On Derivations Of Genetic Algebras

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Abstract. A genetic algebra is a (possibly non-associative) algebra used to model inheritance in genetics. In application of genetics this algebra often has a basis corresponding to genetically different gametes, and the structure constant of the algebra encode the probabilities of producing offspring of various types. In this paper, we find the connection between the genetic algebras and evolution algebras. Moreover, we prove the existence of nontrivial derivations of genetic algebras in dimension two.

1. Introduction

In mathematical genetics, genetic algebras are (possibly non-associative) used to model inheritance in genetic. In application of genetic this algebra often has a basis corresponding to genetically different gametes, and the structure constant of the algebra encode the probabilities of producing offspring of various types. There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called genetic. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. The most comprehensive references for the mathematical research done in this area are [1, 2, 3, 4].

In [1] an evolution algebra $\mathbb{A}$ associated to the free population is introduced and using this non-associative algebra many results are obtained in explicit form, e.g. the explicit description of stationary quadratic operators, and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection.

In [3] a new type of evolution algebra is introduced. This algebra also describes some evolution laws of genetics and it is an algebra E over a field K with a countable natural basis $e_1, e_2, \dots$ and multiplication given by $e_ie_i = \sum_j a_{ij}e_j$, $e_ie_j = 0$ if $i \neq j$. Therefore, e_ie_i is viewed as "self-reproduction". The derivation for evolution algebra E is defined as usual, i.e. a linear operator $d : E \to E$ is called a derivation if

$$d(u \circ v) = d(u) \cdot v + u \cdot d(v)$$

for all $u, v \in E$.

Note that for any algebra, the space $Der(E)$ of all derivations is a Lie algebra with the commutator multiplication.

In the theory of non-associative algebras, particularly, in genetic algebras, the Lie algebra of derivations of a given algebra is one of the important tools for studying its structure. There has been much work on the subject of derivations of genetic algebras ([5],[6],[7]). For evolution algebras the system of equations describing the derivations are given in [3].

In [8] it was showed that the multiplication is defined in terms of derivations, showing the significance of derivations in genetic algebras. Several genetic interpretations of derivation of genetic algebra are given in [9].

The paper is organized as follows. In section 2 we recall some definitions and theorems, which are needed in this paper. In section 3 we describe derivations of three dimensional genetic algebras. section 3 is devoted to show the connection between genetic and evolution algebras in dimension two. In section 4 we prove the existence of nontrivial derivations of genetic algebras in dimension two.

2. Preliminaries

Let $\mathfrak{g}$ be an algebra over the field $\mathbb{K}$. Assume that $\mathfrak{g}$ admits a basis $\{e_1, ..., e_n\}$ such that the multiplication constants $P_{ij,k}$ with respect to this basis, are given by

$$e_i \cdot e_j = \sum_{k=1}^{n} P_{ij,k} e_k.$$

We say that $\mathfrak{g}$ is *a genetic algebra* if the multiplication constants $P_{ij,k}$ satisfy

(i) $P_{ij,k} \geq 0$

(ii) $\sum_{k=1}^{n} P_{ij,k} = 1$.

In that case, the basis $\{e_1, ..., e_n\}$ is called *a natural basis.*
Let $(E, \cdot)$ be an algebra over a field $\mathbb{F}$. If it admits a basis $\{e_1, e_2, \dots\}$ such that

$$e_i \cdot e_j = 0, \quad for\ i \neq j, \qquad e_i \cdot e_i = \sum_k a_{i,k} e_k, \quad for\ any\ i,$$

then E is called an *evolution algebra.* By $\mathbf{A}$ we denote the structural matrix of E, i.e. $\mathbf{A} = (a_{ij})_{1\leq i,j\leq n}$

Recall that the *derivation* on algebra A is a linear operator $d : A \to A$ such that

$$d(u \cdot v) = d(u) \cdot v + u \cdot d(v)$$

for all $u, v \in A$.

