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Comparisons of the nonlinear and the quasilinear model for the bump-on-tail instability with phase decorrelation

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Abstract. The dynamics of discrete global modes in a toroidal plasma interacting with an energetic particle distribution is studied, and in particular when the dynamics of the system using the nonlinear and quasilinear descriptions are macroscopically similar. The dynamics can be described with a nonlinear bump-on-tail model in a two-dimensional phase space of particles. A Monte Carlo framework is developed for this model with an included decorrelation of the wave-particle phase, which is used to model extrinsic stochastisation of the wave-particle interactions. From this description, a quasilinear version of the model is also developed, which is described by a diffusive process in energy space due to the added phase decorrelation. Due to the reduced dimensionality of phase space, the quasilinear description is typically less computationally demanding than the nonlinear description. The purpose of the studies is to find conditions when a quasilinear model sufficiently describes the same phenomena of the wave-plasma interactions as a nonlinear model does. Via numerical and theoretical parameter studies, regimes where the two models overlap macroscopically are found. These regimes exist above a given threshold of the strength of the decorrelation, where coherent phase space structures are destroyed on time scales shorter than characteristic time scales of nonlinear particle motion in phase space close to the wave-particle resonance. Specifically for the quasilinear model, a theoretical value of the time scale of quasilinear flattening is derived and numerically verified.

1. Introduction

Wave-particle interaction plays an important role in plasma physics for heating with waves and for transport caused by microinstabilities. Instabilities appear when the distribution function in energy increases along the characteristic describing the wave-particle interactions. The dynamics is either studied with a quasilinear approximation or with a fully nonlinear model, and comparisons are made to evaluate the applicability of the quasilinear approximation. The nonlinear bump-on-tail model for the modeling of discrete global modes was developed by Berk *et al* [1, 2], and has been extensively studied [1–6] (to name a few references). A parameter quantifying extrinsic stochastisation of the wave-particle interactions has been introduced to this model. Above a certain threshold of this stochastisation parameter, the kinetic equation of the wave-particle interaction can be replaced by a diffusion equation in particle energy, independent of the wave-particle phase. Such a description, with a diffusion coefficient similar to the standard quasilinear diffusion of weakly turbulent plasmas [7, 8], is identified as the quasilinear limit of the nonlinear wave-particle interaction model.

2. Model equations

2.1. Nonlinear Monte Carlo model

Based on an action-angle description of the guiding center Hamiltonian in an axisymmetric toroidal system with slowly varying electromagnetic fields [9], it was shown by Berk, Breizman and collaborators [1, 2] that the Alfvén eigenmode-particle system can be described in a region of phase space locally around the wave-particle resonance using a one-dimensional bump-on-tail model. Following these derivations, using proper variable substitutions, and adding an *ad hoc* collision operator acting on the energetic particles, the wave-particle system can be reduced to the following set of equations:

$$\frac{\partial f}{\partial \tau} + u\frac{\partial f}{\partial \phi} + \mathrm{Re}\left[A\mathrm{e}^{-\mathrm{i}\phi}\right]\frac{\partial f}{\partial u} = \left.\frac{\mathrm{d}f}{\mathrm{d}\tau}\right|_{\mathrm{coll}}, \tag{1}$$

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\int \mathrm{d}\phi\,\mathrm{d}u\, f(\phi,u,\tau)\mathrm{e}^{\mathrm{i}\phi} - \gamma_{\mathrm{d}}A, \tag{2}$$

where $f(\phi, u, \tau)$ is the near-resonance energetic ("bump") distribution, (ϕ, u) is the particle position-energy phase space, τ is a parametrization of the time, $A(\tau)$ is the complex amplitude of the eigenmode ($\arg(A) - \phi$ is the relative wave-particle phase), $\mathrm{d}f/\mathrm{d}\tau|_{\mathrm{coll}}$ is the collision operator acting on the bump distribution, and γ_{d} is an explicit wave damping rate, e.g. due to interactions with a thermal background distribution of particles. Assuming that the amplitude evolves much slower than the phase space evolution of particles and taking $\mathrm{d}f/\mathrm{d}\tau|_{\mathrm{coll}} = 0$, the system approximately becomes a pendulum equation, with particles deeply trapped by the wave field oscillating at a frequency $\omega_{\mathrm{B}} = \sqrt{|A|}$ and a trapped region within $|u| < \sqrt{2(|A| - \mathrm{Im}[A\mathrm{e}^{\mathrm{i}\phi}])}$.

