Nusselt number correlations for simultaneously developing laminar duct flows of liquids with temperature dependent properties

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Abstract. New correlations, suitable for engineering applications, for the mean Nusselt number in the entrance region of circular tubes and square ducts with uniform heat flux boundary conditions specified at the walls are proposed. These correlations are obtained on the basis of the results of a previous parametric investigation on the effects of temperature dependent viscosity and thermal conductivity in simultaneously developing laminar flows of liquids in straight ducts of constant cross-sections. In these studies, a finite element procedure has been employed for the numerical solution of the parabolized momentum and energy equations. Viscosity and thermal conductivity are assumed to vary with temperature according to an exponential and to a linear relation, respectively, while the other fluid properties are held constant. Axial distributions of the mean Nusselt number, obtained by numerical integration from those of the local Nusselt number, are used as input data in the derivation of the proposed correlations. A superposition method is proved to be applicable in order to estimate the Nusselt number by considering separately the effects of temperature dependent viscosity and thermal conductivity. Therefore, for each of the considered cross-sectional geometries, two distinct correlations are proposed for flows of liquids with temperature dependent viscosity and with temperature dependent thermal conductivity, in addition to that obtained for constant property flows.

1. Introduction

In many laminar duct flows, entrance effects on fluid flow and heat transfer must be taken into account, since the total length of the duct is comparable with that of the entrance region. Moreover, the development of the velocity and temperature fields can be strongly affected by temperature dependence of fluid properties. If the fluid is a liquid, the relative variations of viscosity with temperature are the most relevant, while those of thermal conductivity are, in general, much smaller, and those of density and specific heat capacity are almost negligible [1, 2]. As far as velocity distribution and pressure drop are concerned, the main effects of temperature dependent fluid properties can be retained even if only viscosity is considered to vary with temperature, while the other properties are assumed constant [1, 3]. Instead, if heat transfer characteristics, as the Nusselt number, have to be evaluated, also the temperature dependence of thermal conductivity must be taken into account [2].

In previous articles, we carried out systematic analyses of the effects of temperature dependent viscosity and thermal conductivity on forced convection in simultaneously developing laminar flows of liquids in straight ducts with uniform wall heat flux boundary conditions [4, 5]. In particular, we obtained axial distributions of the local value of the peripherally averaged Nusselt number for ducts of different cross-sections. We assumed that viscosity decreases with increasing temperature according to an exponential relation, while thermal conductivity varies linearly with temperature. Suitable dimensionless parameters, namely, the viscosity and thermal conductivity Pearson numbers, were used to quantitatively express temperature dependence of the corresponding properties. A finite element procedure was employed for the step-by-step solution of the parabolised momentum and energy equations in a domain corresponding to the cross-section of the duct [6]. Finally, a superposition method was proved to be applicable in order to obtain accurate estimates of the local Nusselt number by considering separately the effects of temperature dependent viscosity and thermal conductivity [4, 5].

In this paper, first we use previous numerical results for the local Nusselt number for circular and square cross-sectional geometries [5] to obtain by numerical integration the corresponding axial distributions of the mean, i.e., longitudinally averaged, Nusselt number. Then, on the basis of such distributions, we obtain very accurate correlations, suitable for engineering applications. Actually, taking advantage of the above mentioned superposition method, we propose for each of the considered cross-sectional geometries a correlation valid for constant property flows and two distinct correlations for flows of liquids with temperature dependent viscosity and with temperature dependent thermal conductivity. Finally, we demonstrate that, by properly assembling these correlations, accurate estimates of the mean Nusselt number can be obtained in wide ranges of operative conditions. No correlations are presented here for pressure drop parameters since, as demonstrated in previous works [4, 5], the influence of thermal conductivity variation on pressure drop is almost negligible if compared with that of viscosity variations. Therefore, the correlations for pressure drop parameters already proposed by the authors for liquids with temperature dependent viscosity and constant thermal conductivity [7] can be used for all liquids with temperature dependent properties.

2. Statement of the problem

The laminar forced convection in the entrance region of straight ducts of constant cross-sections with uniform wall heat flux $q''_w > 0$ (fluid heating) is studied. Since the liquid heating is assumed to begin at the duct inlet, uniform values of the axial velocity component u and of the temperature t (i.e., $t = t_e$, $u = u_e = \overline{u}$ and $v = w = 0$) are specified as the appropriate inlet conditions, being $\overline{u}$ the average axial velocity and v and w the transverse velocity components. The heat flux (Neumann) boundary condition $q'' = k\, \partial t/\partial n = q''_w$ is applied at the solid walls, where k is the thermal conductivity and n is the outward normal to the boundary.