Theorem 2.1 *[10] Let $d : E \to E$ be a derivation of an evolution algebra E with non-singular* evolution matrix *in basis $\langle e_1, \dots, e_n\rangle$. Then the derivation d is zero.*

Theorem 2.2 *[10] Let E be an evolution algebra with structural matrix $\mathbf{A} = (a_{ij})_{1\leq i,j\leq n}$ in the natural basis $e_1, \dots, e_n$, and rank$\mathbf{A} = n - 1$ such that $e_n e_n = b(e_1 e_1)$. Then the following assertions are hold true:*

(i) If $b = 0$, then the derivations d of E is either zero or it is in one of the following forms up to basis permutation:

$$\begin{pmatrix} 0 & \dots & 0 & d_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & d_{n-1n} \\ 0 & \dots & 0 & 0 \end{pmatrix}, \tag{D_1}$$

where $\sum_{k=1}^{n-1} a_{ik}d_{kn} = 0,\ 1 \le i \le n-1;$

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \frac{d_{nn}}{2^{n-k-1}} & \dots & 0 & d_{k+1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \frac{d_{nn}}{2} & d_{n-1n} \\ 0 & \dots & 0 & 0 & \dots & 0 & d_{nn} \end{pmatrix}, \tag{D_2}$$

where $d_{i+1n} = \frac{a_{in}}{a_{ii+1}}\left(\frac{1}{2^{n-i-1}} - 1\right) d_{nn},\ a_{ii+1} \neq 0,\ k+1 \le i \le n-2, 1 \le k \le n-1$ *and* $d_{k+1n} \in \mathbb{C}$.

(ii) If $b \neq 0$*. Then derivation* d *is either zero or it is in one of the following forms up to basis permutation:*

(i) (D_3), *where* $d_{11} = \dfrac{\delta}{2^{n-s}-1},\ 1 \le s \le n-1$ *and* $\delta^2 = -bd_{1n}^2$;

(ii) (D_4), *where* $d_{22} = \dfrac{1-2^{m-k}}{2^{k-1}}d_{11}, d_{11} = \dfrac{\delta}{2^{m-k+1}-1},\ 1 \le k < m \le n-1$ *and* $\delta^2 = -bd_{1n}^2$;

(iii) (D_5), *where* $d_{11} = \delta$ *and* $\delta^2 = -bd_{1n}^2$.

Here D_3, D_4 *and* D_5 *are respectively given By*

$$\begin{pmatrix} d_{11} & 0 & \dots & 0 & 0 & \dots & 0 & d_{1n} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 2d_{11} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 2^{n-s-1}d_{11} & 0 \\ -bd_{1n} & 0 & \dots & 0 & 0 & \dots & 0 & d_{11} \end{pmatrix} \tag{D_3}$$

$$\begin{pmatrix} d_{11} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{1n} \\ 0 & d_{22} & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 2^{k-1}d_{22} & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 2d_{11} & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 2^{m-k}d_{11} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ -bd_{1n} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{11} \end{pmatrix} \tag{D_4}$$

$$\begin{pmatrix} d_{11} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{1n} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{d_{11}}{2^{n-s-2}} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{d_{11}}{2} & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & d_{11} & 0 \\ -bd_{1n} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & d_{11} \end{pmatrix} \qquad (D_5)$$

3. Derivations Of Three Dimensional Genetic Algebras

In this section, we are going to describe derivations of three dimensional genetic algebras. Let $\{e_1, e_2, e_3\}$ be a basis of three dimensional genetic algebra then the rule of multiplication is defined as follows:

$$(e_i \circ e_j) = \sum_{i,j=1}^{3} P_{ij,k} e_k, \quad k = \overline{1,3}, \tag{1}$$

where $\{P_{ij,k}\}$ are coefficients of heredity, which satisfy the following conditions

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{3} P_{ij,k} = 1, i, j, k \in \{1, 2, 3\}. \tag{2}$$

