

Integral Transform Analysis of Poisson Problems that Occur in Discrete Solutions of the Incompressible Navier-Stokes Equations

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Abstract. The present work presents an alternate method for solving the poisson equation for calculating the pressure field that appears in many discrete numerical solvers of the incompressible Navier-Stokes equations. The methodology is based on a pressure-correction scheme with a mixed approach that employs Integral Transform Technique for the calculation of the pressure field from a given discrete velocity field. Two solution schemes are analyzed, these being the single transformation and the double transformation. The poisson equation is solved with the two different schemes using a prescribed source term to simulate the discrete data that could arise in the solution process of the momentum equation and an numerical results are presented. An error analysis of these results show that the single-transformation scheme is computationally superior to the double transformation, and that good convergence rates can be obtained with few terms in the series. Moreover, it was also verified that the series solution employed for the Poisson equation maintains the original spatial order of the discretization.

1. Introduction

A major difficulty in the numerical simulation of incompressible flows is the fact that velocity and pressure are coupled by the incompressibility constraint. To overcome this difficulty in time-dependent viscous incompressible flows, fractional step methods, which are also referred as projection methods, were developed. These methods can be classified into three classes [1], namely pressure-correction methods, velocity-correction methods, and consistent splitting methods. The most attractive feature of projection methods is that, at each time step, one only needs to solve a sequence of decoupled elliptic equations for the velocity and the pressure, making it very efficient for large-scale simulations. In spite of the advantages of projection methods, the employed decomposition is intrinsically second-order accurate, hence limiting the application of higher-order approximations.

Pressure-correction schemes are time-marching techniques composed of two sub-steps for each time step: the pressure is treated explicitly or ignored in the first sub-step and is corrected in the second one [2]. The most common methodology to obtain a pressure-correction equation involves combining the momentum and continuity equations by taking the divergence of the former and substituting the latter where necessary, effectively generating a Poisson-type equation



for determining the pressure field from a given velocity distribution. Naturally, the correction scheme is performed in an iterative fashion until mass conservation is satisfied at each time-iteration. As a result, a major computational cost associated with pressure-correction methods resides in the required sub-iterations per time step. At each sub-iteration, a Poisson equation needs to be solved, which consumes large amounts of CPU time. To make matters worse, the convergence rate of common iterative algorithms for this purpose, such as the Jacobi and Gauss-Seidel, rapidly decreases as the mesh is refined [3]. However, this issue can generally be remedied by employing Multigrid methods, which are designed to exploit the inherent differences of the error behavior among meshes of different size [4].

In the realm of analytical methods, the Integral Transform Technique [5] has been used for the solution of a variety of problems. The method deals with expansions of the sought solution in terms of infinite orthogonal basis of eigenfunctions, keeping the solution process always within a continuous domain. The resulting system is generally composed of a set of uncoupled differential equations which can be solved analytically. A few works have implemented a mixed approach using the Integral Transform Technique and other discrete schemes. Most of these employ the Generalized version of the technique (GITT) [6], which can be used to non-transformable problems in general, including those with non-linear effects. Cotta and Gerk [7], employed the integral transform method in conjunction with second-order-accurate explicit finite-differences schemes, to handle a class of parabolic-hyperbolic problems. Guedes and Ozisik [8, 9] analyzed unsteady forced convection in laminar flow between parallel plates, solving the problem with a hybrid scheme that combines the GITT with second-order finite differences. More recently, Castelões and Cotta. [10], employed a partial integral transformation strategy in periodic convection in micro-channels and Naveira-Cotta et al. [11] used a similar approach in a conjugate conduction-convection problem. In theses studies, one of the spatial variables was handled automatically by a numerical PDE solver, while the other was handled by integral transformation. Very recently, different from these previous studies, a new approach was employed in [12], where integral transformation and upwind discretization techniques were applied within a same spatial variable.