Since the fluids considered here are liquids, the dynamic viscosity μ is assumed to decrease with increasing temperature according to the widely used exponential formula [3]

$$\mu = \mu_e\ \exp[-\beta\ (t - t_e)] \tag{1}$$

where μ_e is the value of μ at t_e and $\beta = -(\mathrm{d}\mu/\mathrm{d}t)/\mu = \text{const}$ is positive. The thermal conductivity k can be assumed both to increase or decrease with increasing temperature, depending on the fluid considered, according to the linear relation

$$k = k_e\ [1 + \alpha\ (t - t_e)] \tag{2}$$

where k_e is the value of k at t_e and $\alpha = (\mathrm{d}k/\mathrm{d}t)/k_e = \text{const}$ can be both positive or negative. By means of simple manipulations, equations (1) and (2) can be cast in the following dimensionless forms

$$\frac{\mu}{\mu_e} = \exp(-\mathrm{Pn}_\mu\, T) \tag{3}$$

$$\frac{k}{k_e} = 1 + \mathrm{Pn}_k\, T \tag{4}$$

where $T = (t - t_e)\, k_e/(q''_w\, D_h)$ is the dimensionless temperature, D_h is the hydraulic diameter, $\mathrm{Pn}_\mu = \beta\, q''_w\, D_h/k_e$ is the viscosity Pearson number (representing the ratio of the characteristic process temperature difference $q''_w\, D_h/k_e$ to the characteristic temperature difference $1/\beta$ that can produce appreciable viscosity variations) and $\mathrm{Pn}_k = \alpha\, q''_w\, D_h/k_e$ is the thermal conductivity Pearson number (representing the ratio of the characteristic process temperature difference $q''_w\, D_h/k_e$ to the characteristic temperature difference $1/\alpha$ that can produce appreciable thermal conductivity variations).

It is worth noting that, since the density ρ and the specific heat c are constant, the local Reynolds number $\mathrm{Re} = \rho\, \overline{u}\, D_h/\mu$, the local Prandtl number $\mathrm{Pr} = \mu\, c/k$ and the local Péclet number $\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr} = \rho\, c\, \overline{u}\, D_h/k$ all vary with temperature because of the variations of μ, of the ratio μ/k and of k, respectively. Therefore we have $\mu/\mu_e = \mathrm{Re}_e/\mathrm{Re}$ and $k/k_e = \mathrm{Pe}_e/\mathrm{Pe}$, where Re_e and Pe_e are the Reynolds and Péclet numbers evaluated at t_e. Moreover, since the viscosity of liquids decreases with increasing temperature ($\beta > 0$), in the case of fluid heating we have $\mathrm{Pn}_\mu > 0$ and $\mathrm{Re}_e/\mathrm{Re} < 1$, while $\mathrm{Pn}_\mu = 0$ and $\mathrm{Re}_e/\mathrm{Re} = 1$ refer to constant viscosity fluids ($\beta = 0$). Instead, since the thermal conductivity can either increase ($\alpha > 0$) or decrease ($\alpha < 0$) with increasing temperature, we have $\mathrm{Pn}_k > 0$ and $\mathrm{Pe}_e/\mathrm{Pe} > 1$ in the first case and $\mathrm{Pn}_k < 0$ and $\mathrm{Pe}_e/\mathrm{Pe} < 1$ in the second one, while $\mathrm{Pn}_k = 0$ and $\mathrm{Pe}_e/\mathrm{Pe} = 1$ refer to constant thermal conductivity fluids ($\alpha = 0$).

As already pointed out, as the fluid temperature rises along the duct, the viscosity decreases while Re increases. Therefore, to ensure laminar flow conditions, the local Reynolds number Re_b evaluated at the bulk temperature is only allowed to reach the maximum value $(\mathrm{Re}_b)_{max} = 2,000$, corresponding to the maximum value x_{max} of the axial coordinate x, whereupon computations are stopped. Therefore, taking into account equation (3) and the appropriate heat balance for the duct, the following expression for the maximum value X^*_{max} of the dimensionless axial coordinate $X^* = x/(D_h\, \mathrm{Pe}_e)$ can be obtained [4, 5, 7]

$$X^*_{max} = \frac{1}{4\,\mathrm{Pn}_\mu} \ln\left[\frac{(\mathrm{Re}_b)_{max}}{\mathrm{Re}_e}\right] \tag{5}$$

Numerical results of interest for the present study concern the axial distributions of the peripherally averaged local Nusselt number $\mathrm{Nu} = \overline{h} D_h/k_e$, where $\overline{h}$ is the peripherally averaged local convective heat transfer coefficient defined as

$$\overline{h} = \frac{q''_w}{\overline{t}_w - t_b} = \frac{k_e}{D_h\,(\overline{T}_w - T_b)} \tag{6}$$

In equation (6), $\overline{t}_w$ and t_b are the peripherally averaged wall temperature and the mean bulk temperature, respectively, and $\overline{T}_w$ and T_b are their dimensionless forms. According to equation (6), the peripherally averaged local Nusselt number can be expressed as

$$\mathrm{Nu} = \frac{1}{\overline{T}_w - T_b} \tag{7}$$

Axial distributions of the longitudinally averaged Nusselt number, defined as [1]

$$\overline{\mathrm{Nu}} = \frac{1}{x}\int_0^x \mathrm{Nu}\,\mathrm{d}x = \frac{1}{X^*}\int_0^{X^*} \mathrm{Nu}\,\mathrm{d}X^* \tag{8}$$

can be obtained by means of an appropriate numerical integration rule.

