

On the Chern-Simons State in General Relativity and Modified Gravity Theories

Ricardo Rosas-Rodríguez

Instituto de Física y Matemáticas, Universidad Tecnológica de la Mixteca, 69000, Huajuapán de León, Oaxaca, México

E-mail: rrosas@mixteco.utm.mx

Abstract. The Chern-Simons state is one solution to quantum constraints of gravity in the context of general relativity (GR) theory if we use Ashtekar's variables and if one orders the constraints with the triads to the left. Six years ago Krasnov introduced a certain class of modified gravity theories by replacing the cosmological constant by a "cosmological function of the curvature". If this function is a constant we come back to GR. In this note we review how the Chern-Simons state is one solution to the constraints of GR and we state the problem to face if we wish a generalized Chern-Simons state for the modified Krasnov's theories.

1. Introduction

In 2008 K. Krasnov introduced a class of alternative gravity theories with the name of "non-metric gravity", which are known as modified gravity theories today [1] (see also [2]). The essence of this modifies theories is to replace the cosmological constant by a "cosmological function" in the Hamiltonian constraint of the Ashtekar's canonical formulation of general relativity (GR) [3] (see also [4]). The Gauss and diffeomorphism constraints are unchanged. The canonically conjugate variables are still a densitized triad¹ $\tilde{\sigma}_i^a$ and a (complexified) SU(2) connection A_a^i . a, b, c, \dots are spatial indices and i, j, k, \dots are "internal" indices. The modified (Gauss, diffeomorphism and Hamiltonian) constraints of gravity take the following form [5]

$$\mathcal{G}_i = \mathcal{D}_a \tilde{\sigma}_i^a \approx 0, \quad (1)$$

$$\mathcal{H}_b = \tilde{\sigma}^{ai} F_{ab}^i \approx 0, \quad (2)$$

$$\mathcal{H} = \epsilon^{ijk} \tilde{\sigma}^{ai} \tilde{\sigma}^{bj} F_{ab}^k + \phi(\Psi_{\text{tf}}) \epsilon^{ijk} \epsilon_{abc} \tilde{\sigma}^{ai} \tilde{\sigma}^{bj} \tilde{\sigma}^{ck} \approx 0, \quad (3)$$

where \mathcal{D}_a is the covariant derivative with respect to the connection A_a^i , ϵ^{ijk} and ϵ_{abc} are the completely antisymmetric internal and spatial tensors taking values ± 1 , $F_{ab}^i = 2\partial_{[a} A_{b]}^i + \epsilon^{ijk} A_a^j A_b^k$ is the curvature of A_a^i and $\phi(\Psi_{\text{tf}})$ is an arbitrary function of the trace-free part of the tensor

$$\Psi^{ij} := \frac{F_{ab}^i \epsilon^{jkl} \tilde{\sigma}^{ak} \tilde{\sigma}^{bl}}{\epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}^{ai} \tilde{\sigma}^{bj} \tilde{\sigma}^{ck}}. \quad (4)$$

¹ We use the Ashtekar convention: the tilde above a symbol denotes a density of weight 1, under a symbol a density of weight -1.



Note that $\Psi^{ij} \approx \Psi^{(ij)}$ in view of the diffeomorphism constraint (2). GR is the simplest case $\phi = \text{constant} = \Lambda$, in which the quantity Ψ_{tf}^{ij} defined by (4) is nothing but the (self-dual part) of the Weyl curvature tensor [6].

The spacetime covariant theory that leads to the modified constraints (1), (2) and (3) was proposed in [2] with particular arguments about why this theory should be closed under renormalization and its action is given by

$$S = \int B_i \wedge F^i(A) - \frac{1}{2} \left(\Lambda^{ij} - \frac{1}{3} \delta^{ij} \phi(\Lambda) \right) B^i \wedge B^j, \quad (5)$$

where B^i is an SU(2) Lie algebra valued two-form, A^i is the connection one-form, $F^i(A)$ is its curvature, and Λ^{ij} is the Lagrange multiplier field. When $\phi = \text{constant} = \Lambda$ one obtains from (5) the Plebański action for GR with the cosmological constant [7]. The 3 + 1 decomposition of the theory (5) is given in [5].

