{1}{2}}$ belongs to the discrete series. Thus we can choose values of the parameters (λ, b) to produce a 2-point local fermionic oscillator field representation of each of the discrete series Virasoro modules.

Next we consider the case of $c = 1$, i.e., $\lambda = \frac{1}{2}$. In this case we either have

$$L_0^{\frac{1}{2},b} v_n = \frac{(b-n)^2}{2} v_n = \frac{m^2}{4} v_n, \quad \text{for some } m \in \mathbb{Z};$$

i.e.,

$$(b-n)^2 = \frac{m^2}{2}, \quad \text{for some } m \in \mathbb{Z}; \tag{3.28}$$

or the submodule $F_{(n)}^{\otimes \frac{1}{2}}$ is irreducible. It is clear that if for some $n_1 \in \mathbb{Z}$ the parameter b satisfies

$$(b-n_1)^2 = \frac{m^2}{2}, \quad \text{for some } m \in \mathbb{Z}; \tag{3.29}$$

then for all other $n \in \mathbb{Z}$, $n \neq n_1$, we have $(b-n)^2 \neq \frac{m^2}{2}$, for any $m \in \mathbb{Z}$. Hence if the module $F_{(n_1)}^{\otimes \frac{1}{2}}$ is indeed reducible, then for all other $n \in \mathbb{Z}$, $n \neq n_1$, $F_{(n)}^{\otimes \frac{1}{2}}$ will be irreducible. Thus, when $\lambda = \frac{1}{2}$ we either have $(b-n)^2 \neq \frac{m^2}{2}$, for any $n, m \in \mathbb{Z}$ and thus all modules $F_{(n)}^{\otimes \frac{1}{2}}$ are irreducible (we call such b "generic"). Or exactly one the modules $F_{(n_1)}^{\otimes \frac{1}{2}}$ is completely reducible, the others are irreducible. The general structure of these reducible highest Vir modules with central charge 1 is well known from [FF83] (see also [KR87]), but here we can actually describe explicitly the singular vectors generating the submodules. We can without loss of generality assume that for $\lambda = \frac{1}{2}$ the one reducible submodule is $F_{(0)}^{\otimes \frac{1}{2}}$ (i.e., $b^2 = \frac{m^2}{2}$, for some $m \in \mathbb{Z}$), and thus the other $F_{(n)}^{\otimes \frac{1}{2}}$, $n \neq 0$, are irreducible.

Lemma 3.6 *Let* $\lambda = \frac{1}{2}$, $b = \frac{m}{\sqrt{2}}$, $m \in \mathbb{Z}$ *is fixed.*

Case I. *For* $m \geq 0$, *the following vectors in* $F_{(0)}^{\otimes \frac{1}{2}}$, *indexed by* $k \in \mathbb{Z}$, $k \geq 0$, *exhaust all singular vectors for the two-point local* Vir *representation on* $F_{(0)}^{\otimes \frac{1}{2}}$*:*

$\tilde{v}_{m,0} = v_0 = |0\rangle, \quad \textit{for } k = 0;$

$\tilde{v}_{m,k} = \phi^D_{-2(k+m)+1-\frac{1}{2}} \phi^D_{-2(k+m-1)+1-\frac{1}{2}} \cdots \phi^D_{-2m-1-\frac{1}{2}} \phi^D_{-2(k-1)-\frac{1}{2}} \phi^D_{-2(k-2)-\frac{1}{2}} \cdots \phi^D_{-\frac{1}{2}} |0\rangle, \quad \textit{for } k > 0.$

We have

$$L_j^{\frac{1}{2},\frac{m^2}{2}} \tilde{v}_{m,k} = 0, \quad \textit{for any } j > 0, k \geq 0;$$
$$L_0^{\frac{1}{2},\frac{m^2}{2}} \tilde{v}_{m,k} = \frac{1}{4}(m+2k)^2 \tilde{v}_{0,k}.$$

Case II. *For* $m < 0$, *the following vectors in* $F_{(0)}^{\otimes \frac{1}{2}}$, *indexed by* $k \in \mathbb{Z}$, $k \geq -m$, *exhaust all singular vectors for the two-point local* Vir *representation on* $F_{(0)}^{\otimes \frac{1}{2}}$*:*