Let us define a derivation of the genetic algebra. Then one can represent it as

$$d(e_i) = \sum_{j=1}^{3} d_{ij} e_j.$$

So, let us start to calculate

$$d(e_1 \circ e_1) = d(e_1)e_1 + e_1 d(e_1) = 2d(e_1)e_1.$$

Hence,

$$d(e_1 \circ e_1) = d(p_{11,1}e_1 + p_{11,2}e_2 + p_{11,3}e_2) = 2(d_{11}e_1 + d_{12}e_2 + d_{13}e_3)e_1,$$

then

$$p_{11,1}d(e_1) + p_{11,2}d(e_2) + p_{11,3}d(e_3) = 2(d_{11}(e_1.e_1) + d_{12}(e_2.e_1) + d_{13}(e_3.e_1)).$$

Put the values of $d(e_i)$ and $(e_i \circ e_j)$ into above expression and compare the coefficients, we get

$$\begin{cases} p_{11,1}d_{11} + p_{11,2}d_{21} + p_{11,3}d_{31} = 2p_{11,1}d_{11} + 2p_{12,1}d_{12} + 2p_{13,1}d_{13} \\ p_{11,1}d_{12} + p_{11,2}d_{22} + p_{11,3}d_{32} = 2p_{11,2}d_{11} + 2p_{12,2}d_{12} + 2p_{13,2}d_{13} \\ p_{11,1}d_{13} + p_{11,2}d_{23} + p_{11,3}d_{33} = 2p_{11,3}d_{11} + 2p_{12,3}d_{12} + 2p_{13,3}d_{13} \end{cases} \tag{3}$$

In the same manner one gets the following system, for $d(e_2 \circ e_2)$:

$$\begin{cases} p_{22,1}d_{11} + p_{22,2}d_{21} + p_{22,3}d_{31} = 2p_{12,1}d_{21} + 2p_{22,1}d_{22} + 2p_{23,1}d_{23} \\ p_{22,1}d_{12} + p_{22,2}d_{22} + p_{22,3}d_{32} = 2p_{12,2}d_{21} + 2p_{22,2}d_{22} + 2p_{23,2}d_{23} \\ p_{22,1}d_{13} + p_{22,2}d_{23} + p_{22,3}d_{33} = 2p_{12,3}d_{21} + 2p_{22,3}d_{22} + 2p_{23,3}d_{23} \end{cases} \tag{4}$$

From $d(e_3 \circ e_3)$ one finds

$$\left\{\begin{array}{l} p_{33,1}d_{11} + p_{33,2}d_{21} + p_{33,3}d_{31} = 2p_{13,1}d_{31} + 2p_{23,1}d_{32} + 2p_{33,1}d_{33} \\ p_{33,1}d_{12} + p_{33,2}d_{22} + p_{33,3}d_{32} = 2p_{13,2}d_{31} + 2p_{23,2}d_{32} + 2p_{33,2}d_{33} \\ p_{33,1}d_{13} + p_{33,2}d_{23} + p_{33,3}d_{33} = 2p_{13,3}d_{31} + 2p_{23,3}d_{32} + 2p_{33,3}d_{33} \end{array}\right. \tag{5}$$

From $d(e_1 \circ e_2)$ we obtain

$$\left\{\begin{array}{lcl} p_{12,1}d_{11} + p_{12,2}d_{21} + p_{12,3}d_{31} & = & p_{12,1}d_{11} + p_{22,1}d_{12} + p_{23,1}d_{13} + p_{11,1}d_{21} \\ & & +p_{12,1}d_{22} + p_{13,1}d_{23} \\ p_{12,1}d_{12} + p_{12,2}d_{22} + p_{12,3}d_{32} & = & p_{12,2}d_{11} + p_{22,2}d_{12} + p_{23,2}d_{13} + p_{11,2}d_{21} \\ & & +p_{12,2}d_{22} + p_{23,1}d_{23} \\ p_{12,1}d_{13} + p_{12,2}d_{23} + p_{12,3}d_{33} & = & p_{12,3}d_{11} + p_{22,3}d_{12} + p_{23,3}d_{13} + p_{11,3}d_{21} \\ & & +p_{12,3}d_{22} + p_{13,3}d_{23} \end{array}\right. \tag{6}$$