In the absence of sources and sinks, applied by the conditions $\mathrm{d}f/\mathrm{d}\tau|_{\mathrm{coll}} = 0$ and $\gamma_{\mathrm{d}} = 0$, the total energy, expressed as

$$U_{\mathrm{tot}} = \frac{|A|^2}{2} + \int \mathrm{d}\phi\,\mathrm{d}u\, f(\phi,u,\tau)u, \tag{3}$$

is a conserved quantity of the wave-particle system. For this considered case, given an initial low-amplitude perturbation of the eigenmode and a positive derivative of the particle distribution with respect to energy around the wave-particle resonance, the evolution of the amplitude perturbation becomes exponentially growing in time after some initial mixing of the phase space distribution of particles [10]. The *linear growth rate* of the amplitude is given by $\gamma_{\mathrm{L}} = \pi/2 \times \mathrm{d}F(u,0)/\mathrm{d}u|_{u=0}$, where $F(u,\tau) = \int \mathrm{d}\phi\, f(\phi,u,\tau)$. In the nonlinear simulations presented in this paper, a sinusoidal perturbation in ϕ is applied to the initial distribution function, since an initial flat ϕ distribution is in unstable equilibrium, for which the dynamics depend critically on statistical noise in the phase space distribution. An applied perturbation makes the results less dependent on statistical fluctuations.

The quasilinear model is based on the assumption that wave-particle interactions are extrinsically decorrelated, such that coherent interactions only occur on linear time scales. In order to gradually go from the fully nonlinear description to the quasilinear one, a collision operator of the form

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\tau}\right|_{\mathrm{coll}} = D_{\phi}\frac{\partial^2 f}{\partial \phi^2} \tag{4}$$

is introduced, where $D_{\phi} \geq 0$ is a constant, quantifying the strength of phase decorrelation of the particles. This specific form of the collision operator linearly preserves the particle energy distribution, which is typically not the case for a physical stochastisation process. However, its energy conservation property facilitates the comparisons with quasilinear theory a lot, which is explained in Sec. 2.2. Using the Kolmogorov forward equation, the system described by eqs. (1), (2) and (4) can be expressed as a system of stochastic differential equations (SDEs) according to

$$\mathrm{d}\phi_k = u_k\,\mathrm{d}\tau + \sqrt{2D_{\phi}}\,\mathrm{d}W_{\tau,k}, \tag{5a}$$

$$\mathrm{d}u_k = \mathrm{Re}\left[A\mathrm{e}^{-\mathrm{i}\phi_k}\right]\mathrm{d}\tau, \tag{5b}$$

$$\mathrm{d}A = -\left(\sum_k w_k \mathrm{e}^{\mathrm{i}\phi_k} + \gamma_\mathrm{d} A\right)\mathrm{d}\tau, \tag{5c}$$

where the bump distribution $f(\phi, u, \tau)$ is described by a set of discrete entities (markers) with a phase ϕ_k, energy u_k and weight w_k, and $W_{\tau,k}$ are individual independent Wiener processes in τ. An added common weight factor to all particles, such that $w_k \to \alpha w_k$, $\alpha > 0$, can be transformed away using the set of substitutions $A \to \alpha^{2/3}A$, $\tau \to \alpha^{-1/3}\tau$, $(u, \gamma_\mathrm{d}, D_\phi) \to \alpha^{1/3}(u, \gamma_\mathrm{d}, D_\phi)$.

