

# One-dimensional dark solitons in lattices with higher-order nonlinearities

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**Abstract.** The existence of dark solitons in the discrete nonlinear Schrödinger equation with third- and fifth-order nonlinearities is investigated. Effects of discreteness for the homogeneous case are analyzed. Exact analytical solutions are found for wide static dark solitons in the presence of impurities. The bound soliton-defect formations can have a single minimum for attractive impurities or two minima for repulsive impurities. In contrast to the standard cubic nonlinear case, where the positions of the minima do not depend on the nonlinearity, now they are strongly influenced by the quintic nonlinearity. The model plays an important role in numerous physical systems with complicated nonlinear interactions.

## 1. Introduction

Phenomena associated with the interplay between nonlinearity and disorder are subject of intensive studies due to their importance for various physical systems. Nonlinearity induces the formation of spatially localized solitary waves (solitons), while the presence of impurities in inhomogeneous systems may support spatially localized impurity modes. The competition between the two different mechanism of localization leads to a complicated picture of localized states. The interaction of solitons in the continuum limit with linear [1-4] and nonlinear [5-8] point defects is widely investigated. Soliton trapping on impurities in discrete systems has been studied in [9-12]. These investigations are based on the standard nonlinear Schrödinger (NLS) equation where the nonlinearity is of third order. Considerable interest is also devoted to different variants of the NLS equation, such as the saturable NLS equation [13,14], the equation with power-law nonlinearity [6,15], and the cubic-quintic NLS equation [16-19]. The existence and stability of soliton solutions in these systems have been examined. The models appear in many physical problems of the macroscopic nonlinear dynamics of solids, nonlinear optics, Bose-Einstein condensation etc. [20]. Subject of these investigations are mostly the bright solitons. The existence of dark solitons ("holes" on a continuous wave background) in systems with various types of complicated nonlinear interactions is also important.

The present work is devoted to the study of dark solitary excitations in a nonlinear cubic-quintic lattice. The influence of the discreteness and the fifth-order nonlinearity on the soliton dynamics is analyzed. The interaction of the dark solitons with point defects in the static case is investigated.



## 2. Dark solitons in the homogeneous cubic-quintic lattice

We consider the dynamics of quasiparticles (e.g. phonons, excitons, magnons) with the amplitude  $\alpha_n(t)$  at site  $n$  in an inhomogeneous chain. When the density of the quasiparticles is high enough, their interaction should be taken into account and the system is described by the following set of nonlinear ordinary differential equations in the nearest-neighbour approximation:

$$i \frac{\partial \alpha_n}{\partial t} = -b(\alpha_{n+1} + \alpha_{n-1} - 2\alpha_n) + 2g|\alpha_n|^2\alpha_n + 3f|\alpha_n|^4\alpha_n + d\delta_{n,0}\alpha_n, \quad (1)$$

where  $b$  is the exchange interaction,  $g$  and  $f$  are the cubic and quintic nonlinear interactions, respectively.  $d$  characterizes a local linear defect of repulsion or attraction. We shall look for solutions in the form of amplitude-modulated waves

$$\alpha_n(t) = \varphi_n(t) e^{i(kn - \omega t)}, \quad (2)$$

where  $k$  and  $\omega$  are the wave vector and the frequency of the carrier wave (the lattice constant equals unity) and the envelope  $\varphi_n(t)$  is a real slowly varying function of the position and time. In the continuum limit  $\varphi_n(t) \rightarrow \varphi(x, t)$ , equation (1) transforms into the following perturbed NLS equation with cubic-quintic nonlinearities:

$$i \frac{\partial \varphi}{\partial t} = [2b(\cos k - 1) - \omega - d\delta(x)]\varphi - i2b \sin k \frac{\partial \varphi}{\partial x} - b \cos k \frac{\partial^2 \varphi}{\partial x^2} + 2g|\varphi|^2\varphi + 3f|\varphi|^4\varphi. \quad (3)$$

