

# An Analysis of the Spekkens Toy Theory with Connection to Wootters Discrete Phase Space

Mojtaba Aliakbarzadeh<sup>1</sup> and Hishamuddin Zainuddin<sup>1,2</sup>

<sup>1</sup> Laboratory of Computational Science and Mathematical Physics, Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia

<sup>2</sup> Department of Physics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia

E-mail: mojtabaalek@gmail.com

## Abstract.

The toy model of Spekkens is a formalism which can partially describe quantum mechanics. The theory deals with the (epistemic) states of a spin-1/2 particle, or qubits and it is closely related to the discrete phase space formalism of Wootters and collaborators. One can apply the stabilizer formalism for finding similarities of these two models. Noting that MUB basis vectors are obtained by eigenstates of generalized Pauli operators, the MUB basis vectors are thus the set of stabilizer states. Galvao has characterized the set of states with non-negative Wigner function class; they form the convex hull of the stabilizer states used as the MUB basis vectors. By combining both approaches, one can show epistemic states that are analogous to the convex hull of the stabilizer states (used as basis vectors in the MUB set) always make valid nonmaximal knowledge epistemic states.

## 1. Introduction

The Wigner function plays the role of a quasi-probability distribution on continuous phase space. While this real-valued function has many properties of probability distributions, it can take negative values. This negativity can be considered as nonclassical features of such quantum states [6]. In the same way as the original Wigner function, Wootters et. al [2, 3, 4] constructed Wigner functions on discrete phase spaces to describe finite-dimensional quantum systems. For determining this discrete phase space, they labelled the axes of the phase space with finite field (Galois field) of  $N$  elements where  $N$  is power of prime and at the same time with the quantum states of the system.

The discrete phase space of Wootters in  $d$ -dimensional Hilbert space is a  $d \times d$  real array and consists of lines as a set of points. It can be divided to parallel lines which are called striations. Parallel lines in each striation may not be really similar to the geometrical image of parallel lines in Euclidean space, but they have the similar feature that they never intersect with each other at any point. It should be mentioned that non-parallel lines at two different striations intersect at just one point. This feature is the result of using the Galois field to form  $p$  and  $q$  axes of the discrete phase space. For constructing discrete Wigner function Wootters et al. associate vectors of each basis with specific lines of the corresponding striation. This association is arbitrary and each separate choice leads to different class of Wigner function.



There is great interest to characterize quantum states with classical properties in the sense of giving rise to non-negative discrete Wigner function [5, 7]. On the other hand, while Spekkens toy model [1] is impotent to reproduce the violation of Bell inequalities and the existence of a Kochen-Specker theorem, we want to relate its states to the subset of quantum states which has non-negative Wigner functions. Spekkens toy model is based on foundational principle that there is a balance between knowledge and ignorance in a state of maximal knowledge. From this starting point, and a few other assumptions, one can derive the toy theory.

In this paper we try to implement the ontic and epistemic states of the Spekkens model in the discrete phase space formalism. We organize our approach for one and two qubits in following sections.

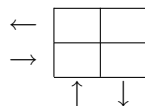
## 2. Elementary systems

In Spekkens toy model disjunction of four ontic states “1”, “2”, “3” and “4” lead to six epistemic states which can be represented graphically as follows (symbol  $\vee$  uses as notation of disjunction)

$$\begin{array}{ll}
 1 \vee 2 \iff |0\rangle \iff \begin{array}{|c|c|c|c|} \hline \blacksquare & \blacksquare & \square & \square \\ \hline \end{array} & 2 \vee 4 \iff |-\rangle \iff \begin{array}{|c|c|c|c|} \hline \square & \blacksquare & \square & \blacksquare \\ \hline \end{array} \\
 3 \vee 4 \iff |1\rangle \iff \begin{array}{|c|c|c|c|} \hline \square & \square & \blacksquare & \blacksquare \\ \hline \end{array} & 2 \vee 3 \iff |+i\rangle \iff \begin{array}{|c|c|c|c|} \hline \square & \blacksquare & \blacksquare & \square \\ \hline \end{array} \\
 1 \vee 3 \iff |+\rangle \iff \begin{array}{|c|c|c|c|} \hline \blacksquare & \square & \blacksquare & \square \\ \hline \end{array} & 1 \vee 4 \iff |-i\rangle \iff \begin{array}{|c|c|c|c|} \hline \square & \blacksquare & \square & \blacksquare \\ \hline \end{array}
 \end{array}$$

as it can be seen these six epistemic states are analogous to the six pure qubit states indicated.