This work is focused on an alternative numerical scheme for solving the unsteady incompressible Navier-Stokes equations with primitive variables in three dimensions. The methodology is based on pressure-correction schemes using a mixed approach based on the Integral Transform Technique. This work proposes the use of the Integral Transform Technique to find the pressure dependence on the discrete velocity field analytically, which potentially eliminates the need for a pressure-velocity decomposition. In this paper, the analysis of the Integral Transform solution of the Poisson equation for pressure field with a given discrete source term is analyzed.

2. Problem Formulation

The governing equations are the traditional form of the Navier-Stokes Equations for incompressible flow:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f}, \quad \text{for } \mathbf{x} \in \mathcal{V} \quad \text{and} \quad t \geq 0, \quad (1a)$$

$$\nabla \cdot \mathbf{v} = 0, \quad \text{for } \mathbf{x} \in \mathcal{V}, \quad (1b)$$

in which equation (1a) is the momentum conservation equation and equation (1b) is the mass conservation equation, also called the incompressibility constraint.

The proposed methodology to solve the Navier-Stokes equations is based on the projection methods for incompressible flows [1]. The first step of the methodology is to apply the divergence operator on equation (1a) and use the continuity equation (1b) to obtain a Poisson equation for

determining the pressure field:

$$\frac{1}{\rho} \nabla^2 p = \nabla \cdot \mathbf{f} - \nabla \mathbf{v} : (\nabla \mathbf{v})^T, \quad \text{for } \mathbf{x} \in \mathcal{V}, \quad (2)$$

where ρ is the fluid density, \mathbf{f} is the body force vector (in acceleration units), \mathbf{v} is the velocity vector, p is the pressure and \mathcal{V} is a general domain volume. With this procedure, the continuity equations can be replaced by equation (2), which needs to be solved together with the momentum equation (1). With the given formulation, one needs to specific pressure boundary conditions at all boundaries. In this work, normal zero gradients for pressure at the boundaries will be used for illustrational purposes:

$$(\nabla p \cdot \mathbf{n})_{\partial \mathcal{V}} = 0, \quad (3)$$

where $\partial \mathcal{V}$ is the boundary of the general domain volume.

2.1. Initial Value Problem Discretization

The next step of the methodology is the time discretization. There are many ways to discretize the problem in time [13], such as methods like Forward Euler, Backward Euler, BDF methods, among others. If a Forward Euler Method is employed, the time-discretized version of equations (1a) and (2) become:

$$\mathbf{v}^{l+1} = \Delta t \left(-\frac{1}{\rho} \nabla p^l + \nu \nabla^2 \mathbf{v}^l - \mathbf{v}^l \cdot \nabla \mathbf{v}^l + \mathbf{f}^l \right) + \mathbf{v}^l, \quad (4a)$$

$$\frac{1}{\rho} \nabla^2 p^l = \nabla \cdot \mathbf{f}^l - \nabla \mathbf{v}^l : (\nabla \mathbf{v}^l)^T, \quad (4b)$$

where, in order to find the velocity \mathbf{v}^{l+1} , it is necessary to know the pressure field p^l .

2.2. Poisson Equation Solution

For this work, the two dimensional case in cartesian coordinates will be considered. For these assumptions, the equation (4b) is simplified to the following form:

$$\frac{\partial^2 p(x, y, t)}{\partial x^2} + \frac{\partial^2 p(x, y, t)}{\partial y^2} = \rho [h(x, y, t) - g(x, y, t)], \quad (5a)$$

$$\left(\frac{\partial p(x, y, t)}{\partial y} \right)_{y=0} = 0, \quad \left(\frac{\partial p(x, y, t)}{\partial y} \right)_{y=H} = 0, \quad (5b)$$

$$\left(\frac{\partial p(x, y, t)}{\partial x} \right)_{x=0} = 0, \quad \left(\frac{\partial p(x, y, t)}{\partial x} \right)_{x=L} = 0, \quad (5c)$$

in which:

$$g(x, y, t) = \left(\frac{\partial u}{\partial x} \right)^2 + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y} \right)^2, \quad h(x, y, t) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}. \quad (5d)$$

The Classical Integral Transform Technique (CITT) [5] is then used for the purpose of solving the Poisson equation (4b). Two approaches are used for this purpose: CITT transforming in one direction (single transformation) and CITT transforming in two directions (double transformation), as described next.

