A Framework for Determining the Maximum Theoretical Power Output for a Given Vibration Energy

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Abstract. This paper outlines a mathematical framework to determine the upper bound on extractable power as a function of the forcing vibrations. In addition to determining the upper bound on power output, the method described provides insight into the dynamic transducer forces required to attain the upper bound. This relationship, between input vibration parameters and transducer force gives a critical first step in determining the optimal transducer architecture for a given vibration input. The method developed is applied to two specific vibration inputs; a single sinusoid, and the sum of two sinusoids. For the single sinusoidal case, the optimal transducer force is found to be that produced by a linear spring, resonant with the input frequency, and a linear viscous damper, with matched impedance to the mechanical damper. The solution to this first case was previously known, but has been used here to validate the methodology. The resulting transducer force for the input described by a sum of two sinusoids is found to be inherently time dependent. This time dependency shows that an active system can outperform a passive system. Furthermore, the upper bound on power output is shown to be twice that obtainable from a linear harvester centred at the lower of the two frequencies.

1. Introduction

Much recent work in vibration energy harvesting has focused on structure and transducer designs to improve power output from vibration sources that are not modeled as a single sinusoidal input. Much of this work has investigated the use of nonlinearities as a way to increase energy output [1-4]. These nonlinearities are usually of the form of a nonlinear spring, such as a Duffing oscillator. These works give useful insight to the potential uses of nonlinearities for harvesting from complex vibration inputs. However these works do not give a clear relationship between the parameters that define the input vibration and the optimal transducer dynamics.

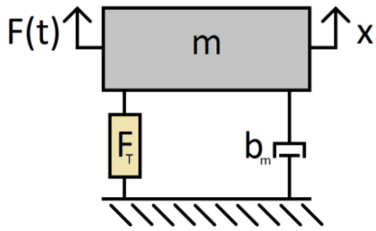
Other researchers have taken the opposite approach, starting with the vibration excitation and investigating the optimal transducer architectures to extract the maximum power. Daqaq et al. [5] showed that for Gaussian white noise the energy generation was not a function of the transducer's potential function. Halvorsen et. al. [6] proved that for an input described by a single frequency harmonic (i.e a single sinusoid) when the proof mass is subjected to viscous damping, the optimal transducer dynamics are those of a velocity damped resonant generator (VDRG) [7]. In both cases, the results are limited to very simple and specific types of forcing vibrations.

Continuing along a similar pathway, this work will present a method to find the unconstrained and globally optimal relationship between the input vibration, and force that must be produced by the transducer. This relationship will also define an upper limit for power generated for a given vibration input. This framework will then be applied to two case studies, a single sinusoid and the sum of two sinusoids.

2. General Model

A simple model for a kinetic harvester with a generic transducer force F_T that acts on the proof mass is shown in Figure 1. This generic transducer may contain both energy dissipative elements for power generation as well as energy conservative restoring elements. In general, the system is subject to a forcing function *F(t)*. The inherent mechanical losses that are found in any real system are approximated by a linear viscous damper described by a single coefficient b_m. This single degree of freedom system is characterized by a single displacement *x*. If the system is excited through base excitation, as is the normal case for an inertial generator, then *F(t)* would be the mass (*m*) multiplied by the base acceleration *A(t)*. In this case the displacement *x* is the relative distance between the proof mass and the base. The system schematic as well as its governing equations are shown in Figure 1.

(a)



(b)

$$m\ddot{x} + b_m\dot{x} + F_T = F(t) \tag{1}$$

$$\dot{x}_1 = x_2 \tag{2}$$

$$\dot{x}_2 = \frac{1}{m}\left(-b_m x_2 - F_T + F(t)\right) \tag{3}$$

Figure 1. (a) A generic inertial generator characterized by a single displacement *x*. Here F_T represents the force produced by an unknown transducer architecture. b_m is the coefficient that characterizes the system's linear viscous damping due to inherent mechanical losses of the system. (b) Governing equations for the generic inertial generator.