3. Numerical procedure

If the effects of axial diffusion can be neglected ($\mathrm{Re}_e > 50$ and $\mathrm{Pe}_e > 50$) and there is no recirculation in the longitudinal direction, steady-state flow and heat transfer in straight ducts of constant cross-sections are governed by the continuity and the parabolized Navier-Stokes and energy equations [8, 9]. These equations are not reported here for lack of space, but they are reported elsewere together with the boundary conditions specified on the boundaries of the computational domain [5, 7].

A finite element procedure for the analysis of the forced convection of fluids with temperature dependent properties in the entrance region of straight ducts [6, 7, 10] is used to solve the model equations. The adopted procedure is based on a segregated approach which implies the sequential solution of the momentum and energy equations on a two-dimensional domain in the case of three-dimensional geometries and on a one-dimensional domain in axisymmetric problems, corresponding to the cross-section of the duct. A marching method is then used to move forward in the axial direction. The pressure-velocity coupling is dealt with using an improved projection algorithm already employed by one of the authors (C.N.) for the solution of the Navier-Stokes equations in their elliptic form [11]. The procedure has already been validated, with reference to both constant and temperature dependent property fluids, by comparing heat transfer and pressure drop results with existing literature data for simultaneously developing laminar flows in straight ducts [6, 10, 12, 13].

4. Numerical results

In this study, two cross-sectional geometries are considered, namely circular and square, chosen as representative of axisymmetric and three-dimensional duct geometries, respectively. The corresponding computational domains, defined taking into account existing symmetries, are one-dimensional and two-dimensional for the circular and the square cross-sections, respectively. Details of finite element domain and time discretisations are reported elsewere [4, 5].

The same value of the Reynolds number $\mathrm{Re}_e = 100$ is assumed in all the computations. Instead, the values $\mathrm{Pr}_e = 5, 20$ and 100 of the Prandtl number at t_e are selected to take into account the behaviours of different liquids. The corresponding values of the Péclet number at t_e are $\mathrm{Pe}_e = 500, 2000$ and 10 000. The values of the dynamic viscosity Pearson number $\mathrm{Pn}_\mu = 0, 1, 2$ and 4 are considered to account for reasonable viscosity temperature dependences. Thus, for the assumed Re_e, the maximum values of the dimensionless axial coordinate given by equation (5) are $X^*_{max} = 0.7489, 0.3745$ and 0.1872 for $\mathrm{Pn}_\mu = 1, 2$ and 4, respectively. For each nonzero value of Pn_μ eight values of Pn_k are selected (four positive and four negative), giving the corresponding values of the ratio $\mathrm{Pn}_\mu/\mathrm{Pn}_k = \beta/\alpha = \pm 10, \pm 20, \pm 40$ and ± 80. Thus, on the whole, the values $\mathrm{Pn}_k = 0, \pm 0.0125, \pm 0.025, \pm 0.05, \pm 0.1, \pm 0.2$ and ± 0.4 are considered.

The effects of temperature dependent properties (viscosity and thermal conductivity) on heat transfer can be illustrated by comparing the local Nusselt number $\mathrm{Nu}_{\mu k}$, obtained for given nonzero values of Pn_μ and Pn_k, with the corresponding local Nusselt number Nu_c, computed for simultaneously developing constant property flows ($\mathrm{Pn}_\mu = \mathrm{Pn}_k = 0$). Therefore, we can assume the value of the ratio $(\mathrm{Nu}_{\mu k} - \mathrm{Nu}_c)/\mathrm{Nu}_c = (\mathrm{Nu}_{\mu k}/\mathrm{Nu}_c) - 1$ as a measure of the effects of both temperature dependent viscosity and thermal conductivity. Axial distributions of the ratio $\mathrm{Nu}_{\mu k}/\mathrm{Nu}_c$ with reference to the dimensionless coordinate X^*, obtained by means of the numerical procedure described in Section 3 for the same cross-sectional geometries and values of dimensionless input parameters considered in this paper, are reported elsewere [4, 5]. The conclusion reached there is that both temperature dependent viscosity and thermal conductivity can have comparable effects on the Nusselt number, according to the values of Pn_μ and Pn_k. Moreover, it has been demonstrated that if the effects of temperature dependent viscosity and thermal conductivity are considered separately to compute Nu_μ (under the assumptions of temperature dependent viscosity and constant thermal conductivity, i.e., $\mathrm{Pn}_\mu > 0$ and $\mathrm{Pn}_k = 0$)

and Nu_k (under the assumptions of temperature dependent thermal conductivity and constant viscosity, i.e., $\mathrm{Pn}_k \neq 0$ and $\mathrm{Pn}_\mu = 0$), a superposition method is applicable in order to obtain approximate values of the Nusselt number $\mathrm{Nu}_{\mu k}$, according to the relation

$$\mathrm{Nu}_{\mu k} - \mathrm{Nu}_c \cong (\mathrm{Nu}_\mu - \mathrm{Nu}_c) + (\mathrm{Nu}_k - \mathrm{Nu}_c) \tag{9}$$