3. CITT with Single Transformation

In order to establish the transformation pair, the pressure field is written as function of an orthogonal eigenfunctions obtained from the following auxiliary eigenvalue problem known as the Helmholtz classical problem, where $X_n(x)$ are the eigenfunctions and λ_n are the eigenvalues.

$$\frac{d^2 X_n(x)}{dx^2} + \lambda_n^2 X_n(x) = 0, \quad X'(0) = 0, \quad X'(L) = 0, \quad (6)$$

which has the following solution:

$$X_n(x) = \cos(x\lambda_n), \quad \text{with} \quad \lambda_n = \frac{\pi n}{L}, \quad \text{for} \quad n = 0, 1, 2, 3, \dots \quad (7)$$

Now, the transformation pair can be defined:

$$\text{Transformation} \Rightarrow \bar{p}_n(y, t) = \int_0^L p(x, y, t) X_n(x) dx, \quad (8a)$$

$$\text{Inversion} \Rightarrow p(x, y, t) = p_0(y, t) + \sum_{n=1}^{\infty} \frac{X_n(x) \bar{p}_n(y, t)}{N_n}, \quad (8b)$$

where the norms N_n are defined by:

$$N_n = \int_0^L X_n^2 dx \quad (9)$$

which gives $N_n = L/2$ for $n \neq 0$ and $N_n = L$ for $n = 0$.

The final solution is given by a decomposition of the pressure field in two parts: the average pressure in the x direction p_{avg} , and the modified pressure p_{mod} :

$$p(x, y, t) = p_{\text{avg}}(y, t) + p_{\text{mod}}(x, y, t), \quad (10)$$

in which p_{avg} comes from the solution of the eigenproblem when $\lambda = 0$ and p_{mod} comes from the solution when $\lambda \neq 0$, in other words:

$$p_{\text{avg}}(y, t) = \frac{p_0(y, t)}{N_0}, \quad p_{\text{mod}}(x, y, t) = \sum_{n=1}^{\infty} \frac{X_n(x) \bar{p}_n(y, t)}{N_n}. \quad (11)$$

3.1. Solution for p_{mod} ($\lambda \neq 0$)

The integral transformation of the governing differential equation is derived by applying the operator $\int_0^L (\bullet) X_n dx$ to equation (5), obtaining the following transformed Poisson equation:

$$\frac{\partial^2 \bar{p}_n(y, t)}{\partial y^2} - \lambda_n^2 \bar{p}_n(y, t) = \rho \bar{h}_n(y, t) - \rho \bar{g}_n(y, t), \quad (12a)$$

$$\left(\frac{\partial \bar{p}_n(y, t)}{\partial y} \right)_{y=0} = 0, \quad \left(\frac{\partial \bar{p}_n(y, t)}{\partial y} \right)_{y=H} = 0, \quad (12b)$$

where \bar{g}_n and \bar{h}_n are the transformed versions of g and h :

$$\bar{g}_n(y, t) = \int_0^L g(x, y, t) X_n dx, \quad \bar{h}_n(y, t) = \int_0^L h(x, y, t) X_n dx, \quad (13)$$

and equation (12) admits a closed-form analytical solution:

$$\begin{aligned} \bar{p}_n(y, t) = & e^{-y\lambda_n} \left(-\frac{1}{2} \cosh(y\lambda_n) \operatorname{csch}(H\lambda_n) e^{(H+y)\lambda_n} \int_0^H \frac{\rho e^{-y\lambda_n}}{\lambda_n} (\bar{h}_n(y, t) - \bar{g}_n(y, t)) dy + \right. \\ & \frac{1}{4} (e^{2y\lambda_n} + 1) (\coth(H\lambda_n) - 1) \int_0^H \frac{\rho e^{y\lambda_n}}{\lambda_n} (\bar{g}_n(y, t) - \bar{h}_n(y, t)) dy + \\ & \frac{1}{2} \int_0^y \frac{\rho e^{y'\lambda_n}}{\lambda_n} (\bar{g}_n(y', t) - \bar{h}_n(y', t)) dy' + \\ & \left. \frac{1}{2} e^{2y\lambda_n} \int_0^y \frac{\rho e^{-y'\lambda_n}}{\lambda_n} (\bar{h}_n(y', t) - \bar{g}_n(y', t)) dy' \right) \quad (14) \end{aligned}$$

To find the actual solution for modified pressure p_{mod} , the inversion formula is used, equation (11). By observing equations (13), one notices integrals of the discrete variables u , v , f_x and f_y . In order to compute these integrals, the following integral separation is proposed:

$$\int_0^L \Lambda(u, v, f_x, f_y) dx = \sum_{q=1}^{i_{\max}} \int_{x_{q-1}}^{x_q} \Lambda(u, v, f_x, f_y) dx \quad (15)$$

where Λ is a general function of u , v , f_x and f_y . Then, to compute the integrals analytically, a Taylor expansion is used to expand the variables u , v , f_x and f_y in each subdomain:

3.2. Solution for p_{avg} ($\lambda = 0$)

In order to obtain the transformed differential equation for $\lambda = 0$, a similar process is done, leading to the following the transformed equation:

$$\frac{\partial^2 \bar{p}_0(y, t)}{\partial y^2} = \rho \bar{h}_0(y, t) - \rho \bar{g}_0(y, t), \quad (16a)$$

$$\left(\frac{\partial \bar{p}_0(y, t)}{\partial y} \right)_{y=0} = 0, \quad \left(\frac{\partial \bar{p}_0(y, t)}{\partial y} \right)_{y=H} = 0, \quad (16b)$$

where:

$$\bar{g}_0(y, t) = \int_0^L g(x, y, t) dx \quad \text{and} \quad \bar{h}_0(y, t) = \int_0^L h(x, y, t) dx. \quad (17)$$

The equation (16) admits a general analytical solution in the following form:

$$\bar{p}_0(y, t) = \int_0^y \int_0^{y''} \rho (\bar{h}_0(y', t) - \bar{g}_0(y', t)) dy' dy'' + c_1 y + c_2. \quad (18)$$

By applying the boundary conditions, one arrives to the following equations:

$$c_1 = 0, \quad 0 = \int_0^H \rho (\bar{h}_0(y, t) - \bar{g}_0(y, t)) dy + c_1, \quad (19)$$

such that c_1 must be zero and the integral also must be zero:

$$\int_0^H \rho (\bar{h}_0(y, t) - \bar{g}_0(y, t)) dy = 0. \quad (20)$$

One can easily prove that the integral above indeed equals zero due to the momentum conservation law. With this information at hand, the solution of the transformed differential equation is achieved:

$$\bar{p}_0(y, t) = \int_0^y \int_0^{y''} \rho (-\bar{g}_0(y', t) + \bar{h}_0(y', t)) dy' dy'' \quad (21)$$

Then, the same integral separation (equation (15)) and Taylor series expansions are used to derive analytically the coefficients \bar{h}_0 and \bar{g}_0 :

3.3. Discrete Derivatives

In order to solve the pressure problem, the discrete derivatives of u , v , f_x and f_y must be calculated. In this work, a second order central differencing scheme is used inside the domain and second-order the backward/forward (depending of the boundary) differencing scheme is used at the boundaries.