In [5] and [8] it is shown that the constraints (1), (2) and (3) form a first-class algebra, i.e., that the Poisson bracket of any two constraints vanishes on the constraint surface. Therefore these constraints generate gauge transformations of the modified theories and the count of the number of physical degrees of freedom is simple: we have nine kinematical configurational variables A_a^i , together with seven constraints, which give two physical degrees of freedom.

The constraint algebra for the 2+1-dimensional modified gravity theories is studied in [9].

The aim of this note is to present the problem to face if we wish a generalization of the Chern-Simons state to the modified Krasnov's theories.

In the next section we show how the Chern-Simons state is one solution to the constraints of GR. In Sect. 3 we show the problem to face if we wish a generalized Chern-Simons state for the modified Krasnov's theories. We finish this letter with some concluding remarks.

2. Chern-Simons State for GR

If we ignore the condition of square integrability and think of states as arbitrary functions $\psi[A]$, to quantize, we replace the classical 'position' and 'momentum' variables A_a^i and E_i^a by the operators

$$\hat{A}_a^i \psi[A] = A_a^i \psi[A], \quad \hat{E}_i^a \psi[A] = \frac{\delta}{\delta A_a^i} \psi[A]$$

which have commutation relations analogous to the Poisson bracket relations. With these operators in hand we can then make the Hamiltonian, diffeomorphism and Gauss law constraints into operators. There are different operator ordering to choose from, but we choice the triads to the left:

$$\begin{aligned} \hat{\mathcal{G}}_i &= \hat{D}_a \hat{E}_i^a \\ \hat{\mathcal{H}}_b &= \hat{E}^{ai} \hat{F}_{ab}^i \\ \hat{\mathcal{H}} &= \epsilon^{ijk} \hat{E}^{ai} \hat{E}^{bj} \hat{F}_{ab}^k - \frac{\Lambda}{6} \epsilon^{ijk} \epsilon_{abc} \hat{E}^{ai} \hat{E}^{bj} \hat{E}^{ck} \end{aligned}$$

where Λ is the cosmological constant and

$$\hat{F}_{ab}^i \psi[A] = F_{ab}^i \psi[A]. \quad (6)$$

We define the physical state space $\mathcal{H}_{\text{phys}}$ to be the space of functions $\psi[A]$ that satisfy the constraints in quantum form, i.e.,

$$\mathcal{H}_{\text{phys}} = \{ \psi : \hat{\mathcal{G}}_i \psi = \hat{\mathcal{H}}_b \psi = \hat{\mathcal{H}} \psi = 0 \}. \quad (7)$$

Then, the problem is to find functions ψ in $\mathcal{H}_{\text{phys}}$.

The Chern-Simons theory [11] gives rise to a solution of all three constraint equations, if we choice $\Lambda \neq 0$.

We define the Chern-Simons state Ψ_{CS} to be the following function

$$\Psi_{CS}[A] = e^{-\frac{6}{\Lambda}S_{CS}[A]} \quad (8)$$

where

$$\begin{aligned} S_{CS}[A] &= \int_{\Sigma} \text{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \\ &= \int_{\Sigma} \epsilon^{abc} \text{tr} \left(A_a \partial_b A_c + \frac{2}{3} A_a A_b A_c \right) \end{aligned} \quad (9)$$

is the Chern-Simons action.

Ψ_{CS} lies in the physical state space for quantum gravity with cosmological constant:

$$\hat{\mathcal{G}}_i \Psi_{CS} = \hat{\mathcal{H}}_b \Psi_{CS} = \hat{\mathcal{H}} \Psi_{CS} = 0. \quad (10)$$

Since the Chern-Simons action is invariant under gauge transformations and diffeomorphisms, two of these three equations are obvious: $\hat{\mathcal{G}}_i \Psi_{CS} = \hat{\mathcal{H}}_b \Psi_{CS} = 0$. The fact that $\hat{\mathcal{H}} \Psi_{CS} = 0$ is not so obvious. We offer a direct argument to show this.