First we consider the homogeneous chain ( $d = 0$ ). For nonvanishing boundary conditions  $|\varphi| \rightarrow \varphi_0$  at  $x \rightarrow \pm\infty$  equation (3) possesses a dark soliton solution of the form [21]

$$\begin{aligned} \varphi(x, t) &= \varphi_0 \sqrt{B} \sinh \xi (1 + B \sinh^2 \xi)^{-1/2}, \quad \xi = \frac{x - vt}{L} \\ L^2 &= \frac{b \cos k}{g \varphi_0^2 + 3f \varphi_0^4}, \quad B = 1 - \frac{f \varphi_0^2}{g + 3f \varphi_0^2}, \\ \omega &= 2b(\cos k - 1) + 2g \varphi_0^2 + 3f \varphi_0^4, \quad v = 2b \sin k. \end{aligned} \quad (4)$$

$L$  and  $v$  are the width and the velocity of the soliton.  $B$  determines the deviation of the amplitude from its form for  $f = 0$  and has to be positive ( $B > 0$ ). The maximum intensity of the soliton is  $|\varphi(\pm\infty)|^2 = \varphi_0^2$ , while the minimum intensity is  $|\varphi(0)|^2 = 0$ .

For  $f = 0$  ( $B = 1$ ) the solution (4) turns into the well-known dark soliton of the cubic NLS equation:

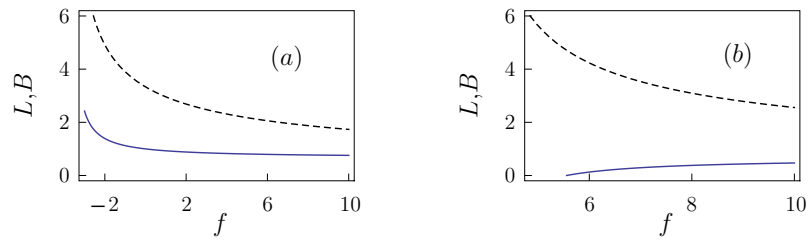
$$\begin{aligned} \varphi(x, t) &= \varphi_0 \tanh \frac{x - vt}{L} \\ L^2 &= \frac{b \cos k}{g \varphi_0^2}, \quad \omega = 2b(\cos k - 1) + 2g \varphi_0^2, \quad v = 2b \sin k. \end{aligned} \quad (5)$$

When quintic nonlinearity is present ( $f \neq 0$ ) the conditions  $L > 0$  and  $B > 0$  i.e.

$$\frac{b \cos k}{g + 3f \varphi_0^2} > 0, \quad \frac{g + 2f \varphi_0^2}{g + 3f \varphi_0^2} > 0 \quad (6)$$

have to be fulfilled. The inequalities (6) determine the range of values of the coefficients  $b$ ,  $g$ , and  $f$  for which the solution (4) exists. For example if  $b \cos k > 0$  the allowed values of  $f$  are  $f > -g/3 \varphi_0^2$  for  $g > 0$  and  $f > -g/2 \varphi_0^2$  for  $g < 0$ . In contrast to the solution of the cubic NLS equation (5) the parameter  $g$  can have positive as well as negative values. For  $g > 0$  we can choose  $f$  positive which leads to  $B < 1$  or  $f$  negative ( $B > 1$ ). For  $g$  negative we can choose only positive values of  $f$  ( $B < 1$ ). The dependence of the quantities  $L$  and  $B$  on the quintic nonlinear coefficient  $f$  for different values of  $g$  is shown on figure 1.

Note that the soliton velocity  $v$  does not depend on the nonlinearity as in the case of bright solitons [19].