4. CITT with Double Transformation

In a similar manner as done for the single transformation scheme, one needs to establish the transformation pair. In order to obtain that for this approach, two eigenvalue problems are defined. The eigenvalue problem associated with the x direction is the same as given by equations (6), whereas the problem associated with the y direction is given by:

$$\frac{d^2 Y_n(y)}{dy^2} + \beta_m^2 Y_n(y) = 0, \quad Y'(0) = 0, \quad Y'(H) = 0, \quad (22)$$

which has the following solution:

$$Y_n(x) = \cos(y\beta_m), \quad \text{with} \quad \beta_m = \frac{\pi m}{H}, \quad \text{for} \quad m = 1, 2, 3, \dots, \quad (23)$$

and the transformation pair can then be defined as:

$$\text{Transformation} \Rightarrow \bar{p}_{n,m}(t) = \int_0^H \int_0^L p(x, y, t) X_n(x) Y_m(y) dx dy, \quad (24)$$

$$\text{Inversion} \Rightarrow p(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\bar{p}_{n,m}(t) X_n(x) Y_m(y)}{N_{x_n} N_{y_m}}, \quad (25)$$

where the norms N_{x_n} and N_{y_m} are defined by:

$$N_{x_n} = \int_0^L X_n^2 dx \quad \text{and} \quad N_{y_m} = \int_0^H Y_m^2 dy \quad (26)$$

Applying the operator $\int_0^H \int_0^L (\bullet) X_n Y_m dx dy$ to the Poisson equation, the following transformed Poisson equation is obtained:

$$-(\lambda_n^2 + \beta_m^2) \bar{p}_{n,m}(t) = \rho (\bar{h}_{n,m}(t) - \bar{g}_{n,m}(t)). \quad (27)$$

This equation is a simple uncoupled algebraic systems where $\bar{g}_{n,m}$ and $\bar{h}_{n,m}$ are written as:

$$\bar{g}_{n,m}(t) = \int_0^H \int_0^L g(x, y, t) X_n Y_m dx dy, \quad \bar{h}_{n,m}(t) = \int_0^H \int_0^L h(x, y, t) X_n Y_m dx dy, \quad (28)$$

and has a direct solution given by:

$$\bar{p}_{n,m}(t) = \frac{\rho (\bar{g}_{n,m}(t) - \bar{h}_{n,m}(t))}{\lambda_n^2 + \beta_m^2}, \quad (29)$$

such that the final solution is obtained by using the inversion formula, eq. (25).

The greatest advantage of this approach is that it requires a lot less analytical effort and the final solution is simpler and more compact. But the final solution has a double summation that can increase computational cost. In order to minimize this cost, one can use a reordering scheme, switching from the double summation to a single one. This is done by associating pairs (n, m) with a single index k in by organizing pairs that promote lower values of $(\lambda_n^2 + \beta_m^2)$ in ascending order. With the reordering scheme one arrives at the following expression for the pressure distribution:

$$p(x, y, t) = \sum_{k=0}^{\infty} \rho \frac{\bar{g}_{n(k),m(k)}(t) - \bar{h}_{n(k),m(k)}(t)}{\lambda_{n(k)}^2 + \beta_{m(k)}^2} \frac{X_{n(k)}(x) Y_{m(k)}(y)}{N_{x_{n(k)}} N_{y_{m(k)}}}. \quad (30)$$

5. Results

The results presented in this paper are intended to evaluate the feasibility of using a CITT poisson solver with the discrete source terms that appear in algorithms that use pressure-based projection methods, comparing both single and double transformation schemes. For all cases herein presented $L = 1$, $H = 1$, $\rho = 1$ and $\mu = 1$ were used and the source term of the Poisson equation (5) was assumed to be of the following form:

$$\left[\rho(h(x, y, t) - g(x, y, t)) \right]_{i,j} = -\sin\left(\frac{2\pi y_j}{H}\right) - \sin\left(\frac{2\pi x_i}{L}\right), \quad (31)$$

which was chosen in this form to ensure that relation (20) is satisfied.