with an accuracy which can be considered satisfactory in most situations. With reference to the ratios $\mathrm{Nu}_{\mu k}/\mathrm{Nu}_c$, $\mathrm{Nu}_\mu/\mathrm{Nu}_c$ and $\mathrm{Nu}_k/\mathrm{Nu}_c$, equation (9) can be recast in the form [4, 5]

$$\frac{\mathrm{Nu}_{\mu k}}{\mathrm{Nu}_c} \cong \left(\frac{\mathrm{Nu}_{\mu k}}{\mathrm{Nu}_c}\right)' = \frac{\mathrm{Nu}_\mu}{\mathrm{Nu}_c} + \frac{\mathrm{Nu}_k}{\mathrm{Nu}_c} - 1 \tag{10}$$

where $(\mathrm{Nu}_{\mu k}/\mathrm{Nu}_c)'$ is the approximate value of $\mathrm{Nu}_{\mu k}/\mathrm{Nu}_c$ given by the superposition method. Therefore, the values of the differences $(\mathrm{Nu}_\mu/\mathrm{Nu}_c)-1$ and $(\mathrm{Nu}_k/\mathrm{Nu}_c)-1$ approximately measure the separate effects of temperature dependent viscosity and thermal conductivity, respectively.

In this work, previous numerical results concerning axial distributions of the ratios $\mathrm{Nu}_{\mu k}/\mathrm{Nu}_c$, $\mathrm{Nu}_\mu/\mathrm{Nu}_c$ and $\mathrm{Nu}_k/\mathrm{Nu}_c$ [5] have been used to obtain, by means of a suitable numerical integration rule according to equation (8), the corresponding distributions of the ratios $\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c$, $\overline{\mathrm{Nu}}_\mu/\overline{\mathrm{Nu}}_c$ and $\overline{\mathrm{Nu}}_k/\overline{\mathrm{Nu}}_c$. Then, it has been verified that the following relation, obtained by combining equations (8) and (9), still holds true with an accuracy which can be considered satisfactory in most situations

$$\frac{\overline{\mathrm{Nu}}_{\mu k}}{\overline{\mathrm{Nu}}_c} \cong \left(\frac{\overline{\mathrm{Nu}}_{\mu k}}{\overline{\mathrm{Nu}}_c}\right)' = \frac{\overline{\mathrm{Nu}}_\mu}{\overline{\mathrm{Nu}}_c} + \frac{\overline{\mathrm{Nu}}_k}{\overline{\mathrm{Nu}}_c} - 1 \tag{11}$$

In equation (11), $(\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c)'$ represents the approximate value of $\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c$ given by the superposition method. Axial distributions of the ratio $(\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c)'$ for the highest values of Pn_μ, i.e., $\mathrm{Pn}_\mu = 2$, and $\mathrm{Pn}_\mu = 4$, different values of Pn_k and $\mathrm{Pr}_e = 20$ are reported in figures 1 and 2 for circular tubes and square ducts, respectively, to allow the comparison with the corresponding distributions of $\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c$. As can be seen, even for the highest values of Pn_μ and the highest/lowest values of Pn_k the dashed curves, representing axial distributions of the ratio $(\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c)'$, are very close to the solid ones giving the numerical solutions for $\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c$,

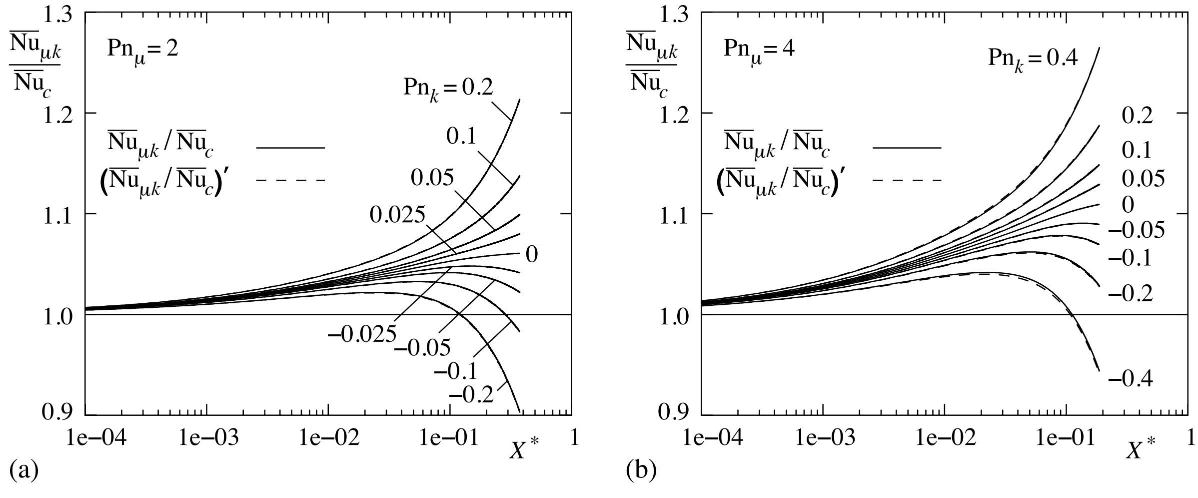

Figure 1. Comparison of axial distributions of the ratios $\overline{\mathrm{Nu}}_{\mu,k}/\overline{\mathrm{Nu}}_c$ and $\left(\overline{\mathrm{Nu}}_{\mu,k}/\overline{\mathrm{Nu}}_c\right)'$ for simultaneously developing laminar flows in circular tubes with $\mathrm{Pr}_e = 20$, different values of Pn_k and: (a) $\mathrm{Pn}_\mu = 2$ and (b) $\mathrm{Pn}_\mu = 4$.