A discrete time approximation is used for the numerical simulations. Assuming that the wave amplitude evolves on time scales much longer than the individual ϕ_k and u_k, A can be treated as an independent variable in eq. (5b), splitting the complete system into individual two-dimensional systems of SDEs (one for each particle), and two ODEs for the wave amplitude (the real and the imaginary component). Using an Itō-Taylor numerical scheme with a strong convergence of order 1.5 [11], the discrete stepping of particles in phase space is described by

$$\Delta\phi_k = \sqrt{2D_\phi}\Delta W_k + u_k\Delta\tau + \frac{1}{2}\mathrm{Re}\left[A\mathrm{e}^{-\mathrm{i}\phi_k}\right]\Delta\tau^2, \tag{6a}$$

$$\Delta u_k = \mathrm{Re}\left[A\mathrm{e}^{-\mathrm{i}\phi_k}\left(\Delta\tau - \mathrm{i}\sqrt{2D_\phi}\Delta Z_k + \frac{1}{2}\left[\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}\tau} - D_\phi - \mathrm{i}u_k\right]\Delta\tau^2\right)\right], \tag{6b}$$

where ΔW_k and ΔZ_k are Itō integrals of the Wiener processes. These can be sampled as $\Delta W_k = \xi_{1,k}\sqrt{\Delta\tau}$, $\Delta Z_k = (\xi_{1,k} + \xi_{2,k}/\sqrt{3})\Delta\tau^{3/2}/2$, where $\xi_{i,k}$ are independent normally distributed random variables of unit variance and zero mean. For the finite time stepping of the wave amplitude A, the standard fourth order Runge-Kutta method is used.

2.2. Quasilinear Monte Carlo model

The Brownian motion of the phase, in eq. (5a), induces a decorrelation of the wave-particle interaction in eq. (5b). When this phase decorrelation is strong, the evolution of the energy may be approximated by a random walk with a *quasilinear diffusion coefficient*, defined as

$$D_u(u) = \frac{A^2 D_\phi}{2(D_\phi^2 + u^2)}. \tag{7}$$

This diffusion coefficient is similar to the standard quasilinear diffusion of a weakly turbulent plasma [7,8]. Since the quasilinear wave-particle interaction is independent of ϕ and $\arg(A)$, the dimensionality of particle phase space is reduced from two to one, and the amplitude is a real quantity in the quasilinear description. The corresponding kinetic equation of the quasilinear description is given by

$$\frac{\partial F}{\partial\tau} = \frac{\partial}{\partial u}\left(D_u(u)\frac{\partial F}{\partial u}\right) = -\frac{\partial}{\partial u}\left(\frac{\mathrm{d}D_u}{\mathrm{d}u}F\right) + \frac{\partial^2}{\partial u^2}\left(D_u F\right). \tag{8}$$

The specific form of the added phase decorrelation preserves the total energy of the system of the nonlinear model, as given by eq. (3). Assuming this is true also for the purely quasilinear model, it is straightforward to derive the quasilinear amplitude equation from eqs. (3) and (8):

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = -\frac{1}{A}\int \mathrm{d}u\,\frac{\mathrm{d}D_u}{\mathrm{d}u}F(u,\tau) - \gamma_\mathrm{d}A = \frac{1}{A}\int \mathrm{d}u\,D_u(u)\frac{\partial F}{\partial u} - \gamma_\mathrm{d}A. \tag{9}$$

On time scales much shorter than the time scale for quasilinear flattening of the distribution (see eq. (20)) the distribution function evolves slowly, and can be approximated with a constant in time. Inserting this approximation into eq. (9) yields an effective growth rate of the wave amplitude,

$$\gamma_{\text{eff}} = \frac{D_\phi}{2} \int \mathrm{d}u \, \frac{\mathrm{d}F_0}{\mathrm{d}u} \frac{1}{D_\phi^2 + u^2}, \tag{10}$$

that includes effects of a finite width of the energy distribution, which was not considered in the derivation of γ_{L} in the nonlinear description.