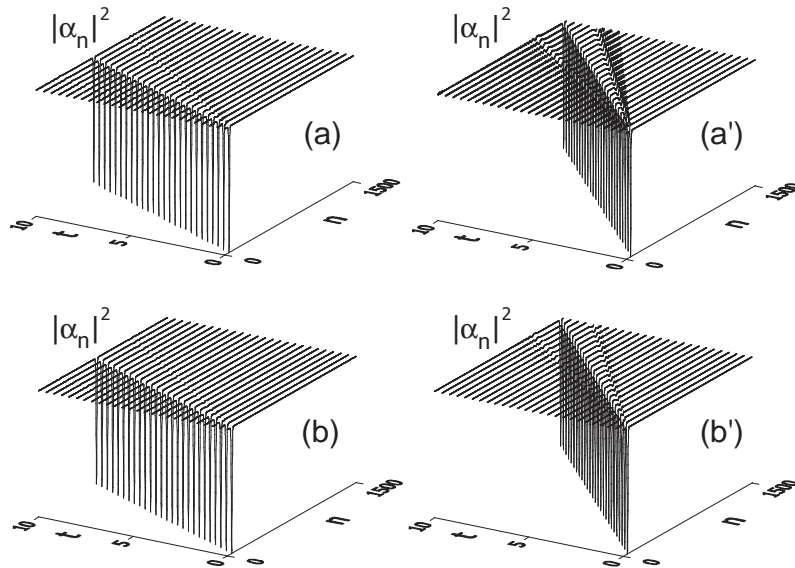
**Figure 1.** Dependence of the quantities  $L$  (dashed curves) and  $B$  (solid curves) on the nonlinearity coefficient  $f$ . (a):  $g = 1$ ; (b):  $g = -1$ .

### 3. Solitons in the homogeneous quintic lattice

In this section we shall consider the case when only the quintic nonlinearity is present ( $g = 0, f \neq 0$ ). Then  $B = 2/3$  and the dark soliton solution has the form

$$\alpha_n(t) = \varphi_0 \sqrt{2} \sinh \xi (3 + 2 \sinh^2 \xi)^{-1/2} e^{i(kn - \omega t)}, \quad \xi = \frac{n - vt}{L}$$

$$L^2 = \frac{b \cos k}{3f \varphi_0^4}, \quad \omega = 2b(\cos k - 1) + 3f \varphi_0^4, \quad v = 2b \sin k. \quad (7)$$

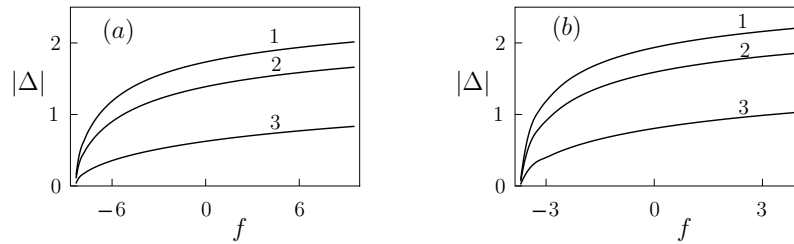


**Figure 2.** Propagation of dark solitons with  $\varphi_0 = 0.3$ ,  $L = 5$ ,  $b = 1$ . (a):  $g = 1$ ,  $f = -2.14$ ; (a'):  $g = 1$ ,  $f = -2.54$ ; (b):  $g = 0$ ,  $f = 1.57$ ; (b'):  $g = 0$ ,  $f = 1.16$ .  $k = 0.316$  for (a) and (b);  $k = 0.787$  for (a') and (b'). The time is in units of  $100/b$ .