$\tilde{v}_{m,-m} = v_0 = |0\rangle, \quad \textit{for } k = -m;$

$\tilde{v}_{m,k} = \phi^D_{-2(k+m)+1-\frac{1}{2}} \phi^D_{-2(k+m-1)+1-\frac{1}{2}} \cdots \phi^D_{-1-\frac{1}{2}} \phi^D_{-2(k-1)-\frac{1}{2}} \cdots \phi^D_{2(m-1)-\frac{1}{2}} \phi^D_{2m-\frac{1}{2}} |0\rangle, \quad \textit{for } k > -m.$

We have

$$L_j^{\frac{1}{2},\frac{m^2}{2}} \tilde{v}_{m,k} = 0, \quad \textit{for any } j > 0, k \geq -m;$$
$$L_0^{\frac{1}{2},\frac{m^2}{2}} \tilde{v}_{m,k} = \frac{1}{4}(m+2k)^2 \tilde{v}_{0,k}.$$

The proof is by direct calculation and we omit it.

Proposition 3.7 ***Case I.*** *Let $\lambda = \frac{1}{2}$, b is generic, i.e., $b \neq n_1 + \frac{m}{\sqrt{2}}$ for any $m, n_1 \in \mathbb{Z}$. Then as Virasoro modules with central charge $c = 1$*

$$F_{\bar{0}}^{\otimes \frac{1}{2}} = \oplus_{\substack{n\in\mathbb{Z}\\ n \ even}} M\big(1, \frac{(b-n)^2}{2}\big); \quad F_{\bar{1}}^{\otimes \frac{1}{2}} = \oplus_{\substack{n\in\mathbb{Z}\\ n \ odd}} M\big(1, \frac{(b-n)^2}{2}\big).$$

Case II. *Let $\lambda = \frac{1}{2}$, $b = n_1 + \frac{m}{\sqrt{2}}$ for some unique $m, n_1 \in \mathbb{Z}$ with n_1 **even**. Then as Virasoro modules with central charge $c = 1$*

$$F_{\bar{0}}^{\otimes \frac{1}{2}} = \left(\oplus_{\substack{n\in\mathbb{Z}\\ n \ even\\ n\neq n_1}} M\big(1, \frac{(m-\sqrt{2}n)^2}{4}\big) \right) \bigoplus \left(\oplus_{\substack{n\geq 0\\ n\geq -m}} M\big(1, \frac{(m+2n)^2}{4}\big) \right);$$
$$F_{\bar{1}}^{\otimes \frac{1}{2}} = \oplus_{\substack{n\in\mathbb{Z}\\ n \ odd}} M\big(1, \frac{(m-\sqrt{2}n)^2}{4}\big).$$

Case III. *Let $\lambda = \frac{1}{2}$, $b = n_1 + \frac{m}{\sqrt{2}}$ for some unique $m, n_1 \in \mathbb{Z}$ with n_1 **odd**. Then as Virasoro modules with central charge $c = 1$*

$$F_{\bar{0}}^{\otimes \frac{1}{2}} = \oplus_{\substack{n\in\mathbb{Z}\\ n \ even}} M\big(1, \frac{(m-\sqrt{2}n)^2}{4}\big);$$
$$F_{\bar{1}}^{\otimes \frac{1}{2}} = \left(\oplus_{\substack{n\in\mathbb{Z}\\ n \ odd\\ n\neq n_1}} M\big(1, \frac{(m-\sqrt{2}n)^2}{4}\big) \right) \bigoplus \left(\oplus_{\substack{n\geq 0\\ n\geq -m}} M\big(1, \frac{(m+2n)^2}{4}\big) \right).$$