The system of equations given by eqs. (8) and (9) can be written as a system of SDEs:

$$\mathrm{d}u_k = -\frac{A^2 D_\phi u_k}{(D_\phi^2 + u_k^2)^2} \, \mathrm{d}\tau + A \sqrt{\frac{D_\phi}{D_\phi^2 + u_k^2}} \, \mathrm{d}W_{\tau,k}, \tag{11a}$$

$$\mathrm{d}A = \left(\sum_k \frac{A D_\phi w_k u_k}{(D_\phi^2 + u_k^2)^2} - \gamma_{\text{d}} A \right) \mathrm{d}\tau, \tag{11b}$$

and the 1.5 order strong Itō-Taylor numerical scheme [11] yields the following stepping algorithm in u:

$$\Delta u_k = \frac{A}{2} \sqrt{\frac{D_\phi}{D_\phi^2 + u_k^2}} \Bigg\{ 2\Delta W_k - \Lambda_k u_k \left(\Delta W_k^2 + \Delta\tau \right) - \Lambda_k^2 \left(D_\phi^2 - 3u_k^2 \right) \left(\frac{1}{3} \Delta W_k^3 + \Delta Z_k^+ \right)$$
$$+ \Lambda_k^2 u_k^2 \Delta Z_k^- + \Lambda_k^3 u_k \left(7D_\phi^2 - 9u_k^2 \right) \Delta\tau^2 + \frac{2}{A} \frac{\mathrm{d}A}{\mathrm{d}\tau} \left(\Delta Z_k^- + \Lambda_k u_k \Delta\tau^2 \right) \Bigg\}, \tag{12}$$

where $\Delta W_k = \xi_{1,k} \sqrt{\Delta\tau}$, $\Delta Z_k^\pm = (\xi_{1,k} \pm \xi_{2,k}/\sqrt{3}) \Delta\tau^{3/2}/2$, $\Lambda_k \equiv A\sqrt{D_\phi/(D_\phi^2 + u_k^2)^3}$, and $\xi_{i,k}$ are independent and normally distributed random variables of unit variance.

As apparent from eq. (10), the growth rate of the amplitude in the quasilinear model theoretically coincides with the growth rate in the nonlinear model in the limit $D_\phi \to 0$. However, as D_ϕ approaches zero, the maximum allowed $\Delta\tau$ for convergence of the numerical algorithm is reduced. This limitation for low D_ϕ is not present in the nonlinear model. The first order term in eq. (12) differs from the half order term with a factor $\sim \Lambda_k D_\phi \sqrt{\Delta\tau} \sim A/D_\phi^2 \times \sqrt{D_\phi \Delta\tau}$ at most, which must be $\ll 1$ for good convergence.

2.3. Analytic solutions to the quasilinear model

Solving the system of equations of the quasilinear model, eqs. (8) and (9), is possible in principle by first separating the spatial and temporal parts of the kinetic equation. The eigenfunction solutions of the spatial part are integrals over parabolic cylinder functions. The non-trivial eigenfunctions diverge as $\chi \to +\infty$ or $\chi \to -\infty$, which makes them impractical to use as spatial basis functions for the distribution function. When inserting a general set of eigenfunction solutions into the quasilinear amplitude equation, eq. (9), an infinite dimensional system of equations results, which might also be impractical.

An alternative approach is to use an approximate form of the diffusion coefficient that is valid on limited time scales. A model with a parabolic form of the diffusion coefficient,

$$D_u(u) = \begin{cases} \frac{A^2}{2D_\phi} \left(1 - \frac{u^2}{u_{\text{w}}^2} \right) & : \ |u| \le u_{\text{w}}, \\ 0 & : \ |u| > u_{\text{w}}, \end{cases} \tag{13}$$

where $u_{\text{w}} \sim D_\phi$ is the effective width of the diffusion in u, turns out to be practically soluble from an analytical point of view. From now on, the diffusion coefficient in eq. (13) is referred to as parabolic diffusion, whereas the coefficient in eq. (7) is referred to as the Lorentzian diffusion. Inserting the parabolic form of the diffusion coefficient into eq. (8) yields

$$\frac{\partial F}{\partial \tau} = \frac{A^2}{2D_\phi u_{\mathrm{w}}^2}\frac{\partial}{\partial x}\left(\left(1-x^2\right)\frac{\partial F}{\partial x}\right), \quad |x| \leq 1, \tag{14}$$

where $x \equiv u/u_{\mathrm{w}}$. It should be noticed that the eigenfunctions of the right hand side are the Legendre polynomials, which form an orthogonal set in $|x| \leq 1$. Decomposing the distribution function in the region $|u| \leq u_{\mathrm{w}}$ into Legendre polynomials according to