In the case of discrete phase space for a single qubit (which is equivalent to a single spin-1/2 particle), Wootters et al. [4] associate the horizontal axes of phase space to eigenstates of  $\sigma_z$  i.e. states  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , and the vertical axes to eigenstates of  $\sigma_x$  i.e. states  $|\leftarrow\rangle$  and  $|\rightarrow\rangle$



The two choices for the eigenstates of  $\sigma_y$  are equally natural. Note that, in the case of more qubits, there are multiple ways of making these associations which will lead to different Wigner functions for the same fixed set of mutually unbiased basis (MUBs) [2, 3, 4].

Comparing discrete phase space with the epistemic states of a qubit, we can have following associations:

$$\begin{array}{ll}
 1 \vee 2 \iff |0\rangle \iff \begin{array}{|c|c|} \hline \blacksquare & \square \\ \hline \blacksquare & \square \\ \hline \end{array} & 2 \vee 4 \iff |-\rangle \iff \begin{array}{|c|c|} \hline \blacksquare & \blacksquare \\ \hline \square & \square \\ \hline \end{array} \\
 3 \vee 4 \iff |1\rangle \iff \begin{array}{|c|c|} \hline \square & \blacksquare \\ \hline \square & \blacksquare \\ \hline \end{array} & 2 \vee 3 \iff |+i\rangle \iff \begin{array}{|c|c|} \hline \square & \square \\ \hline \blacksquare & \blacksquare \\ \hline \end{array} \\
 1 \vee 3 \iff |+\rangle \iff \begin{array}{|c|c|} \hline \square & \square \\ \hline \blacksquare & \blacksquare \\ \hline \end{array} & 1 \vee 4 \iff |-i\rangle \iff \begin{array}{|c|c|} \hline \blacksquare & \square \\ \hline \square & \blacksquare \\ \hline \end{array}
 \end{array}$$

If we name the four ontic states of Spekkens model as:

1	2	3	4
---	---	---	---

it should correspond to the four states of Wootters model

$\leftarrow$	2	4
$\rightarrow$	1	3
	$\uparrow$	$\downarrow$

We can conclude ontic state “1” relates to state  $\uparrow\rightarrow$ , “2” to  $\uparrow\leftarrow$ , “3” to  $\downarrow\rightarrow$  and “4” to  $\downarrow\leftarrow$ , so we can assume states  $\uparrow\rightarrow$ ,  $\uparrow\leftarrow$ ,  $\downarrow\rightarrow$  and  $\downarrow\leftarrow$  as ontic states of Spekkens model.

1	2	3	4
$\uparrow\rightarrow$	$\uparrow\leftarrow$	$\downarrow\rightarrow$	$\downarrow\leftarrow$

### 3. Pairs of elementary systems

Extending this process for two-qubit case, we obtain the following picture where the rows represent the different ontic states of the first system (identified by subscript 1) and the columns represent the different ontic states of second systems (identified by subscript 2).

$(\downarrow\leftarrow)_1$				
$(\downarrow\rightarrow)_1$				
$(\uparrow\leftarrow)_1$				
$(\uparrow\rightarrow)_1$				
	$(\uparrow\rightarrow)_2$	$(\uparrow\leftarrow)_2$	$(\downarrow\rightarrow)_2$	$(\downarrow\leftarrow)_2$

The phase space for 2 qubits has  $N = 2^2 = 4$  dimensions which brings about a  $4 \times 4$  array of points for its phase space. Wootters associated the axes of the phase space by the discrete variables  $p$  and  $q$ , which take values in  $GF(N) = GF(4)$  and generated lines and striations in this discrete phase space labeled by the Galois field elements. Also conceiving qubits as spin states of spin-1/2 particle, he associated the horizontal axis of the phase space by the (composite) states  $|\uparrow\uparrow\rangle$ ,  $|\uparrow\downarrow\rangle$ ,  $|\downarrow\uparrow\rangle$ ,  $|\downarrow\downarrow\rangle$ , and the vertical axis by the states  $|\rightarrow\rightarrow\rangle$ ,  $|\rightarrow\leftarrow\rangle$ ,  $|\leftarrow\rightarrow\rangle$ ,  $|\leftarrow\leftarrow\rangle$  as shown here

