# Search for vacuum solutions of conformally flat stationary noncircular spacetime

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**Abstract.**
The main goal of this work is to present the first general vacuum solutions, for a conformally flat stationary and non-circular spacetime. The solutions found are static and have an axis of symmetry. In addition, we discuss the geometrical consequences related to the non-circular property of the solutions.

## 1. Introduction
The non-circular stationary spacetimes were first pointed out by Gourgoulhon and Bonazzola in the context of neutron stars with a strong toroidal magnetic field where such field is a consequence of the associated non-circular stress-energy tensor [1]. In this regards, Ioka and Sasaki introduced a general formalism based in the magnetohydrodynamics to study the general case [2], and then analyzed relativistic stars possessing both, poloidal and toroidal magnetic fields as well as a meridional flow [3]. Recently, attention has been paid to the study of neutron stars with meridional circulations [4]. In the exact solutions framework, the most general form of the conformally flat stationary and non-circular spacetimes was developed in [5]. This subject has been extensively studied by several authors and great part of the geometrical properties are now well understood [6, 7, 8].

On the theoretical side, the exact solutions comprise an active line of research in General Relativity where it was argued that the physically interesting spacetimes belong to the family of Lewis-Papapetrou metrics (L-P).The L-P spacetimes are characterized by the presence of two commuting Killing vectors one **k**, timelike and other **m**, spacelike; thus spacetime becomes to cyclic stationary.  The cyclic simmetry is the most general which is related with the spacelike Killing vector, however, there is a symmetry related with both Killing vectors which isn't highlighted, we refer to the circular symmetry. Specifically, a stationary and cyclically symmetric spacetime is circular if the 2-surfaces orthogonal to the Killing fields are integrable. This is equivalent to satisfy the Frobenius integrability conditions or, alternatively, that the gravitational field is not only locally independent of the time and the rotation angle, but also invariant under the simultaneous inversion of the time and the angle. The circularity property

imposes geometrical restrictions on the structure of the spacetime metrics, as a consequence the Killing vectors are restricted to be on a plane expanded by them. In the case where the circularity is not compulsory, the metric functions do not depend on the coordinates of the Killing vectors, and therefore the spacetime has the most general structure and the metric functions depend only of the two coordinates. These spacetimes are stationary non-circular and cyclic-symmetric.

In the Section 2 some geometrical facts and the vacuum solutions for the conformally flat stationary and non-circular spacetimes are presented, finally in Section 3 we present some conclusion and discussions .

## 2. Non-circular spacetimes

Usually, when one refers to a stationary axisymmetric spacetime, one immediately associates a Lewis-Papapetrou type metric to it. However, as it is remarked by some authors (see [9] and reference there in) in many works, the axial symmetry is not inherent to the general structure of a L-P metric. As it is well known this is an arrangement of fixed points which is invariant under the action of SO(3) group transformations. The circular symmetry is a property not emphasized enough like axial symmetry, and both are the restricted cases of non-circular and cyclic symmetries, respectively [10].

In this work we consider the search of vacuum solutions of non-circular stationary spacetimes; in particular, for the Petrov type 0 or conformally flat which was reported in [5] and is described by the metric,

$$ds^2 = e^{-2Q}\Bigg[ -k(d\tau^2 + 2xd\tau d\sigma + \epsilon d\sigma^2) + 2k(K_0 + K_1 x)d\sigma dy$$

$$+\frac{dx^2}{(C_0 + C_1 x)(x^2 - \epsilon) - k(K_0 + K_1 x)^2} + (C_0 + C_1 x)dy^2 \Bigg], \qquad (1)$$

where $Q = Q(x,y)$ and $K_0$, $K_1$, $C_0$ y $C_1$ are the integration constants. The parameters are $k = \pm 1$ and $\epsilon = \pm 1$, and basically its values define the temporal and angular coordinates in [9] a detailed study is given. In what follows and without lose of generality we use $k = 1$.