An energy balance of the system is used in order to find an expression for the energy generated by the transducer as a functison of the input. By examining the energy balance of the system in steady state we can neglect the kinetic energy of the mass as well as the possible potential energy stored in the transducer. This is due to the fact that these energy storage elements are restorative, thus they do not represent a net energy input or output to the system while it is in steady state. The energy that can be generated as a function of system parameters and input is given in (4):

$$E_{gen} = \int [F(t)x_2 - b_m x_2^2]\, \mathrm{d}t \tag{4}$$

For more generalized results we can look at the square of the power generated to examine a continuous positive definite functional, thereby allowing us to find the critical points in the magnitude of the energy generated.

$$J = \int [F(t)x_2 - b_m x_2^2]^2 \mathrm{d}t \tag{5}$$

If the velocity of the proof mass x_2 is treated as the control parameter, the critical points of the functional, which represents the energy generated by the transducer, can be found through the stationary condition of the Euler-Lagrange equation [8]. Taking the derivative of *I*, the integrand of (5), with respect to x_2 and solving for x_2 yields three optimal velocity paths shown in Table 1. Here $\star$ denotes the critical path with respect to the energy generated. These three relationships for $x_2^\star$ represent the critical velocity paths, given a vibration input *F(t)* to the system that will result in a minimum or maximum energy output. The second derivative determines if each solution is a maximum or minimum. By substituting the optimal velocity relationships into the governing differential equations (2) – (3), an expression for the displacement of the proof mass x_1 as well as the transducer force F_T can be expressed as a function of the system properties and the input vibration force. A summary of these solutions is given in Table 1, which shows the optimal velocity paths, position paths, and transducer forces required to produce those paths.

Table 1. Summary of critical path relationships for a general input.

Critical Velocity Path	Critical Positon Path	Critical Transducer Force	Type
$x_2^\star = \frac{F(t)}{2b_m}$	$x_1^\star = \int \frac{F(t)}{2b_m}\mathrm{d}t$	$F_T^\star = -\frac{m\dot{F}(t)}{2b_m} + \frac{F(t)}{2}$	Maximum
$x_2^\star = \frac{F(t)}{b_m}$	$x_1^\star = \int \frac{F(t)}{b_m}\mathrm{d}t$	$F_T^\star = -\frac{m\dot{F}(t)}{b_m}$	Minimum
$x_2^\star = 0$	$x_1^\star = 0$	$F_T^\star = F(t)$	Minimum

3. Model Applied to a Single Sinusoid Input

It is difficult to see the relevance of the equations in Table 1 in their general form. To help illustrate these relationships a simple example of a single frequency sinusoidal input will be examined. As the solution for this example is known, it further serves to validate the method. We will look at the relationship that maximizes the energy output of the system.

For the maximum power condition $x_2^\star = \frac{F(t)}{2b_m}$, letting $F(t) = A\,m\,sin(\omega t)$ results in the following relationships:

$$x_1^\star = -\frac{A\,m}{2b_m\omega}\cos(\omega t) \tag{6}$$

$$x_2^\star = \frac{A\,m}{2b_m}\sin(\omega t) \tag{7}$$

$$F_T{}^\star = -\frac{A\,\omega\,m^2}{2b_m}\cos(\omega t) + \frac{A\,m}{2}\sin(\omega t) \tag{8}$$

Substituting for $x_1^\star$ and $x_2^\star$ yields:

$$F_T^\star = \omega^2 m\,x_1 + b_m x_2 \tag{9}$$

The minimum power conditions will not be further investigated as they are not relevant to the goal of maximizing extracted power from a vibration source. Equations (6) – (9) can be interpreted as follows: the optimal transducer model is a linear spring ($k = \omega^2 m$) and a linear viscous damper which represents an electromechanical transducer. The constant of the linear spring is found to be resonant with the vibration input and the impedance of the electrical damper is found to be matched to the impedance of the mechanical damper b_m. This result is widely known and has been previously reported [6-7]. Thus, the application of this framework yielded the known optimal transducer architecture, without placing any a priori assumptions on the form of the transducer.

