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Possible New Positronium Bound State

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Abstract. The Two-Body Dirac equations of constraint dynamics applied to QED yields an exact Sommerfeld-like solution for the spectrum of ${}^1J_J$ singlet positronium states which agrees with standard perturbative results through order α^4. At short distance the radial part of the wave function $u = r\psi$ has two solutions with probabilities near the origin behaving like $\psi^2 d^3r = u^2 drd\Omega = r^{(1\pm\sqrt{(2J+1)^2-4a^2})}drd\Omega$. For $J \neq 0$ only the first sign is allowable but both signs for $J = 0$ are well behaved. The first sign in that case corresponds to ordinary positronium (with a binding energy of about 6.8 eV). The second sign corresponds to a new positronium state with a binding energy of about 300 keV and a root-mean-square radius on the order of a Compton wavelength. The ordinary $1S$ positronium state decays into this new $1S$ state by two photon emission with c.m. energy of about 300 keV. The peculiar $1S$ state then annihilates promptly into two photons with c.m. energy of about 700 keV. Thus the existence of this new positronium state would be a distinctive 4 photon decay signature of ordinary singlet positronium.

In this paper we report on the prediction of a new positronium bound state with a very large binding energy of about 300 keV. It results from a metastable two-photon decay of the usual positronium bound state which has a binding energy of about 6.8 eV. It has a size of about the Compton wave length of an electron. Once it is formed it annihilates promptly into 2 photons with a c.m. energy of about 700 keV. Thus the existence of this new positronium bound state would be a distinctive 4 photon decay signature of the usual singlet bound state.

The prediction arises from an exact solution of the Two Body Dirac equations (TBDE) of constraint dynamics applied to QED solutions for singlet $S-$ states [1], [2]. In the references just quoted one can examine the details of the predictions. Here we present the highlights. One finds that the TBDE can be recast in a Schrödinger-like form reminiscent of that for the hydrogen atom. The radial form of that equation displays an invariant form of the Coulomb potential and its relativistic square,

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$$
\begin{aligned}
\left\{-\frac{d^2}{dr^2}+2\varepsilon_w\mathcal{A}(r)-\mathcal{A}^2(r)\right\}u &= b^2u,\\
\mathcal{A}(r) &= -\frac{\alpha}{r},\ \alpha\text{ -fine structure constant},\\
w &= c.m.\text{ total energy},\\
b^2 &= \varepsilon_w^2-m_w^2;\ \text{two-body relativistic eigenvalue},\\
\varepsilon_w &= (w^2-2m^2)/2w,\ \text{invariant energy of relative motion},\\
m_w &= m^2/w, \text{invariant reduced mass of relative motion }.(1)
\end{aligned}
$$

This equation has the short distance ($r << \alpha/2\varepsilon_w$) behavior

$$\left\{-\frac{d^2}{dr^2}-\frac{\alpha^2}{r^2}\right\}u=0, \tag{2}$$

with solutions

$$
\begin{aligned}
u_+ &\sim r^{\lambda_++1},\ \text{usual}\\
u_- &\sim r^{\lambda_-+1},\ \text{peculiar}\\
\lambda_\pm &= (-1\pm\sqrt{1-4\alpha^2})/2.
\end{aligned} \tag{3}
$$

With these behaviors, the probability for the relative location of the electron and positron is

$$\psi_\pm^2 d^3r=\frac{u_\pm^2}{r^2}r^2drd\Omega=u_\pm^2drd\Omega=r^{(1\pm\sqrt{1-4a^2})}drd\Omega. \tag{4}$$

This is clearly finite for both signs. Thus both behaviors are physically acceptable. If we have a nonzero angular momentum with $J\neq 0$ such that $J(J+1)-\alpha^2>0$ then the second (peculiar) solution would not be physically acceptable. The peculiar solution would also not be acceptable if the electron and positron were not point particles ($\mathcal{A}\neq-\alpha/r$). Thus, finding these peculiar states would strongly support that the electron be a point particle.