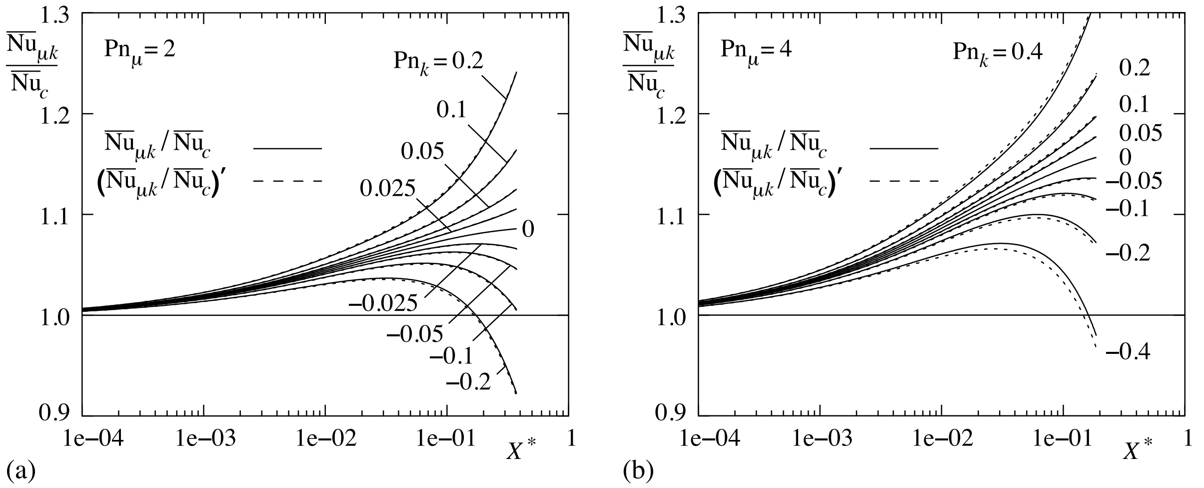

Figure 2. Comparison of axial distributions of the ratios $\overline{\mathrm{Nu}}_{\mu,k}/\overline{\mathrm{Nu}}_c$ and $\left(\overline{\mathrm{Nu}}_{\mu,k}/\overline{\mathrm{Nu}}_c\right)'$ for simultaneously developing laminar flows in square ducts with $\mathrm{Pr}_e = 20$, different values of Pn_k and: (a) $\mathrm{Pn}_\mu = 2$ and (b) $\mathrm{Pn}_\mu = 4$.

thus confirming the validity of equation (11). Moreover, the maximum positive and negative relative errors in the approximation of $\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c$ by means of $(\overline{\mathrm{Nu}}_{\mu k}/\overline{\mathrm{Nu}}_c)'$ are $\varepsilon^+_{max} = 0.15\%$ and $\varepsilon^-_{max} = -0.28\%$ for circular ducts and $\varepsilon^+_{max} = 0.42\%$ and $\varepsilon^-_{max} = -1.39\%$ for square ducts in the range $10^{-4} \leq X^* \leq X^*_{max}$ with $\mathrm{Pr}_e = 5$, 20 and 100 and different values of Pn_μ and Pn_k. Therefore, we can conclude that the agreement between computed and approximate results is very good.

5. Correlations for the mean Nusselt number

Taking advantage of equation (11), a correlation for $\overline{\mathrm{Nu}}_{\mu k}$ has been obtained by assembling the different correlations for $\overline{\mathrm{Nu}}_c$, $\overline{\mathrm{Nu}}_\mu/\overline{\mathrm{Nu}}_c$ and $\overline{\mathrm{Nu}}_k/\overline{\mathrm{Nu}}_c$ obtained from computed results for flows of liquids with constant property ($\mathrm{Pn}_k = \mathrm{Pn}_\mu = 0$), temperature dependent viscosity ($\mathrm{Pn}_k = 0$ and $\mathrm{Pn}_\mu > 0$) and temperature dependent thermal conductivity ($\mathrm{Pn}_\mu = 0$ and $\mathrm{Pn}_k \neq 0$), respectively.