4. Model Applied to a Multiple Sinusoid Input

A source consisting of two sinusoids at different frequencies is a common vibration input. This type of vibration occurs in rotating machinery where two unbalanced masses rotate at different rates fixed relative to one another or in a system where multiple harmonics of a structure are well represented.

The RMS power output scales with A^2 for the standard linear system. Thus, the case where the amplitudes of the two sinusoids are equal will be examined. In the case where one sinusoid has an amplitude much greater than the other, it is reasonable to assume that the maximum power generation will be achieved by creating a linear harvester tuned to the frequency corresponding to the maximum value of A^2/ω. In the case where the two sinusoids are of similar, but different amplitudes, the following analysis is relevant. The expression for this double sinusoidal input is given as $F(t) = A\,m\,(\sin(\omega t) + \sin(n\omega t))$. Here, $n \in (0\ \infty)$ represents the multiple difference between the two frequency components.

Examining now only the vibration input which results in the maximum energy output, the optimal velocity signal for an input of two sinusoids is obtained in (10).

$$x_2^\star = \frac{A\,m}{2b_m}(\sin(\omega t) + \sin(n\omega t)) \tag{10}$$

Using the equations in Table 1, the relationships for the optimal position path and corresponding transducer force to achieve the velocity response as shown in (10) can be written as:

$$x_1^{\star} = -\frac{A\,m}{2\omega b_m}\left(\cos(\omega t) + \frac{1}{n}\cos(n\omega t)\right) \tag{11}$$

$$F_T^{\star} = \frac{A\,m}{2}\left(sin(\omega t) + sin(n\omega t)\right) - \frac{A\,\omega\,m^2}{2b_m}\left(cos(\omega t) + ncos(n\omega t)\right) \tag{12}$$

Substituting (11) and (10) for x_1 and x_2 into (12), where available, yields:

$$F_T^{\star} = b_m x_2 + \omega^2 m\, x_1 + TD \tag{13}$$

where *TD* is the time dependent component of the transducer force that cannot be directly substituted for by the systems states x_1 and x_2.

$$TD = \frac{A\,\omega\,m^2}{2b_m}\left(\frac{1}{n} - n\right)\cos(n\omega t) \tag{14}$$

The time dependent component of the transducer force shows that the true unconstrained optimal transducer force for an input vibration of this form cannot be realized with a passive system. Thus, in principle an active system can outperform a passive system of any type, linear or non-linear. However, this would assume that the time dependent restoring force implemented is conservative.

An energy balance is used to determine the nature of the time dependent force. It must be determined if the force does work, adding energy to the system over time, takes energy from the system, or does no net work on the system. The net energy into the proof mass from the time dependent force can be calculated by integrating the force over the displacement for a period T of the entire signal. Using equations (10) and (14) yields:

$$E_{TD}\left(\frac{2\pi}{\omega}\kappa\right) = \frac{A^2 m^3 (1-(-2+n)n-n(1+n)\mathrm{Cos}[2\kappa(-1+n)\pi]+(-1+n^2)\cos(2\kappa n\pi)^2+(-1+n)n\cos(2\kappa(1+n)\pi))}{8b^2n^2} \tag{15}$$

where κ is an integer value such that $n * \kappa \in \mathbb{Z}$. The total period for any input of this form is $T=2\pi/\omega\,\kappa$.