From $d(e_1 \circ e_3)$ one has

$$\left\{\begin{array}{lcl} p_{13,1}d_{11} + p_{13,2}d_{21} + p_{13,3}d_{31} & = & p_{13,1}d_{11} + p_{23,1}d_{12} + p_{33,1}d_{13} + p_{11,1}d_{31} \\ & & +p_{12,1}d_{32} + p_{13,1}d_{33} \\ p_{13,1}d_{12} + p_{13,2}d_{22} + p_{13,3}d_{32} & = & p_{13,2}d_{11} + p_{23,2}d_{12} + p_{33,2}d_{13} + p_{11,2}d_{31} \\ & & +p_{12,2}d_{32} + p_{13,1}d_{33} \\ p_{13,1}d_{13} + p_{13,2}d_{23} + p_{13,3}d_{33} & = & p_{13,3}d_{11} + p_{23,3}d_{12} + p_{33,3}d_{13} + p_{11,3}d_{31} \\ & & +p_{12,3}d_{32} + p_{13,3}d_{33} \end{array}\right. \tag{7}$$

From $d(e_2 \circ e_3)$ one finds

$$\left\{\begin{array}{lcl} p_{23,1}d_{11} + p_{23,2}d_{21} + p_{23,3}d_{31} & = & p_{12,1}d_{21} + p_{23,1}d_{22} + p_{33,1}d_{23} + p_{12,1}d_{31} \\ & & +p_{22,1}d_{32} + p_{23,1}d_{33} \\ p_{23,1}d_{12} + p_{23,2}d_{22} + p_{23,3}d_{32} & = & p_{12,2}d_{21} + p_{23,2}d_{22} + p_{33,2}d_{23} + p_{12,2}d_{31} \\ & & +p_{22,2}d_{32} + p_{23,2}d_{33} \\ p_{23,1}d_{13} + p_{23,2}d_{23} + p_{23,3}d_{33} & = & p_{12,3}d_{21} + p_{23,3}d_{12} + p_{33,3}d_{23} + p_{12,3}d_{31} \\ & & +p_{22,3}d_{32} + p_{23,3}d_{33} \end{array}\right. \tag{8}$$

Now, let us consider the following example

Example 3.1 *Let*

$$p_{ii,k} = \left\{\begin{array}{lcl} 0 & : & i \neq k \\ 1 & : & i = k \end{array}\right.$$

And when $i \neq j$ we have the following matrix

$$p_{ij,k} = \left(\begin{array}{ccc} a & a & 0 \\ 0 & 0 & 1 \\ 1-a & 1-a & 0 \end{array}\right), a \in [0,1]$$

Now, we are going to find that the derivations of genetic algebra corresponding to $\{p_{ij,k}\}$. Let us first substitute the values of $p_{ij,k}$ into (3),(4),(5),(6),(7), and (9), Hence, we obtain

$$\begin{cases} -d_{11} - 2ad_{12} - 2ad_{13} = 0 \\ d_{12} = 0 \\ d_{13} - (2(1-a))d_{12} - (2(1-a))d_{13} = 0 \\ d_{21} - 2ad_{21} = 0 \\ -d_{22} - 2d_{23} = 0 \\ d_{23} - (2(1-a))d_{21} = 0 \\ d_{31} - 2ad_{31} = 0 \\ -d_{32} = 0 \\ -d_{33} - (2(1-a))d_{31} = 0 \\ (1-a)d_{31} - d_{21} - ad_{22} - ad_{23} = 0 \\ ad_{12} + (1-a)d_{32} - d_{12} - d_{13} = 0 \\ ad_{13} + (1-a)d_{33} - (1-a)d_{11} - (1-a)d_{22} - (1-a)d_{23} = 0 \\ (1-a)d_{31} - d_{31} - ad_{32} - ad_{33} = 0 \\ ad_{12} + (1-a)d_{32} - d_{12} - ad_{33} = 0 \\ ad_{13} - (1-a)d_{11} - d_{13} - (1-a)d_{32} = 0 \\ d_{21} - ad_{21} - ad_{31} = 0 \\ -d_{32} - d_{33} = 0 \\ -(1-a)d_{21} - (1-a)d_{31} = 0 \end{cases} \tag{9}$$