First, the functional derivative of S_{CS} is

$$\frac{\delta}{\delta A_c^k} S_{CS}[A] = -\frac{1}{2} \epsilon^{abc} F_{abk}.$$

Then,

$$\begin{aligned} \frac{\delta}{\delta A_c^k} \Psi_{CS}[A] &= \frac{\delta}{\delta A_c^k} e^{-\frac{6}{\Lambda}S_{CS}[A]} = \frac{3}{\Lambda} \epsilon^{abc} F_{abk} e^{-\frac{6}{\Lambda}S_{CS}[A]} \\ &= \frac{3}{\Lambda} \epsilon^{abc} F_{abk} \Psi_{CS}[A]. \end{aligned}$$

It follows that

$$\epsilon_{abc} \frac{\delta}{\delta A_c^k} \Psi_{CS}[A] = \frac{6}{\Lambda} F_{abk} \Psi_{CS}[A].$$

Finally

$$\hat{\mathcal{H}} \Psi_{CS} = \epsilon^{ijk} \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^j} \left(F_{abk} - \frac{\Lambda}{6} \epsilon_{abc} \frac{\delta}{\delta A_c^k} \right) \Psi_{CS} = 0.$$

The work of Kodama [12] indicate that the Chern-Simons state is a quantized version of anti-deSitter space, a simple solution of the vacuum Einstein equations with $\Lambda \neq 0$.

3. Chern-Simons State for Modified Gravity Theories

For Krasnov's modified gravity theories we define the modified Chern-Simons state as

$$\Psi_{MCS}[A] = e^{-\frac{6}{\phi(\Psi^{ij})}S_{CS}[A]} \quad (11)$$

where $S_{CS}[A]$ is the Chern-Simons action again. Note that Ψ_{MCS} is yet a gauge and Lorentz scalar, thus, we have yet $\hat{\mathcal{G}}_i \Psi_{MCS} = \hat{\mathcal{H}}_b \Psi_{MCS} = 0$, directly. The trouble again is to show that $\hat{\mathcal{H}} \Psi_{MCS} = 0$.

In this case, we have

$$\begin{aligned} \frac{\delta}{\delta A_c^k} \Psi_{MCS} &= \frac{\delta}{\delta A_c^k} e^{-\frac{6}{\phi(\Psi^{ij})} S_{CS}} \\ &= - \left[\frac{6}{\phi(\Psi^{ij})} \frac{\delta}{\delta A_c^k} S_{CS} + S_{CS} \frac{\delta}{\delta A_c^k} \frac{6}{\phi(\Psi^{ij})} \right] e^{-\frac{6}{\phi(\Psi^{ij})} S_{CS}} \\ &= \left[\frac{6}{\phi(\Psi^{ij})} \left(\frac{1}{2} \epsilon^{abc} F_{abk} \right) + S_{CS} \frac{6}{\phi^2(\Psi^{ij})} \frac{\delta \phi}{\delta A_c^k} \right] \Psi_{MCS}. \end{aligned} \tag{12}$$

Thus

$$\epsilon_{abc} \frac{\delta}{\delta A_c^k} \Psi_{MCS} = \frac{6}{\phi(\Psi^{ij})} F_{abk} \Psi_{MCS} + S_{CS} \frac{6\epsilon_{abc}}{\phi^2(\Psi^{ij})} \frac{\delta \phi}{\delta A_c^k} \Psi_{MCS},$$

and it is not obvious that $\hat{\mathcal{H}} \Psi_{MCS} = 0$.

4. Concluding Remarks

We can see that $\hat{\mathcal{H}} \Psi_{MCS} = 0$ does not follow directly for modified gravity theories as for GR.

We must to point out that perhaps a different modified Chern-Simons state is need to be defined in order to solve the scalar constraint, but at the present we have not idea of the form of this state.

In this letter we only wish to show the trouble that emerge if we define the modified Chern-Simons state as in (11).

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