For the constraints of $\kappa \in \mathbb{Z}$ and $n * \kappa \in \mathbb{Z}$, (15) reduces to zero. This shows that the time dependent force acts as a conservative element, not doing any work to the system over time. The upper limit for energy output from the optimal transducer can be shown analytically. This can be accomplished in a similar manner to the derivation of the average power output for the single sinusoid case. Knowing that from the result of (13) and (15) the power output from the transducer is dissipated by the force of a linear viscous damper, the instantaneous power dissipated through this element can be written as:

$$P = F * v = b_m {x_2^{\star}}^2 \tag{16}$$

Here $x_2^{\star}$ is the optimal velocity shown in (10). Integrating the instantaneous power output over time yields the total energy generated by the transducer for all $n \in (1, \infty)$.

$$E_n = \frac{A^2\, m^2 T}{4b_m} \tag{17}$$

Note that for all n ≠ 1, the rms value of the excitation force is Am. However, for the special case in which n = 1, the rms value of the excitation force is $\sqrt{2}Am$, and thus the upper bound on the energy generated would be double that shown in (17). However, if the rms value of the driving force is normalized to Am for the special case of n = 1, (17) will still hold. The result is that the upper bound on power output is not a function of n. Intuitively this means that if the transducer force given by (12) and (13) can be generated, all of the power from both sinusoids could, in theory, be captured.

In order to gain additional insight into (17) a numerical study was performed in which the energy output over a sufficiently long period was measured for various values of n. The output of this study is shown in Figure 2. In one case, the optimal transducer force is applied to the proof mass. In the second case, the system is characterized by a linear oscillator whose resonance is the lower of the two frequencies present in the forcing vibrations. The output is normalized to the energy generated by either system at n = 1. As n deviates from 1, the power output from the linear system quickly drops to ½. However, the power output from the optimal system remains constant at 1.

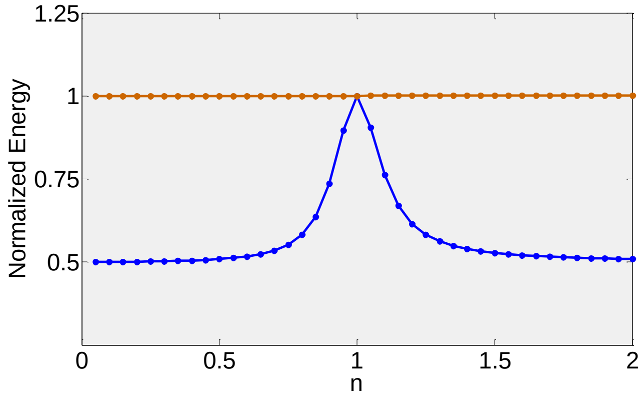


Figure 2. Numeric simulations of the energy output of the optimal transducer as compared to a linear harvester. The energy production has been normalized by the energy output of both systems at $n = 1$.

5. Conclusion

This paper has outlined a framework necessary to relate the form of an input vibration to an optimal transducer force. In creating this framework no assumptions of the transducer architecture were made. This framework was then applied to two case studies. The first was a vibration input of a single sinusoid. The optimal transducer was found to be a linear viscous damper with matched impedance, and a linear spring, resonant with the input frequency. This solution can be expressed as a function of the system's states so is considered a passive system. While the solution of this case study seems trivial, it demonstrates the method and validates it against a known case.

The second application was an input of the sum of two sinusoids at different frequencies. The optimal transducer force found was dependent on the difference between the two frequencies. In all cases the optimal transducer force consists of a linear viscous damper with matched impedance, a linear spring, and a time dependent force component. This time dependent component was found to act as a conservative force, like a time dependent spring. The framework was used to find the upper limit for power generation. This limit was found to be twice the power output of a linear system harvesting only from the lower of the two frequency components.

This basic framework could be applied to vibration inputs of various forms to determine the upper bound of power generation for that type of vibration, and the optimal transducer architecture. If a transducer architecture is assumed, a Duffing oscillator for example, this methodology can be applied to determine how close to the assumed solution is to the upper bound.

6. Acknowledgements

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7. References

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