Now, let us solve the above system. It is clear that $d_{12} = d_{32} = 0$. Therefore, from $-d_{32} - d_{33} = 0$ one gets $d_{33} = 0$. Now, from $(1-a)d_{31} - d_{31} - ad_{32} - ad_{33} = 0$ and $-d_{33} - (2(1-a))d_{31} = 0$. we drive $d_{31} = 0$. Also from $-(1-a)d_{21} - (1-a)d_{31} = 0$ and $d_{21} - 2ad_{21} = 0$ we obtain $d_{21} = 0$. Therefore, from $d_{23} - (2(1-a))d_{21} = 0$ one finds $d_{23} = 0$. Consequently, $d_{22} = 0$ since $-d_{22} - 2d_{23} = 0$. From $ad_{12} + (1-a)d_{32} - d_{12} - d_{13} = 0$ we obtain $d_{13} = 0$. Then $-d_{11} - 2ad_{12} - 2ad_{13} = 0$ implies that $d_{11} = 0$. So, the derivation is zero
It is interesting to know the existence of nontrivial derivations. So, we will show the existence of nontrivial derivations in section five.

4. Relation Between Genetic and Evolution algebras

From the definition of genetic algebra and evolution algebra one can ask the following question: Is there a transformation of genetic algebra to some evolution algebra?
In this section, we are going to answer to this question in dimension two. Let $(p_{ij,k})$ be a structure matrix of the genetic algebra in dimension two. Namely, if e_1, e_2 are the basis of genetic algebra i.e.

$$e_i \circ e_j = \sum_{k=1}^{2} p_{ij,k} e_k. \tag{10}$$

Now, we want to change the given basis e_1, e_2 to a new basis f_1, f_2 such that the algebra generated by f_1, f_2 becomes an evolution algebra.

Theorem 4.1 *Let $\mathfrak{g}$ be a genetic algebra generated by the basis e_1, e_2 with structure constant $(p_{ij,k})$. Then $\mathfrak{g}$ is an evolution algebra with respect to new basis f_1, f_2 if and only if one of the following conditions are satisfied:*

(i) $p_{21,1} = p_{12,1} \neq \frac{p_{11,1} + p_{22,1}}{2}$

(ii) $p_{11,1} = p_{12,1} = p_{22,1}$

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Proof. Assume that the change of bases is given by $f_u = \sum_{j=1}^{2} t_{uj}e_j$. One can see that

$$\begin{aligned} f_1 \circ f_2 &= \Big(\sum_{j=1}^{2} t_{1j}e_j\Big) \circ \Big(\sum_{l=1}^{2} t_{2l}e_l\Big) \\ &= \sum_{j,\ l=1}^{2} t_{1j}t_{2l} \circ \sum_{k=1}^{2} p_{jl,k}e_k \\ &= \sum_{k=1}^{2}\Big(\sum_{j,\ l=1}^{2} t_{1j}t_{2l}p_{jl,k}\Big)e_k \end{aligned} \tag{11}$$

Due to the definition evolution algebra we conclude that an algebra generated by f_1, f_2 is evolution if and only if $f_1 \circ f_2 = 0$. So, from (11) we infer that $f_1 \circ f_2 = 0$ if and only if $\sum_{j,\ l=1}^{2} t_{1j}t_{2l}p_{jl,k} = 0$. Let us redefine the matrix (t_{il}) as follows $(t_{jl}) = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$, and explicitly rewrite the expression $\sum_{j,\ l=1}^{2} t_{1j}t_{2l}p_{jl,k} = 0$. Hence, one finds

$$p_{11,1}xu + p_{12,1}xv + p_{21,1}yu + p_{22,1}yv = 0 \tag{12}$$

$$(1 - p_{11,1})xu + (1 - p_{12,1})xv + (1 - p_{21,1})yu + (1 - p_{22,1})yv = 0 \tag{13}$$

By adding (12) and (13) we obtain

$$xu + xv + yu + yv = 0.$$

So, $(u + v)(x + y) = 0$. This means $y = -x$ or $u + v = 0$. without loss of generosity we assume $y = -x$