Treating the $-\alpha^2/r^2$ terms as a negative angular barrier, the solutions can be obtained analytically for both the usual and peculiar bound states ${}^1S_0$. The corresponding sets of eigenvalues for the total invariant c.m. energies $w_{\pm n}$ in terms of the principle quantum number n are

$$w_{\pm n}=m\sqrt{2+2/\sqrt{1+\alpha^2/(n\pm\sqrt{1/4-\alpha^2}-1/2)^2}}, \tag{5}$$

For the usual states, the bound state eigenvalues w_{+n} agree with standard QED perturbative results through order α^4,

$$w_{+n}=2m-m\alpha^2/4n^2-m\alpha^4/2n^3(1-11/32/n)+O(\alpha^6),\ n=1,2,3,... \tag{6}$$

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For the set of peculiar states, the ground state ($n = 1$) has mass

$$w_{-1} = m\sqrt{2 + 2/\sqrt{1 + \alpha^2/(1/2 - \sqrt{1/4 - \alpha^2})^2}} \sim \sqrt{2}m\sqrt{1 + \alpha}. \tag{7}$$

This represents a very tightly bound state, one with a binding energy on the order of 300 keV for an e^+e^- state and a root mean square radius on the order of a Compton wave length instead of an angstrom.

We now give a brief overview of the constraint Two Body Dirac Equations [3], [4], and how they lead to the above form given in Eq. (1) and the tentative new bound state. In the covariant constraint formalism for two spin-one-half particles interacting by through four-vector potentials there is a Dirac-like equation for each particle,

$$\begin{aligned} \mathcal{D}_1\psi &\equiv (\gamma_1 \cdot (p_1 - \tilde{A}_1) + m_1)\psi = 0 \\ \mathcal{D}_2\psi &\equiv (\gamma_2 \cdot (p_2 - \tilde{A}_2) + m_2)\psi = 0. \end{aligned} \tag{8}$$

The tildas symbolize the restrictions on the potentials that come from the mathematical requirement that the constraints be compatible[5],

$$[\mathcal{D}_1, \mathcal{D}_2]\psi = 0. \tag{9}$$

and the connection to quantum field theory by way of what has been called the quantum mechanical transform[6] of the Bethe-Salpeter equation [7]. This leads to the four vector potentials $\tilde{A}_1$, $\tilde{A}_2$ that depend on an invariant function $\mathcal{A}(x_\perp)$ and its gradient as well as energy, momentum, and spin dependence through the gamma matrices,

$$\tilde{A}_i = \tilde{A}_i(\mathcal{A}(x_\perp), \partial\mathcal{A}(x_\perp), w, p_1, p_2, \gamma_1, \gamma_2). \tag{10}$$

The invariant $\mathcal{A}$ is a function of the separation four-vector,

$$\begin{aligned} x_\perp^\mu &= (x_1 - x_2)_\nu(\eta^{\mu\nu} - P^\mu P^\nu/P^2), \\ P &= p_1 + p_2, \\ P \cdot x_\perp &= 0. \end{aligned} \tag{11}$$

perpendicular to the total four-momentum. For QED in lowest order,

$$\begin{aligned} \mathcal{A}(x_\perp) &= -\frac{\alpha}{r}, \\ r &\equiv \sqrt{x_\perp^2},\ x_\perp = (0, \mathbf{r}) \text{ in c.m. } (P = (w, \mathbf{0})). \end{aligned} \tag{12}$$

The relative momentum is space-like

$$P \cdot p\psi = 0, \tag{13}$$

so that combined with Eq. (12) we see that the compatibility condition leads to a covariant three dimensional formalism. For equal masses $p = (p_1 - p_2)/2$.

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The wave function that satisfies the simultaneous equations (8) has 16 components given in terms of 4 four component spinors

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4)\text{- sixteen component spinor.} \tag{14}$$