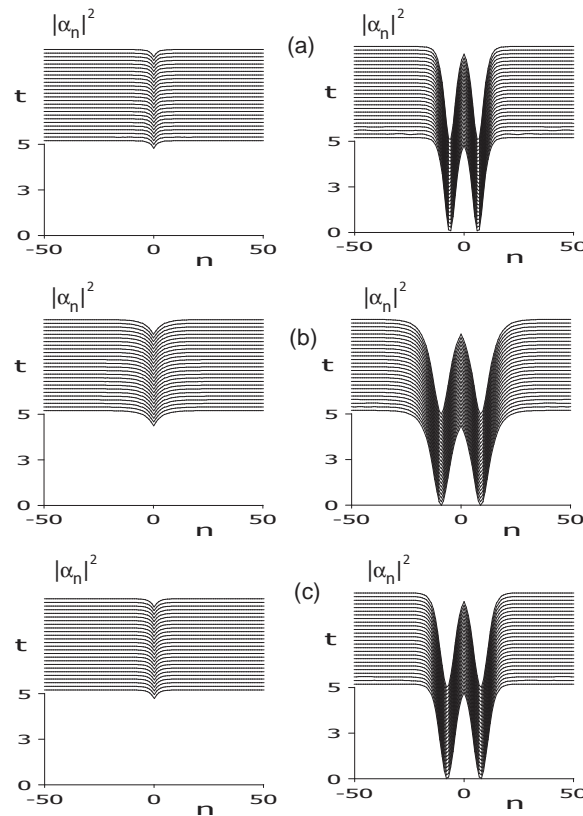
The propagation of the dark solitons (4) and (7) is shown on figure 2 for different  $k$  values (velocities). For small values of  $k$  both solutions are stable [figures 2(a) and (b)]. With increase of the velocity the discreteness effects become important. There is a small radiation from the background which for the cubic-quintic interaction is more intense [figures 2(a')] than for the pure quintic one [figures 2(b')]. The radiation increases when the amplitude  $\varphi_0$  increases (other parameters remain the same) as in this case the solitons become narrower and the condition  $L \gg 1$  is not valid any more.

We like to point out that the nonlinear terms in equation (1) can result from different models with complicated interactions. They are also a good approximation in some cases of the saturable nonlinearity of the form

$$c \frac{\alpha_n}{1 + |\alpha_n|^2}, \quad c = \text{const.} \quad (8)$$



**Figure 3.** Dependence of  $\Delta$  on the nonlinear coefficient  $f$  and the defect strength  $d$  for  $b = g = 1$  and (a)  $\varphi_0 = 0.2$ ; (b)  $\varphi_0 = 0.3$ . Curves 1:  $|d| = 0.05$ , curves 2:  $|d| = 0.1$ , and curves 3:  $|d| = 0.5$ .



**Figure 4.** Evolution of the bound soliton-impurity solution (8) with  $\varphi_0 = 0.3$  for (a)  $g = 1, f = 0$ ; (b)  $g = 1, f = -3$ ; (c)  $g = 0, f = 3$ . The left pictures hold for attractive impurities with  $d = 0.05$  ( $\Delta > 0$ ), while the right pictures hold for repulsive impurities with  $d = -0.05$  ( $\Delta < 0$ ).

#### 4. Interaction of dark solitons with impurities in the cubic-quintic lattice

Now we consider the inhomogeneous static case  $d \neq 0$ ,  $k = v = 0$ . We found that in the continuum approximation used ( $L \gg 1$ ) equation (1) has the following bound dark soliton-defect solution:

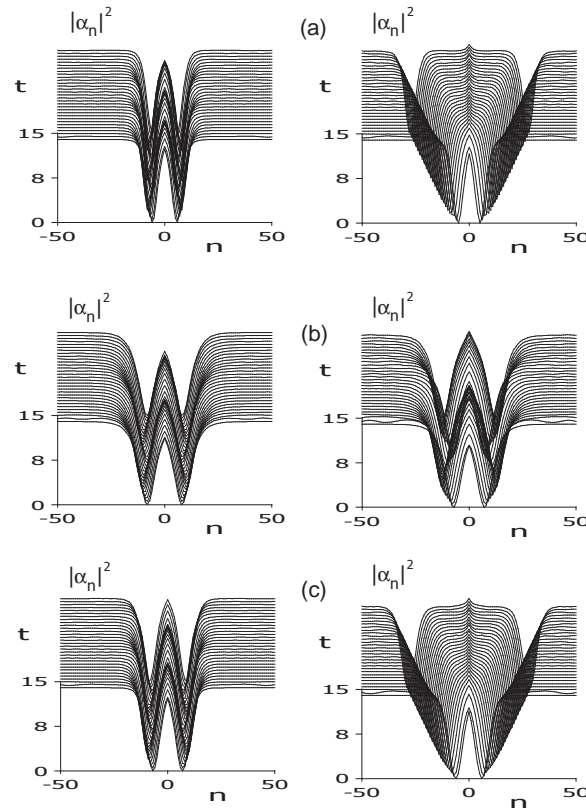
$$\alpha_n(t) = \varphi_0 \sqrt{B} \sinh \xi (1 + B \sinh^2 \xi)^{-1/2} e^{-i\omega t}, \quad \xi = \frac{|n|}{L} + \Delta$$

