

# How nice are critical knots?

## Regularity theory for knot energies

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**Abstract.** In this note we report on some recent developments in *geometric knot theory* which aims at finding links between geometric properties of a given knotted curve and its knot type. The central object of this field are so-called *knot energies* which are defined on closed embedded curves.

First we present three important examples of two-parameter knot energy families, namely O'Hara's energies, the (generalized) integral Menger curvature, and the (generalized) tangent-point energies.

Subsequently we outline the main steps that lead to  $C^\infty$ -regularity of stationary points—especially minimizers—in the non-degenerate sub-critical range of parameters.

Particular attention is devoted to the appearing parallels between these energies which, surprisingly at first glance, are quite similar from an analyst's perspective.

### 1. Introduction

*Is the “cable spaghetti” on the floor really knotted or is it enough to pull on both ends of the wire to completely unfold it?*

This is the basic question that led to the formation of *geometric knot theory*. During the past twenty five years, this new subfield emerged from the search of particularly “nice” shapes of knots (whatever that means), in contrast to classical knot theory [Burde & Zieschang, 2003] making use of fine analytic tools rather than methods involving algebraic topology. The general aim is investigating geometric properties of a given knotted curve in order to gain information on its knot type. In a broader sense, it is part of *geometric curvature energies* which include geometric integrals measuring smoothness and bending for objects that *a priori* do not have to be smooth, also covering the higher-dimensional case.

The first steps into this new area were paved by Fukuhara [1988] who considered knotted polygons. He thought of a new approach to the fundamental question of classical knot theory—to decide whether two given knots belong to the same knot class. His idea was to design a sort of simplification process for complicate knots which of course has to ensure that the simplified



knot still belongs to the same knot class. It should be easier to answer that question for the simplified versions than for the original ones.

So Fukuhara defined an “energy” that basically models *self-avoidance*, i.e. the functional blows up on embedded curves converging to a curve with a self-intersection. Following the negative gradient flow of this functional should simultaneously “untangle” the curve and prevent it from leaving the ambient knot class.

Such a functional can in fact be seen as a measure of “entangledness” and consequently one is interested in finding minimizers within a given knot class, hoping to find the desired “optimal shape”, having strands particularly wide apart. Therefore it is a crucial task to design and investigate self-avoiding functionals that are bounded below, which are referred to as *knot energies* according to O’Hara [2003, Def. 1.1].

Knot energies can help to model repulsive forces of fibres, whenever self-interaction of strands should be avoided. In fact, there is indication for DNA molecules seeking to attain a minimum state of a suitable energy [Moffatt, 1996]. There have been several attempts to employ self-avoiding energies for mathematical models in microbiology, see e.g. Banavar et al. [2003] and references therein for links to polymer science and protein science. Attraction phenomena may also be modeled by a corresponding positive gradient flow [Alt et al., 2009].

### *O’Hara’s energies*

Later on, Fukuhara’s ideas were adapted by O’Hara [1992; 1994] who defined a two-parameter family of knot functionals on smooth curves. Motivated by the Coulomb potential of an equidistributed charge on a smooth curve, he set

$$E^{\alpha,p}(\gamma) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left( \frac{1}{|\gamma(u+w) - \gamma(u)|^\alpha} - \frac{1}{D_\gamma(u+w, u)^\alpha} \right)^p |\gamma'(u+w)| |\gamma'(u)| dw du. \quad (1)$$

Here  $\alpha, p > 0$ , and  $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ . The quantity  $D_\gamma(u+w, u)$  measures the intrinsic distance between  $\gamma(u+w)$  and  $\gamma(u)$  on the curve  $\gamma$ .