$$F(u_{\mathrm{w}}x, \tau) = \frac{1}{u_{\mathrm{w}}}\sum_{n=0}^{\infty} Q_n(\tau)P_n(x), \tag{15}$$

inserting into eqs. (14) and (9), and using the orthogonality property of the Legendre polynomials, it can be shown that each Q_n and the amplitude satisfy the ODEs

$$\frac{\mathrm{d}Q_n}{\mathrm{d}\tau} = -\frac{A^2 Q_n}{2D_\phi u_{\mathrm{w}}^2} n(n+1), \tag{16}$$

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = \frac{A}{D_\phi u_{\mathrm{w}}}\sum_{n=0}^{\infty} Q_n \int_{-1}^{1} \mathrm{d}x\, x P_n(x) = \frac{2Q_1 A}{3D_\phi u_{\mathrm{w}}}. \tag{17}$$

By solving the closed system of equations given by eq. (16) for $n = 1$ and eq. (17), and solving eq. (16) for general n using the obtained $A(\tau)$, one finds the complete analytical solutions:

$$Q_n(\tau) = Q_n(0)\left(\frac{\eta+\psi}{\eta\exp([\eta+\psi]\tau)+\psi}\right)^{n(n+1)/2}, \tag{18}$$

$$A^2(\tau) = \frac{D_\phi u_{\mathrm{w}}^2 \eta(\eta+\psi)}{\eta+\psi\exp(-[\eta+\psi]\tau)}, \tag{19}$$

where $\eta \equiv A^2(0)/D_\phi u_{\mathrm{w}}^2$ and $\psi \equiv 4Q_1(0)/3D_\phi u_{\mathrm{w}}$.

One limitation of the parabolic diffusion model is that it only acts on the distribution function within the region $|u| < u_{\mathrm{w}}$, whereas the Lorentzian diffusion model acts on the whole u space. On time scales where regions $|u| \gtrsim u_{\mathrm{w}}$ of the distribution function are affected by diffusion in the Lorentzian model, the numerical and analytical solutions are expected to diverge. One can select u_{w} such that the initial growth rates of the wave amplitude match in the numerical and the analytical solutions. By doing this, one can obtain similar solutions of the two models on time scales $\tau \lesssim \tau_{\mathrm{QL}}$, where τ_{QL} is the characteristic time scale for quasilinear flattening of the distribution function. Therefore, the analytical solutions can be used to obtain an analytical expression for τ_{QL}.

The time scale of quasilinear flattening can be characterized as the time scale at which $\partial F/\partial u|_{u=0}$ is significantly reduced due to quasilinear diffusion. Only P_n for odd n contribute to the derivative of the distribution function at the resonance. The decay rate of Q_{2n+1} is of the order $2n^2(\eta+\psi)$ according to eq. (18). Hence, the term with the slowest decay that contributes to the derivative is Q_1, which is then expected to be the dominant contribution to $\partial F/\partial u$ at $(u, \tau) = (0, \tau_{\mathrm{QL}})$. The analytical τ_{QL} can be defined as the time at which $Q_1(\tau)/Q_1(0)$ essentially differs from unity. Setting this ratio e.g. to 1/e results in

$$\tau_{\mathrm{QL}} = \frac{1}{\eta+\psi}\left[1+\ln\left(1+\frac{\mathrm{e}-1}{\mathrm{e}}\frac{\psi}{\eta}\right)\right]. \tag{20}$$

Assuming that $\mathrm{d}F_0/\mathrm{d}u \approx \mathrm{d}F_0/\mathrm{d}u|_{u=0}$ on energy scales much larger than D_ϕ, and assuming $\eta \ll \psi$ (equivalent with $A(0) \ll A(\infty) - A(0)$, where $A(\infty)$ is the saturation amplitude of the analytical model) a slightly simpler expression for τ_{QL} than in eq. (20) can be found, namely $\tau_{\mathrm{QL}} = \gamma_{\mathrm{L}}^{-1}[\varsigma + \ln(\sqrt{\gamma_{\mathrm{L}} D_\phi^3}/A(0))]$, where $\varsigma = \ln(3\pi\sqrt{(\mathrm{e}-1)/8}) \approx 1.47$.