In the c.m. frame $\hat{P} = (1, \mathbf{0})$ and $\hat{r} = (0, \hat{\mathbf{r}})$. We defined [1] the 4 component wave functions $\psi_\pm, \eta_\pm$

$$\begin{aligned} \psi_\pm &= \exp(-\mathcal{F}(r) - \mathcal{K}(r)\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})(\psi_1 \pm \psi_4) \\ \eta_\pm &= \exp(-\mathcal{F}(r) - \mathcal{K}(r)\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}})\,(\psi_2 \pm \psi_3) \end{aligned} \tag{15}$$

and performed a Pauli reduction to 4 - uncoupled four-component Schrödinger-like equations. For ψ_+ we found

$$\begin{aligned} & \mathcal{B}^2\psi_+ \equiv \{\mathbf{p}^2 + \Phi(\mathbf{r}, m_1, m_2, w, \sigma_1, \sigma_2)\}\psi_+ \\ = \; & \{\mathbf{p}^2 + 2\varepsilon_w\mathcal{A} - \mathcal{A}^2 + \Phi_D(\mathcal{A}, \nabla\mathcal{A}, \nabla^2\mathcal{A}) + \mathbf{L}\cdot(\sigma_1+\sigma_2)\Phi_{SO}(...) \\ & +\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}}\mathbf{L}\cdot(\sigma_1+\sigma_2)\Phi_{SOT}(...) + \sigma_1\cdot\sigma_2\Phi_{SS}(..) + (3\sigma_1\cdot\hat{\mathbf{r}}\sigma_2\cdot\hat{\mathbf{r}} - \sigma_1\cdot\sigma_2)\Phi_T(..) \\ & +\mathbf{L}\cdot(\sigma_1-\sigma_2)\boldsymbol{\Phi}_{SOD}(..) + i\mathbf{L}\cdot\sigma_1\times\sigma_2\Phi_{SOX}(..)\}\psi_+ \\ = \; & b^2\psi_+ \end{aligned} \tag{16}$$

The Darwin and various spin-dependent terms are dependent on the invariant $\mathcal{A}$ and its derivatives. The attractive spin-spin and repulsive Darwin quasipotentials are quite strong (individually they overwhelm $-\mathcal{A}^2$) but for equal mass spin singlet states we have their exact cancellation $\Phi_D + \sigma_1\cdot\sigma_2\Phi_{SS} = 0$, leading to the bound state radial equation given in Eq. (1)

One finds that the lowest lying usual and peculiar states are not orthogonal with respect to one another. To see this we write their respective radial forms

$$\begin{aligned} u_+(r) &= c_+ r^{\lambda_+ + 1}\exp(-\kappa_+\varepsilon_{w_+}\alpha r), \\ \kappa_+ &= \frac{2}{1+\sqrt{1-4\alpha^2}} = \frac{1}{\lambda_+ + 1}, \\ u_-(r) &= c_- r^{\lambda_- + 1}\exp(-\kappa_-\varepsilon_{w_-}\alpha r), \\ \kappa_- &= \frac{2}{1-\sqrt{1-4\alpha^2}} = \frac{1}{\lambda_- + 1}, \end{aligned} \tag{17}$$

Clearly since they are both zero node solutions we have

$$\langle u_-|u_+\rangle = \int_0^\infty dr u_+(r)u_-(r) \sim (\alpha^{3/2} \sim 1/1000) \neq 0. \tag{18}$$

How do we reconcile this with the expected orthogonality of the eigenfunctions of a self-adjoint operator corresponding to different eigenvalues? One can show that the second derivative is not self-adjoint in this context! But we have that for the quasipotential of the type $-\alpha^2/r^2$ at short distances, both the set of usual states and the peculiar states are physically admissible states. There does not appear to be reasons to exclude one set as being unphysical, given the attractive interaction as it is near the origin.

Only for interactions with sufficient attraction at the origin (so that $-1/4 \le \lambda(\lambda+1) < 0$) can the peculiar states be pulled into existence and appear as eigenstates in the physically acceptable sheet, with regular non-singular radial wave functions at the origin. It is desirable to find ways to admit both types of physical states into a larger Hilbert space to accommodate both sets of states with the mass operator $\mathcal{B}^2$ to be self-adjoint and the states to be part of a complete set. It is reasonable to assign a quantum number which we call "peculiarity" for a states emerging into the physical sheet in this way as physically acceptable states. The introduction of the peculiarity quantum number enlarges the Hilbert space, allows the mass operator $\mathcal{B}^2$ to be self-adjoint, and the set of physically allowed states become a complete set.