Now, let $y = -x$ then the matrix (t_{jl}), after making a simple scale, takes the following form

$$(t_{jl}) = \begin{pmatrix} 1 & -1 \\ u & v \end{pmatrix}$$

And by back substituting into (12) we have

$$p_{11,1}u + p_{12,1}v - p_{21,1}u - p_{22,1}v = 0. \tag{14}$$

Now consider several cases

Case 1. Let $u = 0$, then $v \neq 0$, since $det(t_{jl}) \neq 0$. Therefore, from (14) one has $(p_{12,1} - p_{22,1})v = 0$ which yields $p_{12,1} = p_{22,1}$. So, the matrix (t_{jl}) has the following form

$$(t_{jl}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Case 2. Let $u \neq 0$ and $p_{12,1} = p_{22,1}$. Then from (14) we have $(p_{11,1} - p_{12,1})u = 0$. Therefore, $p_{11,1} = p_{12,1} = p_{12,1}$. So, the matrix (t_{jl}) takes the following form

$$(t_{jl}) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Case 3. Let $u \neq 0$ and $p_{12,1} \neq p_{22,1}$. Then from (14) one finds

$$p_{11,1} + p_{12,1}\Big(\frac{v}{u}\Big) - p_{21,1} - p_{22,1}\Big(\frac{v}{u}\Big) = 0$$

By a simple algebra we get

$$\frac{v}{u} = \frac{p_{21,1} - p_{11,1}}{p_{12,1} - p_{22,1}}.$$

Therefore, $v = \frac{p_{21,1}-p_{11,1}}{p_{12,1}-p_{22,1}} \cdot u$ Then the matrix (t_{jl}), by making simple scale, takes the following form:

$$(t_{jl}) = \begin{pmatrix} 1 & -1 \\ u & \frac{p_{21,1}-p_{11,1}}{p_{12,1}-p_{22,1}}.u \end{pmatrix}$$

If $p_{11,1} + p_{22,1} \neq 2p_{12,1}$, then the matrix (t_{jl}) is non singular,i.e. $det(t_{ij}) \neq 0$, and it teaks the following form

$$(t_{jl}) = \begin{pmatrix} 1 & -1 \\ 1 & \frac{p_{21,1}-p_{11,1}}{p_{12,1}-p_{22,1}} \end{pmatrix}.$$

This completes the proof.

5. Derivations of genetic algebras in dimension two

Its natural to know the existence of nontrivial derivations of genetic algebras. In this section by means of the pervious section we are able to provide some conditions on structure constants of genetic algebra for that algebra exists a nontrivial derivation.

Theorem 5.1 *Let $\mathfrak{g}$ be a genetic algebra with basis $\{e_1, e_2\}$ of dimension two.*

(i) If $p_{21,1} = p_{12,1} \neq \frac{p_{11,1}+p_{22,1}}{2}$. Then any derivation is trivial,i.e. zero.

(ii) If $p_{11,1} = p_{12,1} = p_{22,1}$. Then there exists a nontrivial derivation.

Proof. Let $\mathfrak{g}$ be a genetic algebra. Now, we choose such a basis f_1, f_2 in $\mathfrak{g}$ such that $\mathfrak{g}$ becomes an evolution algebra. Due theorem 4.1, this occurs if $(i), (ii)$ are satisfied.

(i) Let $p_{12,1} = p_{21,1} \neq \frac{p_{11,1}+p_{22,1}}{2}$. Then we have the following transformation matrix

$$(t_{jl}) = \begin{pmatrix} 1 & -1 \\ 1 & \frac{p_{21,1}-p_{11,1}}{p_{12,1}-p_{22,1}} \end{pmatrix}$$