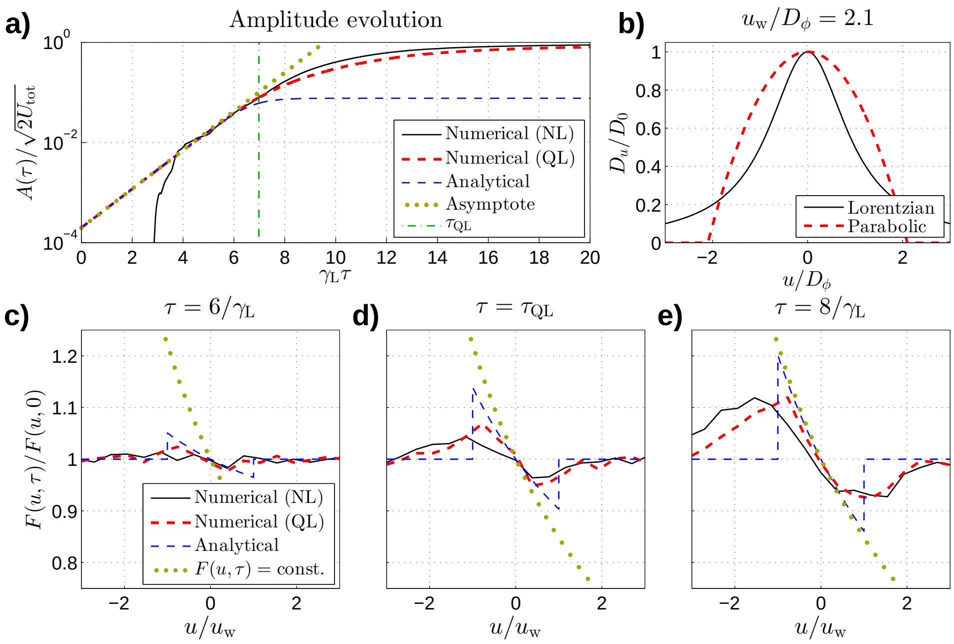

Figure 1. a) Evolution of the wave amplitude using the nonlinear (NL), the quasilinear (QL) and the analytical model. b) Comparison between the Lorentzian and the parabolic diffusion coefficient in the cases presented in (a). $D_0 \equiv A^2/2D_\phi$. c – e) The distribution function around the resonance at three chosen times. The distribution of the NL model is integrated over ϕ.

3. Results

3.1. Quasilinear flattening

Comparisons between the evolutions of the wave amplitude and of the energy distribution functions for simulations of the nonlinear model (NL), the quasilinear (QL) and the analytical model are presented in Fig. 1. An initial triangular bump distribution in energy is used, defined as $F_0(u) = F_0(0)(1 + u/\bar{u})$ for $|u| \le \bar{u}$, and $F_0(u) = 0$ for $|u| > \bar{u}$, which is chosen to minimize possible effects from higher order derivatives of the energy distribution around the resonance. Using eq. (20) it was found that $\gamma_{\rm L}\tau_{\rm QL} = 6.98$ for the specific case in Fig. 1. Unlike the quasilinear model, the immediately initial amplitude evolution of the nonlinear model is not exponential. Rather, there is an initial phase mixing state, with a faster growth of the amplitude due to the added sinusoidal perturbation in ϕ-space of the initial distribution function, as discussed in Sec. 2.1. To resolve this discrepancy, the time is shifted for the nonlinear model, such that it matches the wave amplitude of the numerical quasilinear model at $\tau = \tau_{\rm QL}$. Then the distribution functions of the three models can be compared at given times, which is done in Fig. 1.c – 1.e.