The spacetime described by the metric (1) has two commuting Killing vectors, one **k** timelike and other **m** spacelike and as a results of the non-circularity, are establishes

$$\star(\mathbf{m} \wedge \mathbf{k} \wedge d\mathbf{k}) = e^{2Q}(K_0 + K_1 x),$$
$$\star(\mathbf{k} \wedge \mathbf{m} \wedge d\mathbf{m}) = e^{2Q}(\epsilon K_1 + K_0 x), \qquad (2)$$

where the star stands for the Hodge dual. When the right hand side of the above equation vanish the spacetime is circular and (2) they are know as the circular conditions. So, when $K_0 = 0$ and $K_1 = 0$ we arrive to circular stationary spacetime and cyclic reported in [9]. Furthermore, contrary to what happens in the circular case, the non-circular has an axis of symmetry.

The main goal of this paper is to show exact solutions for the metric (1) in the case of vacuum [11][12]. The starting point are the Einstein field equations,

$$G_{\mu\nu} = 0 \qquad \Leftrightarrow \qquad R_{\mu\nu} = 0. \qquad (3)$$

In order to solve the system of equations thrown by (3), we proceed to integrate the more easy equations, in our case from $G^x_\tau$ and $G^x_\sigma$ we get $Q(x,y) = q(x)$ the next step is to integrate $G^\tau_\sigma$

$$G^\tau_\sigma = e^{2Q}\left\{ [\epsilon(C_0 + C_1 x) + K_0(K_0 + K_1 x)]\frac{\partial Q}{\partial x} - \frac{(K_0 K_1 + C_1 \epsilon)}{2} \right\} = 0,$$

integrate directly the above equation, we obtain

$$q(x) = \frac{1}{2}\ln[(K_0 K_1 + \epsilon C_1)x + K_0^2 + \epsilon C_0] + \frac{1}{2}q_0, \tag{4}$$

where $q_0$ is an integration constant.

Substituting (4) into the equations system (3), now from the components $^y{}_\tau$ and $^y{}_\sigma$ we get

$$e^{q_0}\epsilon(C_0 K_1 - K_0 C_1) = 0, \tag{5}$$
$$e^{q_0}(K_0^3 - \epsilon K_1^2 K_0 - \epsilon^2 K_1 C_1 + \epsilon K_0 C_0) = 0, \tag{6}$$

from (5) one finds

$$K_1 = \frac{C_1}{C_0}K_0, \tag{7}$$

which substituting (7) in (6) one gets

$$\left(\epsilon\frac{C_1^2}{C_0^2} - 1\right)(K_0^2 + \epsilon C_0) = 0, \tag{8}$$

to satisfy the above equation we obtain the vacuum solution for conformally flat stationary and non-circular spacetime. It is remarkable at a glance for to $\epsilon = -1$ one can satisfy for the second term, however, in a deep analysis the second factor should not vanish in order to have a well defined metric.

*2.1. Vacuum solution*

When $\epsilon = 1$ from the equation (8) seemingly we have two possibilities, but actually it is just one, namely

$$i)\ \left(\frac{C_1^2}{C_0^2} - 1\right) = 0 \Rightarrow |C_1| = |C_0|, \tag{9}$$

and then $C$'s, to have the same sign or contrary sign.

For the first case of $(i)$, when $C_1 = C_0$ the Eq.(7) implies $K_1 = K_0$, then we found

$$i.a)\quad q(x) = \frac{1}{2}\ln[q_0(K_0^2 + C_0)(1 + x)]. \tag{10}$$

In the second case, with contrary signs, says $C_1 = -C_0$ implies $K_1 = -K_0$ and then

$$i.b)\quad q(x) = \frac{1}{2}\ln[q_0(K_0^2 + C_0)(1 - x)]. \tag{11}$$

In summary, we have $\epsilon = 1$, $C_0 = C_1$, $K_0 = K_1$ and the conformal factor of the metric (1) for the case $(i.a)$ is read as

$$e^{-2Q} = \frac{1}{q_0(K_0^2 + C_0)(1 + x)}. \tag{12}$$

For the case $(i.b)$ $\epsilon = 1$, $C_0 = -C_1$, $K_0 = -K_1$ and now we have the conformal factor given by

$$e^{-2Q} = \frac{1}{q_0(K_0^2 + C_0)(1-x)}.$$  (13)

In [5] it was shown that (1) is static if the metric functions satisfy $\epsilon = 1$, and $K_0 = \pm K_1$, and a consequence of the integration of the field equations $C_0 = \pm C_1$, which means that this branch of solutions is static. Another fact is the existence of an axis of symmetry at $x = \mp 1$.