A comparison of computational cost is done for the two transformation schemes presented in this work. In order to compare the CITT performance, a fixed mesh is used and different truncation orders for the summations are computed, so only the CITT error is captured. The CITT error is calculated using the following formula:

$$\epsilon_{i,j}^{\text{CITT}}(n_{\max}) = \text{abs}[p_{i,j}(n_{\max}) - p_{i,j}(n_{\max} + 5)] \quad (32)$$

Figure 1 presents a comparison of the error versus CPU time for single and double transformation strategies. As can be seen, the single transformation has better performance results, running the code in less CPU time and generating smaller errors. In other words, the final performance of the CITT single transformation overcomes the bigger effort in the analytical manipulation, which suggests that this is a better approach.

Once the single transformation scheme was shown to have better performance, from now on, the results for the solution of the Poisson equation for the pressure field is carried-out only using the single transformation scheme. With these results, the convergence will be analyzed and discussed. In order to analyze the convergence of the single transformation scheme, an absolute error in the x direction is defined as:

$$\epsilon_{i,j}^x(i_{\max}) = \text{abs}[p_{i,j}(i_{\max}, 1024) - p_{i,j}(2i_{\max}, 1024)], \quad (33)$$

in which ϵ is calculated at the mesh point (i, j) and for a mesh size $(i_{\max}, 1024)$. Similarly, for the y direction:

$$\epsilon_{i,j}^y(j_{\max}) = \text{abs}[p_{i,j}(1024, j_{\max}) - p_{i,j}(1024, 2j_{\max})], \quad (34)$$

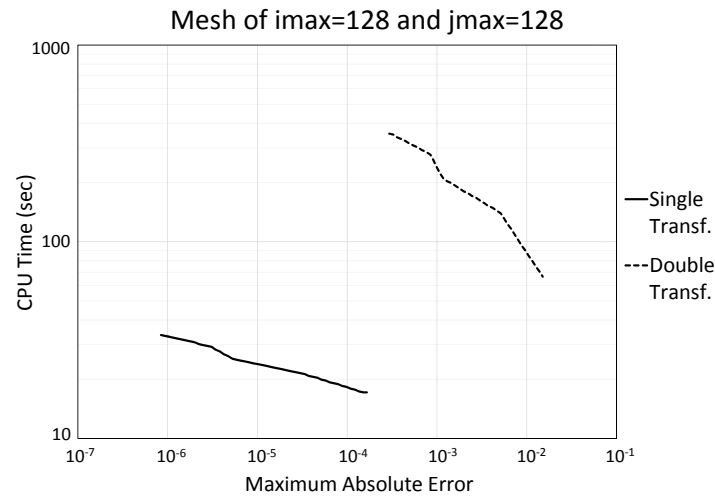


Figure 1. Comparison of the computational cost of CITT using single transformation and CITT using double transformation for a mesh $i_{\max} = 128$ and $j_{\max} = 128$.

where ϵ is calculated at the mesh point (i, j) and for a mesh size $(1024, j_{\max})$. For the calculation of the error in the x -direction, a refined mesh of 1024 divisions in y -direction was used in order to isolate the x error and minimize the effects of the y convergence. The same 1024 mesh divisions in y was used for the error in the x -direction. All meshes utilized in this work have divisions of the form 2^k , thus, one could take maximum advantage of parallel computing.

A great amount of data was generated with the proposed algorithm, and an analysis showed that the error distribution is almost insensitive to the variation of n_{\max} showing that CITT has a very good convergence rate. Hence, for simplicity, the presented results show data for a fixed $n_{\max} = 10$. Figure 2 shows the distribution of the absolute error $\epsilon_{i,j}^x(64)$ while figure 3 shows the distribution $\epsilon_{i,j}^y(64)$. As can be seen from the former, the error oscillates in the x -direction.