5.1. Correlations for the mean Nusselt number for flows of liquids with constant properties

The results obtained for circular and square cross-sectional geometries under the assumption of constant property flow ($\mathrm{Pn}_k = \mathrm{Pn}_\mu = 0$) have been used to obtain correlations for the mean Nusselt number $\overline{\mathrm{Nu}}_c$ as a function of X^* in the form

$$\overline{\mathrm{Nu}}_c = (\mathrm{Nu}_c)_{fd} + \frac{a\,(X^*)^{-m}}{1 + b\,(X^*)^{-n}} \tag{12}$$

where $(\mathrm{Nu}_c)_{fd}$, i.e., the asymptotic value of Nu_c for fully developed conditions, the coefficients a and b and the exponent m only depend on the cross-sectional geometry, while the exponent n also depends on the Prandtl number Pr_e according to the relation

$$n = n_1\,\mathrm{Pr}_e^s - n_2\,\mathrm{Pr}_e \tag{13}$$

whose coefficients n_1 and n_2 and the exponent s only depend on the cross-sectional geometry. The appropriate forms of equations (12) and (13) for circular and square cross-sections, valid

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Table 1. Correlations for $\overline{\mathrm{Nu}}_c$ corresponding to equations (12) and (13) for circular and square cross-sections.

Cross-section	Correlations for $\overline{\mathrm{Nu}}_c$
circular	$\overline{\mathrm{Nu}}_c = 4.3636 + \dfrac{0.065\,(X^*)^{-1.3}}{1 + 0.10\,(X^*)^{-n}}$ $n = 0.761\,\mathrm{Pr}_e^{0.0224} - 0.000109\,\mathrm{Pr}_e$
square	$\overline{\mathrm{Nu}}_c = 3.0874 + \dfrac{0.065\,(X^*)^{-1.3}}{1 + 0.11\,(X^*)^{-n}}$ $n = 0.755\,\mathrm{Pr}_e^{0.0255} - 0.000136\,\mathrm{Pr}_e$

for $5 \le \mathrm{Pr}_e \le 100$ and $10^{-4} \le X^* \le X^*_{max}$, are reported in table 1. Since, for $\mathrm{Pr}_e = 5$, 20 and 100, the maximum positive and negative relative errors in the approximation of $\overline{\mathrm{Nu}}_c$ by means of equation (12) are $\varepsilon^+_{max} = 0.80\%$ and $\varepsilon^-_{max} = -1.29\%$ for circular ducts and $\varepsilon^+_{max} = 2.20\%$ and $\varepsilon^-_{max} = -1.31\%$ for square ducts, we can conclude that the agreement between computed and approximate results is very good.

5.2. Correlations for the mean Nusselt number for flows of liquids with temperature dependent viscosity

The results obtained for circular and square cross-sectional geometries for $\mathrm{Pn}_k = 0$ and $\mathrm{Pn}_\mu > 0$ have been used to obtain correlations for the ratio $\overline{\mathrm{Nu}}_\mu/\overline{\mathrm{Nu}}_c$ as a function of X^* in the form

$$\frac{\overline{\mathrm{Nu}}_\mu}{\overline{\mathrm{Nu}}_c} = 1 + \left[\left(\frac{\mathrm{Nu}_\mu}{\mathrm{Nu}_c}\right)_{fd} - 1\right]\{1 - \exp\left[-a\,(X^*)^m - b\,(X^*)^n\right]\} \tag{14}$$

In equation (14), $(\mathrm{Nu}_\mu/\mathrm{Nu}_c)_{fd}$ is the asymptotic value for fully developed conditions of the ratio $\mathrm{Nu}_\mu/\mathrm{Nu}_c$, which can be evaluated by means of the very accurate formula

$$\left(\frac{\mathrm{Nu}_\mu}{\mathrm{Nu}_c}\right)_{fd} = 1 + d_1\,\mathrm{Pn}_\mu + d_2\,\mathrm{Pn}_\mu^2 \tag{15}$$

whose coefficients d_1 and d_2 depend on the cross-sectional geometry. Besides, still with reference to equation (14), the coefficient b and the exponent n only depend on the cross-sectional geometry, while the coefficient a and the exponent m also depend on the viscosity Pearson number Pn_μ and on the Prandtl number Pr_e, respectively, and can be expressed as

$$a = a_1 + a_2\,\mathrm{Pn}_\mu \tag{16}$$

$$m = m_1\,\mathrm{Pr}_e^{-s} \tag{17}$$

where the coefficients a_1, a_2 and m_1 and the exponent s only depend on the cross-sectional geometry. For engineering application, the appropriate forms of equations (14) to (17) for circular and square cross-sections, valid for $5 \le \mathrm{Pr}_e \le 100$, $1 \le \mathrm{Pn}_\mu \le 4$ and $10^{-4} \le X^* \le X^*_{max}$, are reported in table 2. For $\mathrm{Pr}_e = 5$, 20 and 100, the maximum positive and negative relative errors in the approximation of $\overline{\mathrm{Nu}}_\mu/\overline{\mathrm{Nu}}_c$ by means of equation (14) are $\varepsilon^+_{max} = 0.14\%$ and $\varepsilon^-_{max} = -0.21\%$ for circular ducts and $\varepsilon^+_{max} = 0.31\%$ and $\varepsilon^-_{max} = -0.28\%$ for square ducts. Therefore, we can conclude that the agreement between computed and approximate results is very good.

Table 2. Correlations for $\overline{\mathrm{Nu}}_\mu/\overline{\mathrm{Nu}}_c$ corresponding to equations (14) to (17) for circular and square cross-sections.