We introduce a new peculiarity observable $\hat{\zeta}$ with the quantum number peculiarity ζ such that

$$\begin{aligned} \hat{\zeta}\chi_+ &= \zeta\chi_+ \ \text{ with eigenvalue } \zeta = +1, \\ \hat{\zeta}\chi_- &= \zeta\chi_- \ \text{ with eigenvalue } \zeta = -1, \end{aligned} \tag{19}$$

with the corresponding spinor wave function χ_ζ assigned to the states so that a usual state is represented by the peculiarity spinor χ_+,

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{20}$$

and a peculiar state is represented by the peculiarity spinor χ_-,

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{21}$$

With this introduction, a general wave function can be expanded in terms of the complete set of basis functions $\{u_{+n}, u_{-n}\}$ as

$$\Psi = \sum_{\zeta n} a_{\zeta n} u_{\zeta n} \chi_\zeta, \tag{22}$$

where n represent all the spin and spatial quantum numbers of the state and ζ the peculiarity quantum number. The variational principle applied to

$$\langle \mathcal{B}^2 \rangle = \frac{\langle \Psi | \mathcal{B}^2 | \Psi \rangle}{\langle \Psi | \Psi \rangle}, \tag{23}$$

would lead to

$$\begin{aligned} \mathcal{B}^2 u_{+n}\chi_+ &= -\kappa_{+n}^2 u_{+n}\chi_+, \\ \mathcal{B}^2 u_{-n}\chi_- &= -\kappa_{-n}^2 u_{-n}\chi_-. \end{aligned} \tag{24}$$

It is clear that in this context the usual and peculiar wave functions are orthogonal, $\mathcal{B}^2$ is self-adjoint, and the basis states are complete. We see that the introduction of the peculiarity quantum number resolves the problem of over-completeness property of the basis states and the non-self-adjoint property of the mass operator.

The usual 1^1S_0 ground state of positronium (designated by $1S_u$) turns into the peculiar 1^1S_0 ground state (designated by $1S_p$) by emitting two photons similar to the metastable mechanism by which the $2S$ state of hydrogen decays into the $1S$ state. The Golden Rule in this case takes the form

$$d^3w = 2\pi|T_{fi}|^2 d^3k_1 d^3k_2 d^3p_{1S_p}\delta(E_{1S_u} - E_{1S_p} - \hbar\omega_1 - \hbar\omega_2)\delta(\mathbf{0} - \mathbf{p}_{1S_p} - \mathbf{k}_1 - \mathbf{k}_2), \quad (25)$$

for the emission of photons characterized by $(\mathbf{k}_1, \alpha_1)$ and $(\mathbf{k}_2, \alpha_2)$. In [2] is described in detail the metastable decay mechanism for the usual ground state into the the peculiar state. Our assumption above is that the peculiar and usual states by virtue of the introduction of the operator $\hat{\zeta}$ are orthogonal. As such it would be impossible for the $1S_u$ state to decay into the $1S_p$ state. In that case the usual state would undergo just two photon annihilation. We allow for the possibility that the peculiarity quantum number is not conserved for the full hamiltonian so that there is a nonzero admixture between the two sectors. Calling the admixture amplitude $M_{\zeta\zeta'}$ we find that the branching ratio between the metastable and annihilation channels is $0.075\left|M_{\zeta\zeta'}\right|^2$. We further find that once the peculiar state is formed, it will annihilate into two photons with a c.m. energy of 700 keV with a lifetime on the order of 10^{-21} sec.

To summarize, the TBDE of constraint dynamics applied to QED give a covariant bound state formalism reproducing by nonperturbative treatment the correct spectra through order α^4. Applied to the $^1S_0$ states, new peculiar solutions are uncovered for point e^-e^+ $(\psi_- \to r^{(-1-\sqrt{1-4\alpha^2})/2})$ in addition to the usual solutions $(\psi_+ \to r^{(1-\sqrt{1-4\alpha^2})/2})$. The peculiar bound states, to preserve self adjointness, are distinguished from the usual ones by a new quantum number called peculiarity. The four photon decay signature (with one set of two photons bunching in energy around 300 keV and one set around 700 keV) would strongly indicate the point-like nature of the electron.

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