In general for $k = 1$, $\epsilon = 1$, the metric of the spacetime is rewritten as

$$ds^2 = \frac{1}{q_0(K_0^2 + C_0)(1 \pm x)}\left[ -dt^2 + (x^2 - 1)d\sigma^2 + 2K_0(x \pm 1)d\sigma dy \right.$$
$$\left. + \frac{dx^2}{C_0(x \pm 1)(x^2 - \epsilon) - K_0(x \pm 1)^2} + C_0(x \pm 1)dy^2 \right],$$  (14)

where $dt \to d\tau + xd\sigma$. The metric (14) shows the static structure explicitly and the axis of symmetry.

## 3. Conclusions

In this work we wish to emphasize the non-circular property in the stationary spacetimes, which was unknown for several years, and recently brought to scene. We present general solutions to the Einstein's equations in vacuum for a non-circular stationary spacetime and conformally flat. The solution is found to be static and axisymmetric and it shows an unusual structure because the metric is not diagonal. Unfortunately the Riemann tensor vanishes for the solutions given, which means that are flat spacetimes.

The solutions were obtained by solving a couple of linear equations given by Einstein equations, the integral gives the function of scale factor (4), which after of substituting in the others equations we have the constraints (5) and (6). Like any system of equations these must be satisfied simultaneously, so therefore, our result is just one of the possibilities still lacking to investigate the remaining linearly independent solutions of the field equations, but this will be done in another job.

Similarly to what happens in the circular stationary spacetimes there is a vacuum solution for the static branch and it has an axis of symmetry, as demonstrated in [5]. In fact, the noncircular stationary metrics have not been explored in depth because the technical difficulties, but we find quite interesting that the development of such solutions could be related with several astrophysical phenomena. We think that there are more interesting consequences which can only be studied in non-circular geometries whose study will be done elsewhere.

## References

[1] Gourgoulhon E and Bonazzola S 1993 *Phys. Rev. D* **48**, 2635.
[2] Ioka K and Sasaki M 2003 *Phys. Rev. D* **67**, 12026, arXiv:gr-qc/0302106.
[3] Ioka K and Sasaki M 2004 *Astrophys.J.* **6**00, 296-316, arXiv:astro-ph/0305352.
[4] Birkl R, Stergioulas N and Muller E 2011 *Phys. Rev. D* **8**4, 023003, arXiv:gr-qc/1011.5475.

[5]  Ayón-Beato E, Campuzano C and García A 2006 *Phys. Rev. D***7**4, 024014, arXiv:gr-qc/0506004.

[6]  Vera R 2003 *Class. Quant. Grav.* **2**0, 2785, arXiv:gr-qc/0305108.

[7]  Anabalon A, Deruelle N, Morisawa Y, Oliva J, Sasaki M, Tempo D and Troncoso R 2009 *Class. Quant. Grav.* **2**6, 065002, arXiv:0812.3194[hep-th].

[8]  Anabalon A, Tempo D and Troncoso R 2011 *Int. J. Mod. Phys.* **2**0, 639-647, arXiv:gr-qc/1009.3030.

[9]  García A and Campuzano C 2002 *Phys. Rev. D***6**6, 124018 (2002), arXiv:gr-qc/0205080. [Erratum-ibid 2003 D **6**8, 049901 (2003)].

[10]  García A and Campuzano C 2003 *Circular stationary cyclic symmetric space-times: Conformal flatness* arXiv:gr-qc/0310054.

[11]  Méndez Montes de Oca, José Luis, *Tesis: El problema de Kerr y los espacios estacionarios no-circulares*, Departamento de Física, Facultad de Física e I. A., Universidad Veracruzana 2011.

[12]  Campuzano Vargas, Cuauhtemoc, *Tesis: Solución de vacío para espacios estacionarios con dos simetrías y conformalmente planos*, Facultad de Ciencias Físico Matemáticas, Benemérita Universidad Autónoma de Puebla 2014.