Cross-section	Correlations for $\overline{\mathrm{Nu}}_\mu/\overline{\mathrm{Nu}}_c$
circular	$\dfrac{\overline{\mathrm{Nu}}_\mu}{\overline{\mathrm{Nu}}_c} = 1 + \left[\left(\dfrac{\mathrm{Nu}_\mu}{\mathrm{Nu}_c}\right)_{fd} - 1\right]\left\{1 - \exp\left[-a\,(X^*)^m - 2.0\,(X^*)^{0.6}\right]\right\}$ $\left(\dfrac{\mathrm{Nu}_\mu}{\mathrm{Nu}_c}\right)_{fd} = 1 + 3.39\cdot10^{-2}\,\mathrm{Pn}_\mu - 9.2\cdot10^{-4}\,\mathrm{Pn}_\mu^2$ $a = 2.62 + 0.084\,\mathrm{Pn}_\mu$ $m = 0.48\,\mathrm{Pr}_e^{-0.076}$
square	$\dfrac{\overline{\mathrm{Nu}}_\mu}{\overline{\mathrm{Nu}}_c} = 1 + \left[\left(\dfrac{\mathrm{Nu}_\mu}{\mathrm{Nu}_c}\right)_{fd} - 1\right]\left\{1 - \exp\left[-a\,(X^*)^m + 3.0\,(X^*)^{0.8}\right]\right\}$ $\left(\dfrac{\mathrm{Nu}_\mu}{\mathrm{Nu}_c}\right)_{fd} = 1 + 4.74\cdot10^{-2}\,\mathrm{Pn}_\mu - 1.0\cdot10^{-3}\,\mathrm{Pn}_\mu^2$ $a = 6.79 + 0.044\,\mathrm{Pn}_\mu$ $m = 0.57\,\mathrm{Pr}_e^{-0.050}$

5.3. Correlations for the mean Nusselt number for flows of liquids with temperature dependent thermal conductivity

The results obtained for the considered cross-sectional geometries for $\mathrm{Pn}_\mu = 0$ and $\mathrm{Pn}_k \neq 0$ have been used to obtain correlations for the ratio $\overline{\mathrm{Nu}}_k/\overline{\mathrm{Nu}}_c$ as a function of X^* in the form

$$\frac{\overline{\mathrm{Nu}}_k}{\overline{\mathrm{Nu}}_c} = 1 + a\,\mathrm{Pn}_k + \left(b_1\,\mathrm{Pn}_k + b_2\,\mathrm{Pn}_k^2\right) X^* \tag{18}$$

where the coefficient a can be expressed as

$$a = a_1\,\mathrm{Pr}_e^d\left\{1 - \exp\left[-a_2\,(X^*)^n\right]\right\} \tag{19}$$

In equations (18) and (19) the coefficients a_1, a_2, b_1 and b_2 and the exponent d only depend on the cross-sectional geometry and on the sign of Pn_k, while the exponent n also depends on the Prandtl number Pr_e according to the relation

$$n = n_1\,\mathrm{Pr}_e^{-s} \tag{20}$$

where the coefficient n_1 and the exponent s only depend on the cross-sectional geometry. For engineering application, the appropriate forms of equations (18) to (20) for circular and square cross-sections, valid for $5 \le \mathrm{Pr}_e \le 100$, $0.0125 \le |\mathrm{Pn}_k| \le 0.4$ and $10^{-4} \le X^* \le X^*_{max}$, are reported in table 3. For $\mathrm{Pr}_e = 5$, 20 and 100, the maximum positive and negative relative errors in the approximation of $\overline{\mathrm{Nu}}_k/\overline{\mathrm{Nu}}_c$ by means of equation (18) are $\varepsilon^+_{max} = 0.53\%$ and $\varepsilon^-_{max} = -0.21\%$ for circular ducts and $\varepsilon^+_{max} = 0.30\%$ and $\varepsilon^-_{max} = -0.17\%$ for square ducts. Therefore, we can conclude that the agreement between computed and approximate results is very good.

Table 3. Correlations for $\overline{\mathrm{Nu}}_k/\overline{\mathrm{Nu}}_c$ corresponding to equations (18) and (19) for different cross-sections and signs of Pn_k.