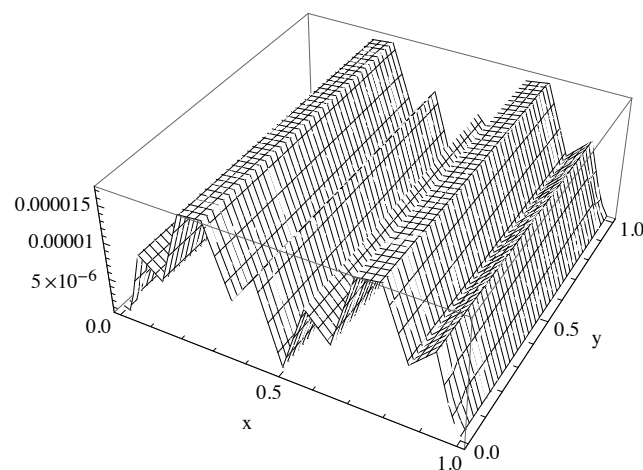


Figure 2. Absolute error variation inside the domain for $n_{\max} = 10$, $\Delta x = 2^{-6}$ and $\Delta y = 2^{-10}$.

This effect is explained by the nature of the Integral Transform Technique, since it is based on oscillatory eigenfunctions and the equation was transformed in the x -direction. Another

point to be highlighted is that the eigenfunctions satisfy the boundary conditions. As a result, the error is zero at $x = 0$ and $x = 1$. The error behavior does not change with the variation Δx , the only notable change being the magnitude of the results, that naturally, decays as Δx is decreased. Figure 3 also displays the error behavior for the mesh in the y -direction, showing that it increases along with the y -coordinate. This phenomenon is due to the exponential feature of

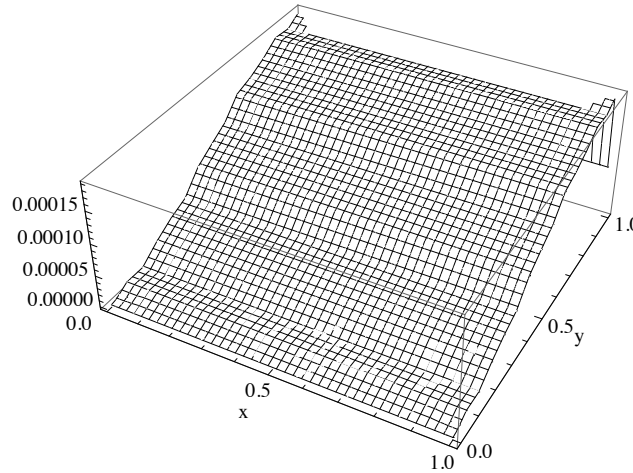


Figure 3. Absolute error variation inside the domain for $n_{\max} = 10$, $\Delta x = 2^{-10}$ and $\Delta y = 2^{-6}$.

the ODE solution obtained after the transformation of the problem. This exponential behavior of the solution magnifies the error while y increases. Similar observations to those pointed out in figure 2, in respect to Δx , can be done here about the variation of Δy .

In order to illustrate the convergence of the solution with the variation of Δx , figure 4 shows a plot of the maximum absolute error with the variation of the mesh size Δx . As one can observe, the convergence order is about 2, which was expected since all approximation made in the mathematical formulation were of this order. Figure 5 shows the convergence of the absolute error with Δy . Although the solution seems to have a higher order for the poorer refined meshes, the order stabilizes at 2 when more refined meshes are implemented.

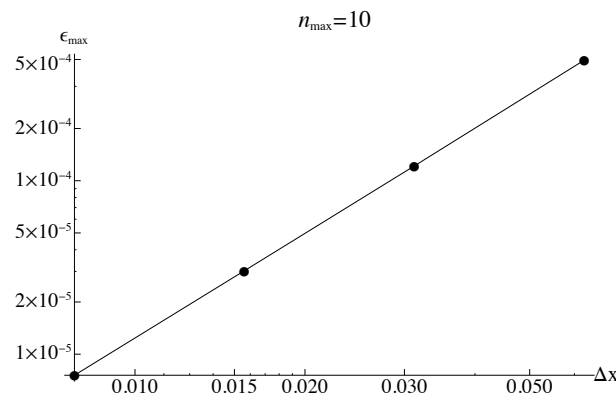


Figure 4. Maximum absolute error vs. Δx for $\Delta y = 2^{-10}$.