The mathematical idea behind this definition is quite easy to understand. To obtain a self-avoiding functional, one intends to penalize pairs of points  $(\gamma(u), \gamma(v))$  having a small Euclidean distance. This motivates the first term  $|\gamma(u+w) - \gamma(u)|^{-1}$  which has to be taken to an adequate power to produce a singularity. On the other hand, this *Ansatz* does not take into account that neighboring points  $\gamma(u), \gamma(u+\varepsilon)$  naturally have a small Euclidean distance. Therefore we have to add some sort of regularization: subtracting the intrinsic distance  $D_\gamma(u, v)$  taken to the same negative power as the Euclidean distance, the singularities stemming from neighboring points are cancelled while those for distant points are not essentially affected. Averaging over all pairs of points is achieved by the integration process, and the factors  $|\gamma'(u+w)| |\gamma'(u)|$  guarantee invariance under reparametrization.

Be aware that not all functionals of the family (1) are in fact knot energies. Some are far too singular, taking the value  $+\infty$  only, while others are not strong enough to ensure self-avoidance. In fact, in order to gain knot energies one has to satisfy  $\alpha p \geq 2$  and  $(\alpha - 2)p < 1$ .

Particular attention was devoted to  $E^{2,1}$  which O’Hara introduced in [1991]. Later on, Freedman, He, and Wang [1994] coined the name *Möbius energy* due to their seminal discovery that it is in fact invariant under Möbius transformations. The immediate consequence that circles are unique minimizers among all closed curves could be generalized by analytical means to the entire family (1) by Abrams, Cantarella, Fu, Ghomi, and Howard [2003].

In the case of the Möbius energy, Freedman, He, and Wang [1994] could establish the existence of minimizers within prime knot classes only—Kusner and Sullivan [1998] even conjecture that this

statement does not hold for composite knots—whereas for  $\alpha p > 2$  the existence of minimizers within any knot class has already been proven by O’Hara [1994].

We briefly mention some further contributions regarding the Möbius energy. Kim and Kusner [1993] constructed extremizing torus knots that are explicitly computable using the calculus of residues, Kusner and Sullivan [1998] performed numerical experiments, Rawdon and co-authors [2006; 2010] established error estimates.

The subsequent article by He on the Euler-Lagrange equation and heat flow for the Möbius energy [2000] was the starting point both for investigating the corresponding gradient flow [Blatt, 2010, 2012b] and the regularity of stationary points [Reiter, 2010, 2012], both for the one-parameter sub-family  $E^{\alpha,1}$ ,  $\alpha \in [2, 3)$ . After a first attempt [Blatt & Reiter, 2008], the identification of the energy spaces [Blatt, 2012a] led to a significant improvement of the latter result [Blatt & Reiter, 2013a; Blatt *et al.*, 2012].

### *Integral Menger curvature*

In the sequel, many other knot energies came up. One important example is the reciprocal of *thickness* which has been investigated by Litherland, Simon, Durumeric, and Rawdon [1999] for  $C^2$ -curves. Diao, Ernst, and Janse van Rensburg [1999] considered a slightly different notion on  $C^1$ -curves. See Ashton *et al.* [2011] for further references.

Gonzalez and Maddocks [1999] provided an alternative characterization—not requiring any initial regularity—by the minimum value of the circumradius function  $R(\gamma(s), \gamma(t), \gamma(u))$  over all triplets of points of  $\gamma$ . Here

$$R(x, y, z) := \frac{|y - z| |y - x| |z - x|}{|(y - x) \wedge (z - x)|} = \frac{|y - z|}{\sin \angle (y - x, z - x)}, \quad x, y, z \in \mathbb{R}^d, \quad (2)$$

denotes the radius of the circle passing through the three points  $x$ ,  $y$ , and  $z$ . The reciprocal of  $R$  is called *Menger curvature*, crediting Menger’s work [1930]. Minimizers of  $\gamma \mapsto \sup_{s \neq t \neq u \neq s} R(\gamma(s), \gamma(t), \gamma(u))^{-1}$  (with fixed length, in a prescribed isotopy class) are referred to as *tight knots* (or *ideal knots*).