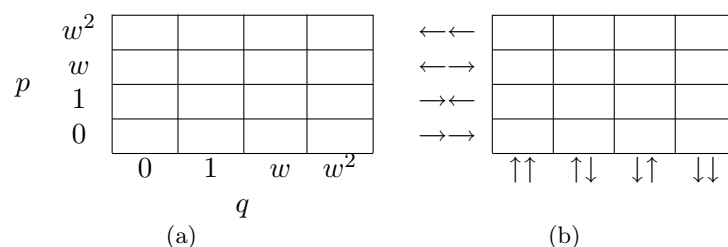


Figure 1: A  $4 \times 4$  array of points of phase space. The axis of phase space associated (a) by the elements of  $GF(N)$  and (b) spin states.

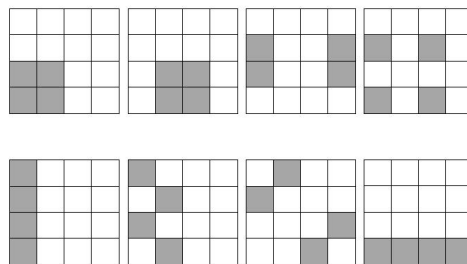
Comparing these two platforms, we can directly obtain following relation between cells of them, which can be used in inverting a state from one model to the other.

$(\downarrow\leftarrow)_1$	11	15	12	16
$(\downarrow\rightarrow)_1$	3	7	4	8
$(\uparrow\leftarrow)_1$	9	13	10	14
$(\uparrow\rightarrow)_1$	1	5	2	6
	$(\uparrow\rightarrow)_2$	$(\uparrow\leftarrow)_2$	$(\downarrow\rightarrow)_2$	$(\downarrow\leftarrow)_2$

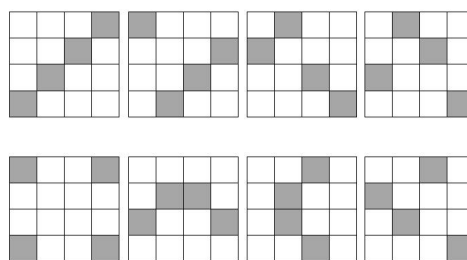
  

$\leftarrow\leftarrow$	13	14	15	16
$\leftarrow\rightarrow$	9	10	11	12
$\rightarrow\leftarrow$	5	6	7	8
$\rightarrow\rightarrow$	1	2	3	4
	$\uparrow\uparrow$	$\uparrow\downarrow$	$\downarrow\uparrow$	$\downarrow\downarrow$

Using this relation one can convert some exemplary states from Spekkens model to discrete phase space model; for instance, the first line of the following picture indicates some states in Spekkens model and the second line are their equivalent states in discrete phase space for uncorrelated composite systems.



For correlated composite systems the equivalent states in discrete phase space are as the second line of following picture.



This relation also can be applied to convert the 5 striations of discrete phase space for 2 qubits to Spekkens epistemic states. Note that there are multiple ways of making associations between lines of striations and states of a fixed MUB. In general, this will lead to different definitions of Wigner function. But here we don't care about these different possibility of Wigner function. Actually we assumed lines of striations as the representative of states of fixed MUB, then we try to find their relation with epistemic states. It is clear that the actual accordance is between epistemic states and MUB vectors, therefore to calculate Wigner function for epistemic states, there are again multiple ways of making associations between lines of striations and states of epistemic toy model.

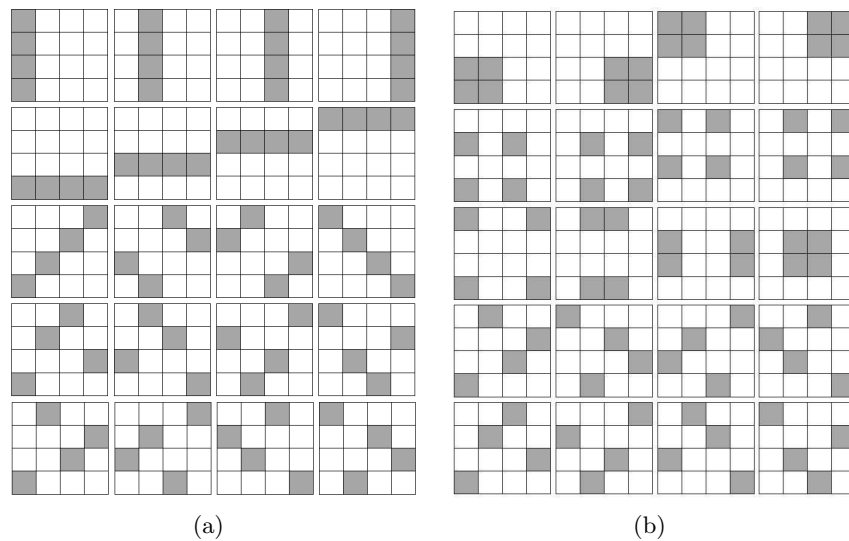
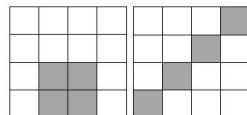