Cross-section	Pn_k	Correlations for $\overline{\mathrm{Nu}}_k/\overline{\mathrm{Nu}}_c$
circular	>0	$\dfrac{\overline{\mathrm{Nu}}_k}{\overline{\mathrm{Nu}}_c} = 1 + a\,\mathrm{Pn}_k + \left(1.97\,\mathrm{Pn}_k - 0.238\,\mathrm{Pn}_k^2\right) X^*$ $a = 0.032\,\mathrm{Pr}_e^{0.031}\left\{1-\exp\left[-28\,(X^*)^n\right]\right\}$ $n = 0.60\,\mathrm{Pr}_e^{-0.024}$
	<0	$\dfrac{\overline{\mathrm{Nu}}_k}{\overline{\mathrm{Nu}}_c} = 1 + a\,\mathrm{Pn}_k + \left(1.90\,\mathrm{Pn}_k - 0.399\,\mathrm{Pn}_k^2\right) X^*$ $a = 0.033\,\mathrm{Pr}_e^{0.035}\left\{1-\exp\left[-28\,(X^*)^n\right]\right\}$ $n = 0.60\,\mathrm{Pr}_e^{-0.024}$
square	>0	$\dfrac{\overline{\mathrm{Nu}}_k}{\overline{\mathrm{Nu}}_c} = 1 + a\,\mathrm{Pn}_k + \left(1.98\,\mathrm{Pn}_k - 0.295\,\mathrm{Pn}_k^2\right) X^*$ $a = 0.052\,\mathrm{Pr}_e^{0.024}\left\{1-\exp\left[-18\,(X^*)^n\right]\right\}$ $n = 0.60\,\mathrm{Pr}_e^{-0.037}$
	<0	$\dfrac{\overline{\mathrm{Nu}}_k}{\overline{\mathrm{Nu}}_c} = 1 + a\,\mathrm{Pn}_k + \left(1.92\,\mathrm{Pn}_k - 0.716\,\mathrm{Pn}_k^2\right) X^*$ $a = 0.053\,\mathrm{Pr}_e^{0.034}\left\{1-\exp\left[-18\,(X^*)^n\right]\right\}$ $n = 0.60\,\mathrm{Pr}_e^{-0.037}$

5.4. Estimation of the mean Nusselt number for flows of liquids with temperature dependent properties by means of the proposed correlations

The correlations proposed in the previous sections can be used to predict the value of $\overline{\mathrm{Nu}}_{\mu k}$. In fact, according to equation (11), we can write

$$\overline{\mathrm{Nu}}_{\mu k} = \overline{\mathrm{Nu}}_c\,\frac{\overline{\mathrm{Nu}}_{\mu k}}{\overline{\mathrm{Nu}}_c} \cong \overline{\mathrm{Nu}}_c\left(\frac{\overline{\mathrm{Nu}}_{\mu k}}{\overline{\mathrm{Nu}}_c}\right)' = \overline{\mathrm{Nu}}_c\left(\frac{\overline{\mathrm{Nu}}_{\mu}}{\overline{\mathrm{Nu}}_c} + \frac{\overline{\mathrm{Nu}}_{k}}{\overline{\mathrm{Nu}}_c} - 1\right) \tag{21}$$

where $\overline{\mathrm{Nu}}_c$, $\overline{\mathrm{Nu}}_\mu/\overline{\mathrm{Nu}}_c$ and $\overline{\mathrm{Nu}}_k/\overline{\mathrm{Nu}}_c$ are given by equations (12), (14) and (18), respectively. The maximum absolute values $|\varepsilon|_{max}$ of the relative error in the approximation of $\overline{\mathrm{Nu}}_{\mu k}$ by means of equation (21) for $\mathrm{Pr}_e = 5$, 20 and 100 and different values of Pn_μ and $\mathrm{Pn}_\mu/\mathrm{Pn}_k$ are reported in table 4 for circular and square cross-sectional geometries. As can be seen, the agreement between computed and approximate results is very good.

6. Conclusions

New correlations, suitable for engineering applications, for the mean Nusselt number in the entrance region of circular tubes and square ducts with uniform heat flux boundary conditions specified at the walls have been proposed. These correlations have been obtained on the basis of the results of a previous parametric investigation on the effects of temperature dependent viscosity and thermal conductivity in simultaneously developing laminar flows of liquids in straight ducts of constant cross-sections. In these studies, a finite element procedure has

Table 4. Maximum absolute values $|\varepsilon|_{max}$ (%) of the relative error in the approximation of $\overline{\mathrm{Nu}}_{\mu k}$ by means of equation (21) for simultaneously developing flows in circular and square ducts with different viscosity and thermal conductivity Pearson numbers.

Cross-section	Pn_μ	$\mathrm{Pn}_\mu/\mathrm{Pn}_k$							
		-10	-20	-40	-80	80	40	20	10
circular	1	1.26	1.26	1.26	1.26	1.25	1.26	1.25	1.25
	2	1.25	1.23	1.23	1.22	1.22	1.22	1.21	1.21
	4	1.29	1.23	1.20	1.19	1.17	1.16	1.15	1.11
square	1	1.67	1.67	1.67	1.67	1.67	1.66	1.66	1.66
	2	1.59	1.60	1.61	1.61	1.61	1.61	1.61	1.62
	4	1.45	1.40	1.43	1.44	1.46	1.47	1.57	1.76

been employed for the numerical solution of the parabolized momentum and energy equations. Viscosity and thermal conductivity have been assumed to vary with temperature according to an exponential and to a linear relation, respectively, while the other fluid properties are held constant. The temperature dependences of viscosity and thermal conductivity have been quantitatively expressed by the corresponding Pearson numbers. Axial distributions of the mean Nusselt number, obtained by numerical integration from those of the local Nusselt number, have been used as input data in the derivation of the proposed correlations. A superposition method has been proved to be applicable in order to estimate the Nusselt number by considering separately the effects of temperature dependent viscosity and thermal conductivity. Therefore, for each of the considered cross-sectional geometries, two distinct correlations have been proposed for flows of liquids with temperature dependent viscosity and with temperature dependent thermal conductivity, in addition to that obtained for constant property flows.

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