The saturation level of the analytical model is much lower than that of the nonlinear and the quasilinear model. This is due to the fact that the wave can only exhaust energy from a localized region $|u| < u_{\rm w}$ around the resonance using the parabolic diffusion model. Although differences are large between the analytical and numerical solutions during the saturation phase, they approximately agree for times up to the analytical time scale of quasilinear flattening, $\tau_{\rm QL}$ (aside from the initial phase mixing state of the nonlinear solution), which can be seen in Fig. 1.a. For $\tau > \tau_{\rm QL}$, the exponential evolution of the wave amplitude gradually ceases for both numerical solutions. This conclusion is consistent with the results presented in Fig. 1.c – 1.e. For $\tau = 6/\gamma_{\rm L} < \tau_{\rm QL}$, the energy distribution deviates from the initial distribution with a few percent at most. For $\tau \ge \tau_{\rm QL}$, deviations from the initial distribution start to become significant around the resonance, which here corresponds to the process of quasilinear flattening.

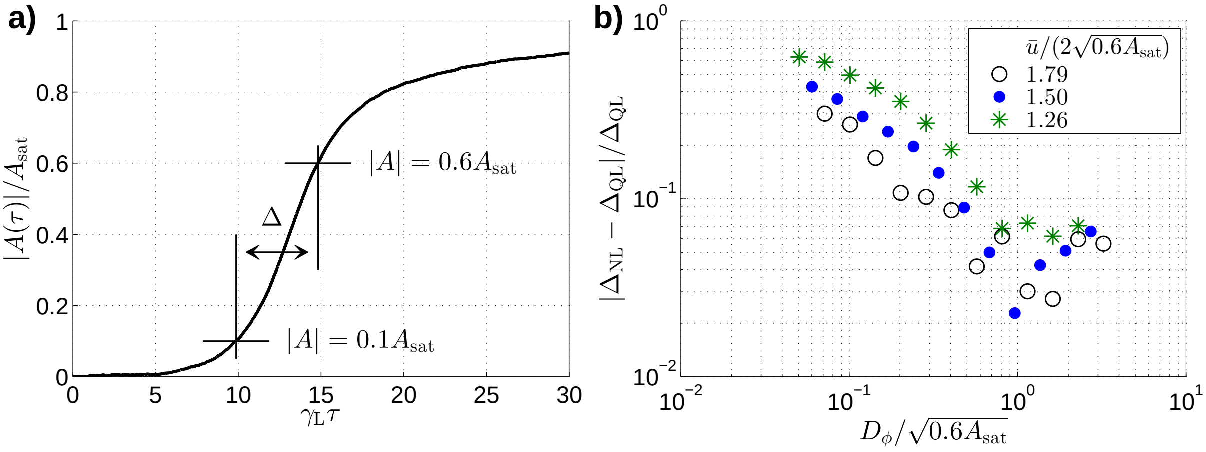

Figure 2. a) The definition of the quantity Δ. $A_{\rm sat}$ is the theoretical value of $|A(\tau)|$ for $\tau \to \infty$. b) The relative difference of Δ between nonlinear simulations and corresponding quasilinear simulations as a function of $D_\phi/\sqrt{|A|}$ at the end of the interval. A triangular initial energy distribution is used, with a full width of $2\bar{u}$, and $\gamma_{\rm d} = 0$.

3.2. Comparisons between the nonlinear and the quasilinear model

In order to conclude in which parameter regimes the numerical nonlinear and the numerical quasilinear model macroscopically agree, one has to find a quantity that primarily depends on the nonlinear dynamics of the system and compare this quantity for a wide set of nonlinear and quasilinear simulations. One nonlinear process is the saturation of the wave amplitude, which can be characterized by a saturation time scale and a value of the saturated amplitude. The latter is however trivial to determine in the presence of phase decorrelation. Turning off the wave damping ($\gamma_{\rm d} = 0$), the saturated amplitude corresponds to a state where the energy difference of the initial particle distribution and a final state of a symmetric energy distribution around the resonance is absorbed by the wave. In the presence of wave damping, the saturated state is simply zero, since the chosen collision operator lacks sources.