$$L^2 = \frac{b}{g\varphi_0^2 + 3f\varphi_0^4}, \quad B = 1 - \frac{f\varphi_0^2}{g + 3f\varphi_0^2}, \quad \omega = 2g\varphi_0^2 + 3f\varphi_0^4 \quad (9)$$

with the following relation for the determination of  $\Delta$

$$\tanh \Delta (1 + B \sinh^2 \Delta) = 2b/dL. \quad (10)$$

For  $\Delta > 0$  the function  $|\alpha_n(t)|$  has a single minimum at  $n = 0$ . For  $\Delta < 0$  there is a maximum at  $n = 0$  between two minima at  $n = \pm\Delta L$ .



**Figure 5.** Evolution of the dark soliton bound on a repulsive impurity ( $d = -0.05$ ) with initial  $\Delta_0$  and  $\varphi_0 = 0.3$  for (a)  $g = 1$ ,  $f = 0$ ; (b)  $g = 1$ ,  $f = -3$ ; (c)  $g = 0$ ,  $f = 3$ . The left pictures hold for  $\Delta_0 = 0.9\Delta$ , while the right pictures hold for  $\Delta_0 = 0.8\Delta$ .

Figure 3 shows the dependence of  $\Delta$  on  $f$ . Similar bound soliton solutions have been obtained for point defects in the cubic NLS equation [3,22]. For  $f = 0$  ( $B = 1$ ) we obtain from (10) the simple expression for  $\Delta$

$$\sinh 2\Delta = 4b/dL. \quad (11)$$

The evolution of the dark soliton bound on an impurity for different values of the nonlinear coefficients is shown on figure 4. For constant amplitude  $\varphi_0$  and impurity strength  $|d|$  the parameters of the form (9) are  $L = 3$ ,  $|\Delta| = 1.94$  [figure 4(a)],  $L = 8$ ,  $|\Delta| = 1.18$  [figure 4(b)] and  $L = 4$ ,  $|\Delta| = 2.07$  [figure 4(c)]. The single minimum form of the bound state corresponds to attraction on the impurity (left pictures), while the two minima form corresponds to repulsion on the impurity (right pictures).

We analyze the stability of the initial form with a  $\Delta_0$  which differs from the exact value  $\Delta$ . For attractive impurities ( $\Delta > 0$ ) the solution is stable at  $\Delta_0 \neq \Delta$ . For repulsive impurities ( $\Delta < 0$ ) the solution is unstable (figure 5). It begins to oscillate for small perturbations ( $\Delta_0 = 0.9\Delta$ , figure 5, left pictures) or splits into two solitons which propagate with opposite velocities for larger perturbations and narrow solitons [ $\Delta_0 = 0.8\Delta$ , figure 5(a) and (c), right pictures]. The process depends strongly on the soliton width  $L$ . For wider solutions ( $L = 8$ ) a relative large perturbation ( $\Delta_0 = 0.8\Delta$ ) leads still to oscillations [figure 5(b), right] in contrast to the cases with  $L = 3$  [figure 5(a), right] and  $L = 4$  [figure 5(c), right].

## 5. Conclusion

We have investigated the existence and stability of dark solitons in a discrete nonlinear chain with cubic-quintic nonlinearities. The interaction of solitons with impurities was studied as well. We have obtained analytical bound soliton-impurity solutions for wide solitons in the static case. The stability of the dark solitons is studied numerically and the influence of the different nonlinear coefficients is shown.

## Acknowledgments

This work is partially supported by the EU FP7 funded project INERA (Grant agreement no: 316309).

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