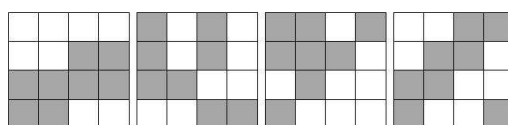
Figure 2: Converting (a) the 5 striations of discrete phase space for 2 qubits to (b) Spekkens epistemic states.

It is worth mentioning that, these five bases of epistemic states completely matches the sets of five mutually unbiased partitionings (MUPs) that Spekkens defined in his paper [1]. Similar to the set of MUBs for two qubits which has three fully separable bases and two non-separable ones [8], here also we have three sets of uncorrelated states and two sets of correlated states. But these MUP states don't cover all the possible correlated and uncorrelated states (there are some correlated and uncorrelated states which are not in this set) for example like



which are analogous to states  $|0\rangle + |i\rangle$  and  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  respectively but they don't belong to the MUB bases vectors. For finding these analogue states we apply the method of Matthew Pusey [9] where he suggested a way to compare the qubit stabilizer formalism and Spekkens toy theory. He introduced the toy stabilizer theory which can be mapped to and compared with qubit stabilizer theory. Using this innovative stabilizer notation he listed analogous states to Spekkens states.

We can see one of the interesting results of the comparison between the epistemic toy model and discrete phase space is in the case of epistemic states of nonmaximal knowledge. The states of nonmaximal knowledge in the Spekkens model are actually mixed states and they can be considered as convex combination of pure states. Spekkens gave some examples of epistemic states of nonmaximal knowledge that violate the principle in some way



We found that the convex combination of MUP states always make valid nonmaximal-knowledge epistemic states. It is consistent with Galvao et. al [5, 7] theorem that the convex hull of MUB basis vectors leads to non-negative discrete Wigner function; they defined the set of pure and mixed states  $C_d$  for  $d$ -dimensional Hilbert space having non-negative discrete Wigner functions. They claim that the only pure states having non-negative Wigner functions are stabilizer states, it is clear the only pure states in  $C_d$  are the MUB projectors which are stabilizer states (simultaneous eigenstates of Pauli operators). One should consider that those epistemic states out of MUP are similar to Pure stabilizer states out of  $C_d$  which lead to non negative Wigner function.

**Acknowledgments** This work is carried out under the ERGS grant 1-2013-552718, Ministry of Education, Malaysia.

## References

- [1] Spekkens R W 2007 Evidence for the epistemic view of quantum states: A toy theory *Physical Review A* **75**(3) 032110
- [2] Wootters W K 1987 A Wigner-function formulation of finite-state quantum mechanics *Annals of Physics* **176**(1) 1-21
- [3] Wootters W K 2004 Picturing qubits in phase space *IBM Journal of Research and Development* **48**(1) 99-110
- [4] Gibbons K S, Hoffman M J and Wootters W K 2004 Discrete phase space based on finite fields *Physical Review A* **70**(6) 062101
- [5] Galvao E F 2005 Discrete Wigner functions and quantum computational speedup *Physical Review A* **71**(4) 042302
- [6] Kenfack A and yczkowski K 2004 Negativity of the Wigner function as an indicator of non-classicality *Journal of Optics B: Quantum and Semiclassical Optics* **6**(10) 396
- [7] Cormick C, Galvao E F, Gottesman D, Paz J P and Pittenger A O 2006 Classicality in discrete Wigner functions *Physical Review A* **73**(1) 012301
- [8] Romero J L, Bjork G and Klimov A B 2005 Structure of the sets of mutually unbiased bases for N qubits *Physical Review A* **72**(6) 062310
- [9] Pusey M F 2012 Stabilizer notation for Spekkens toy theory. *Foundations of Physics* **42**(5) 688-708