The quantity that has been chosen for comparison is the saturation time Δ, here defined as the time between the two states with amplitudes $|A| = 0.1A_{\rm sat}$ and $0.6A_{\rm sat}$ (see Fig. 2.a). In Fig. 2.b, the relative difference between Δ using the nonlinear and the quasilinear numerical models is shown. Effects of the width of the distribution function are studied by performing simulations with different values of $\bar{u}/\sqrt{0.6A_{\rm sat}}$, which is the ratio between the initial full width of the particle distribution in energy and the width of the trapped particle region by the wave field at the end of the Δ-interval. The quantity on the x-axis of Fig. 2.b, $D_\phi/\sqrt{0.6A_{\rm sat}}$, compares the bounce time of particles deeply trapped by the wave field ($\omega_{\rm B}^{-1} = |A|^{-1/2}$, referred to as the nonlinear time scale) with the time scale of phase decorrelation (D_ϕ^{-1}) at the end of the Δ-interval.

As shown in Fig. 2.b, the quasilinear model is able to predict the saturation time scale when the decorrelation time is shorter or similar to the nonlinear time scale ($D_\phi/\sqrt{0.6A_{\rm sat}} \gtrsim 1$). However, when the decorrelation time is long in comparison, e.g. when $D_\phi/\sqrt{0.6A_{\rm sat}} \lesssim 0.1$, the measured relative difference of Δ is larger than 20 %. The decrease of the relative error for increasing decorrelation strength ceases for shorter or similar decorrelation times, which might depend on numerical errors. From Fig. 2.b, one may also observe a better agreement between the models for distribution functions with a wider initial distribution around the resonance relative to the width of the trapped region, i.e. larger $\bar{u}/(2\sqrt{0.6A_{\rm sat}})$. One interpretation is that a large fraction of the complete structure of the distribution function becomes nonlinearly displaced by the wave field when the initial particle distribution is narrow, such that $\delta f/f$ becomes large on short time scales. Another interpretation could be that the discontinuity of the triangular distribution on the positive edge (at $u = \bar{u}$) becomes visible for the wave when the width of the trapped region is similar to $\bar{u}$, which might strongly affect the nonlinear behavior of the system.

4. Conclusions

In this paper, a nonlinear Monte Carlo model and a corresponding quasilinear model for describing the dynamics of discrete global modes interacting with energetic particles in a toroidal plasma in the presence of phase decorrelation have been compared. This study is performed mainly by computing macroscopic quantities in selected parameter regimes using a quasilinear approximation and a fully nonlinear description. There exist parameter regimes where the nonlinear and the quasilinear descriptions approximately coincide macroscopically. These regimes are mainly when the time scale for the destruction of macroscopic phase space structures (due to the added phase decorrelation) are much shorter than the characteristic time scale of phase space evolution of particles around the wave-particle resonance. However, due to the reduced dimensionality of phase space relative to the nonlinear model, there are certain phenomena depending on nonlinear phase space structures that the quasilinear model cannot describe.

Two partly related phenomena common for both the nonlinear and the quasilinear descriptions have been studied for comparison: quasilinear flattening (i.e., local flattening of the energy distribution around the resonance due to quasilinear energy diffusion) and saturation time scales of the wave amplitude in the presence of phase decorrelation. Analytical solutions to a problem similar to the quasilinear description were derived to obtain a theoretical value of the time scale of quasilinear flattening. When compared with numerical simulations using the quasilinear and the nonlinear descriptions, they were found to approximately match the theoretical time scale, both when looking at the deviations from an exponential growth of the wave amplitude and at the flattening of the energy distribution.

The saturation time scale was studied by comparing the time difference (referred to as Δ) between the states where the wave amplitude had reached $10\,\%$ and $60\,\%$ of the saturated amplitude in the presence of phase decorrelation, using the nonlinear and the quasilinear numerical model. It was found that the value of Δ is similar for the two models when the phase decorrelation is faster than the nonlinear bounce time (from the trapping of the wave field). However, when the decorrelation time scale is reduced, the differences between the quasilinear and the nonlinear model are significant. The energetic distribution of the presented numerical models has a finite width in energy, which affects the macroscopic behavior of the system. Wider initial distributions relative to the trapped particle region in the nonlinear model also give better agreement with the quasilinear model in general. This can be explained by the fact that a large fraction of the complete structure of the distribution function becomes nonlinearly displaced by the wave field when the initial particle distribution is narrow relative to the trapped particle region, such that $\delta f/f$ becomes large on short time scales for these cases.

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