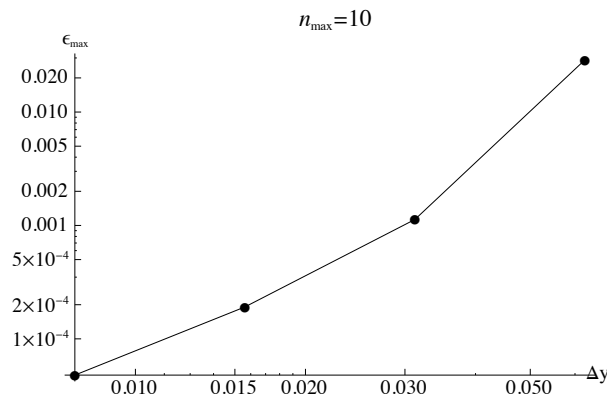


Figure 5. Maximum absolute error vs. Δy for $\Delta x = 2^{-10}$.

6. Summary and Conclusions

The main goal of the proposed work was the development of a new methodology using the Integral Transform Technique in a semi-analytical fashion for the Poisson equation, allowing an explicit expression for the pressure p , that then can be input in the momentum equations and thus be solved using a numerical technique for initial value problem. The solution of the Poisson equation using this semi-analytical approach was accomplished in this work using two different schemes: CITT single transformation and CITT double transformation. The comparison between both schemes showed that the double transformation has poorer performance in comparison with the single transformation scheme. The error convergence rate was shown and it was possible to see that the spatial approximation order matched the expected value of 2 in both x and y directions. It was also observed that the Integral Transform Technique had a very good performance, converging with very few terms in the series.

7. Acknowledgements

The authors would like to acknowledge the financial support provided by CAPES, CNPq, FAPERJ, and the Lemann Foundation.

References

- [1] Guermond J L, Mineev P and Shen J 2006 *Comput. Methods Appl. Mech. Engrg.* **195** 6045
- [2] Ferziger J H and Perić M 1999 *Computational Methods for Fluid Dynamics* 2nd ed (New York, NY: Springer – Verlag)
- [3] Versteeg H K and Malalasekera W 2007 *An Introduction to Computational Fluid Dynamics, The Finite Volume Method* 2nd ed (Pearson Prentice Hall)
- [4] Wesseling P 1992 *An Introduction to Multigrid Methods* (New York: John Wiley & Sons)
- [5] Mikhailov M D and Özişik M N 1984 *Unified Analysis and Solutions of Heat and Mass Diffusion* (New York: John Wiley & Sons)
- [6] Cotta R M 1993 *Integral Transforms in Computational Heat and Fluid Flow* (Boca Raton, FL: CRC Press)
- [7] Cotta R M and Gerk J E V 1994 *Numerical Heat Transfer - Part B* **25** 442–448
- [8] Guedes R O C and Ozisik M N 1994 *International Journal Heat and Mass Transfer* **37** 2699–2706
- [9] Guedes R O C and Ozisik M N 1994 *Int. J. Heat and Fluid Flow* **15** 116–121
- [10] Castellões F V and Cotta R M 2006 *Progress in Computational Fluid Dynamics* **6** 321–326
- [11] Naveira C P, Lachi M, Cotta R M and Padet J 2009 *International Journal of Heat and Mass Transfer* **52** 112–123
- [12] Chalhub D J M N, Sphaier L A and de B Alves L S 2013 *Numerical Heat Transfer - Part B* **63** 167–187
- [13] Moin P 2001 *Fundamentals of Engineering Numerical Analysis* (New York, NY: Cambridge University Press)