We want to finish by showing an application of the above decomposition to calculating directly a positive sum (fermionic) representation of the characters of the Virasoro modules $M\left(\frac{1}{2}, \frac{1}{2}\right)$ and $M\left(\frac{1}{2}, 0\right)$. We can pick any b and use the Proposition above, together with the observation that we have a relation between the highest weights for $L_0^{1/2}$ and the highest weights for $L_0^{\lambda,b}$, via the connection represented by (3.22). In particular for $\lambda = \frac{1}{2}$, $b = 0$ we have $L^{\frac{1}{2},0}(z^2) = L^1(z^2)$; and from (3.22) we have that

$$L_0^{1/2} = 2L_0^1 + \frac{1}{2}h_0^D. \tag{3.30}$$

Hence we have from Proposition 3.7, Case II with $b = 0$:

$$ch_q M\big(\frac{1}{2}, \frac{1}{2}\big) = tr_{F_{\bar{1}}^{\otimes \frac{1}{2}}} q^{L_0^{1/2}} = tr_{F_{\bar{1}}^{\otimes \frac{1}{2}}} (q^2)^{L_0^1} q^{\frac{1}{2}h_0^D} = \sum_{\substack{n\in\mathbb{Z}\\ n \text{ odd}}} ch_{q^2} M\big(1, \frac{n^2}{2}\big) q^{\frac{n}{2}} = \sum_{\substack{n\in\mathbb{Z}\\ n \text{ odd}}} \frac{q^{n^2 + \frac{n}{2}}}{\prod_{i=1}^\infty (1-q^{2i})}.$$

Here we have used the well known character formula $ch_q M\big(1, \frac{n^2}{2}\big) = \frac{q^{n^2/2}}{\prod_{i=1}^\infty (1-q^i)}$, for $n \in \mathbb{Z}, n \neq 0$, and we recover the formula (3.21) we obtained earlier.

Similarly, using Case II, $b = 0$ and the formula for $ch_q M\big(1, \frac{m^2}{4}\big)$ (see e.g. [RC85]):

$$ch_q M\big(1, \frac{m^2}{4}\big) = \frac{1}{\prod_{i=1}^\infty (1-q^i)} \big(q^{m^2/4} - q^{(m+2)^2/4}\big), \quad \text{for } m \in \mathbb{Z},$$

we get

$$\begin{aligned} ch_q M\big(\frac{1}{2},0\big) &= tr_{F_{\bar{0}}^{\otimes\frac{1}{2}}} q^{L_0^{1/2}} = tr_{F_{\bar{0}}^{\otimes\frac{1}{2}}} (q^2)^{L_0^1} q^{\frac{1}{2}h_0^D} = \sum_{\substack{n\in\mathbb{Z}\\ n \text{ even}\\ n\neq 0}} ch_{q^2} M\big(1,\frac{n^2}{2}\big) q^{\frac{n}{2}} + \sum_{n\geq 0} ch_{q^2} M\big(1,n^2\big) \\ &= \sum_{\substack{n\in\mathbb{Z}\\ n \text{ even}\\ n\neq 0}} \frac{q^{n^2+\frac{n}{2}}}{\prod_{i=1}^\infty (1-q^{2i})} + \sum_{n\geq 0} \frac{q^{2n^2}-q^{2(n+1)^2}}{\prod_{i=1}^\infty (1-q^{2i})} = \sum_{\substack{n\in\mathbb{Z}\\ n \text{ even}\\ n\neq 0}} \frac{q^{n^2+\frac{n}{2}}}{\prod_{i=1}^\infty (1-q^{2i})} + \frac{1}{\prod_{i=1}^\infty (1-q^{2i})} \\ &= \sum_{\substack{n\in\mathbb{Z}\\ n \text{ even}}} \frac{q^{n^2+\frac{n}{2}}}{\prod_{i=1}^\infty (1-q^{2i})}. \end{aligned}$$

Note that we could have used a generic b, for instance we have

$$L^{\frac{1}{2},-\frac{1}{4}}(z^2) = L^1(z^2) + \frac{1}{4z^3}h^D(z) + \frac{1}{32z^4}, \qquad L_n^{\frac{1}{2},-\frac{1}{4}} = \frac{1}{2}L_{2n}^{1/2} + \frac{1}{32}\delta_{n,0},$$

but the resulting character formulas would have been the same.

We want to remark that this application of multi-local Virasoro field representations to calculating character formulas can be extended more generally to the discrete Virasoro series, not only to the Ising case of $c = \frac{1}{2}$. The main new idea is to obtain a decomposition of (the vector space of) the irreducible modules represented by a one-point local Virasoro field of discrete central charge $c = 1 - \frac{6}{(m+2)(m+3)}$ into irreducible modules represented by a multi-local Virasoro field of charge 1 (or higher).

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