One can see that $f_1 = e_1 - e_2,\ f_2 = e_1 - \alpha e_2$ where $\alpha = \frac{p_{21,1}-p_{11,1}}{p_{12,1}-p_{22,1}}$. Then by a simple algebra we find

$$\begin{aligned} f_1^2 &= \left(\frac{\gamma\alpha}{1+\alpha} - \frac{\beta}{1+\alpha}\right) f_1 + \left(\frac{\gamma}{1+\alpha} + \frac{\beta}{1+\alpha}\right) f_2 \\ f_2^2 &= \left(\frac{x\alpha}{1+\alpha} - \frac{y}{1+\alpha}\right) f_1 + \left(\frac{x}{1+\alpha} + \frac{y}{1+\alpha}\right) f_2 \end{aligned}$$

Where $\gamma = p_{11,1} + p_{22,1} - 2p_{12,1}, \beta = p_{11,2} + p_{22,2} - 2p_{12,2}, x = p_{11,1} + 2\alpha p_{12,1} + \alpha^2 p_{22,1}$ and $y = p_{11,2} + 2\alpha p_{12,2} + \alpha^2 p_{22,2}$. So, we have the following structural matrix of evolution algebra

$$\mathbf{A} = \begin{pmatrix} \gamma\alpha - \beta & \gamma + \beta \\ x\alpha - y & x + y \end{pmatrix}$$

Suppose that $det(A) = 0$, then $det(A) = (\alpha + 1)(\gamma y - \beta x) = 0$. If $\alpha = -1$ then one can find $p_{12,1} = p_{21,1} = \frac{p_{11,1}+p_{22,1}}{2}$, which is a contradiction to our case. So, $\frac{\gamma}{\beta} = \frac{x}{y}$, i.e. $\frac{p_{11,1}+p_{22,1}-2p_{12,1}}{1-p_{11,1}+1-p_{22,1}-2+2p_{12,1}} = \frac{x}{y}$ therefore, $x = -y$ then

$$p_{11,1} + 2\alpha p_{12,1} + \alpha^2 p_{22,1} = -1 + p_{11,1} - 2\alpha(1 - p_{12,1}) - \alpha^2(1 - p_{22,1}),$$

this means that $(\alpha + 1)^2 = 0$, which is impossible. Hence, $det(A) \neq 0$. Then by theorem 2.1 any derivation is zero in this case.

(ii) Let $p_{11,1} = p_{12,1} = p_{22,1}$ then we have the following matrix

$$(t_{jl}) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore, one finds $f_1 = e_1 - e_2,\ f_2 = e_1$. Then by simple algebra we get

$$\begin{aligned} f_1^2 &= -\beta f_1 + (\gamma + \beta) f_2 \\ f_2^2 &= -p_{11,2} f_1 + f_2, \end{aligned}$$

where $\gamma = p_{11,1} + p_{22,1} - 2p_{12,1}, \beta = p_{11,2} + p_{22,2} - 2p_{12,2}$. So, we have the following structure matrix of evolution algebra:

$$\mathbf{A} = \begin{pmatrix} -\beta & \gamma + \beta \\ -p_{11,2} & 1 \end{pmatrix}$$

But $\gamma = p_{11,1} + p_{22,1} - 2p_{12,1} = 0 = -\beta$. Therefore, the $det(A) = 0$. Then according to Theorem 2.2, we have a nontrivial derivation. This completes the proof.

Corollary 5.2 *Let $\mathfrak{g}$ be a genetic algebra with basis $\{e_1, e_2\}$. Then the nontrivial derivations genetic algebra take the following form*

$$\overline{d} = \begin{pmatrix} 0 & d_{12} \\ -d_{12} & 0 \end{pmatrix}$$

Proof. By Theorem 2.2 we have the following nontrivial derivation evolution algebra

$$d = \begin{pmatrix} 0 & d_{12} \\ 0 & 0 \end{pmatrix}.$$

Since in the Theorem 5.1 we have $f_1 f_1 = 0$, then the nontrivial derivation evolution algebra

$$d = \begin{pmatrix} 0 & d_{12} \\ 0 & 0 \end{pmatrix}.$$

takes the following form

$$d_1 = \begin{pmatrix} 0 & 0 \\ d_{12} & 0 \end{pmatrix}.$$

and by using Theorem 5.1(ii) one has

$$\overline{d} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ d_{12} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & d_{12} \\ -d_{12} & 0 \end{pmatrix}$$

This completes the proof.

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