Existence of tight knots has been proven in [Gonzalez *et al.*, 2002a; Cantarella *et al.*, 2002; Gonzalez & de la Llave, 2003], for regularity questions we refer to [Schuricht & von der Mosel, 2003, 2004; Cantarella *et al.*, 2011]. Despite of the lack of an explicit analytical characterization of the shape of a (non-trivial) tight knot, there are several contributions to discretization and numerical visualization, see [Ashton *et al.*, 2011; Cantarella *et al.*, 2005; Carlen & Gerlach, 2012; Carlen *et al.*, 2005; Gerlach, 2010; Gonzalez *et al.*, 2002b; Smutny, 2004]. An interesting packing problem, namely *maximizing* length for prescribed thickness on the two-dimensional sphere  $\mathbb{S}^2$ , has been tackled by Gerlach and von der Mosel [2011a; 2011b].

The difficulties regarding regularity of tight knots is essentially due to taking infima which is a non-smooth operation. It is reasonable to replace the infima by integration which leads to the functional

$$\mathcal{M}_p(\gamma) := \iiint_{(\mathbb{R}/\mathbb{Z})^3} \frac{ds \, dt \, du}{R(s, t, u)^p}, \quad (3)$$

that appears as  $U_{p,3}[\cdot]$  in [Gonzalez & Maddocks, 1999] amongst several intermediate versions. The element  $\mathcal{M}_2$  is referred to as *total Menger curvature*. Léger [1999] could show that one-dimensional Borel sets with finite total Menger curvature are 1-rectifiable, for  $\mathcal{M}_p$ ,  $p \neq 2$ , see Lin and Mattila [2001]. Mel’nikov and Verdera [1995; 1995; 2001] provided a new proof of the  $L^2$ -boundedness of the Cauchy integral operator on Lipschitz graphs exploiting an identity involving the Menger curvature. A more general setting on metric spaces is discussed in Hahlomaa [2008].

A characterization of the Cauchy integral in terms of analytic capacity and uniform rectifiability is given in Mattila et al. [1996]. For the relation to Painlevé's problem we refer to Dudziak [2010] and references therein.

The functional (3) has been thoroughly investigated by Strzelecki, Szumańska and von der Mosel [2010; 2012] in the case  $p > 3$  which we will presume throughout this paragraph. Their main result is a geometric Morrey-Sobolev embedding theorem [2010, Thm 1.2], stating that the arc-length reparametrization of any finite-energy curve being a local homeomorphism is  $C^{1,1-3/p}$  and its image is  $C^1$ -diffeomorphic to the circle. It is a consequence of a more general geometric Morrey space embedding theorem [2010, Thm 1.3]. The former result can be strengthened as follows: any arc-length parametrized curve being a local homeomorphism has finite energy if and only if it is an  $W^{2-2/p,p}$ -embedding [Blatt, 2013]. The functional  $\mathcal{M}_p$  is self-repulsive, i.e. a knot energy, and possesses minimizers within any knot class [Strzelecki et al., 2012, Cor. 2.3].

Hermes [2012] computed the first variation and performed several numerical experiments on the gradient flow. The regularizing behaviour of this energy extends to general measurable sets  $X \subset \mathbb{R}^n$  (instead of a curve  $\gamma$ ), see Scholtes [2012] and references therein. For higher-dimensional objects one first has to find an appropriate notion for  $R$ . We refer to [Lerman & Whitehouse, 2011, 2009] (for  $\mathcal{M}_2$ ), [Strzelecki & von der Mosel, 2005, 2006, 2011a] and [Blatt & Kolasiński, 2011; Kolasiński, 2012a,b; Kolasiński et al., 2012; Kolasiński & Szumańska, 2011].

Introducing powers in the definition of the circumcircle

$$R^{(p,q)}(x, y, z) := \frac{(|y-z||y-x||z-x|)^p}{|(y-x) \wedge (z-x)|^q} = \frac{|y-z|^p |y-x|^{p-q} |z-x|^{p-q}}{\sin \angle(y-x, z-x)^q} \quad (2')$$

gives rise to a full two-parameter family of *generalized integral Menger curvature* functionals [Blatt & Reiter, 2013b]

$$\text{intM}^{(p,q)} := \iiint_{\mathbb{R}/\mathbb{Z} \times [-\frac{1}{2}, \frac{1}{2}]^2} \frac{|\gamma'(u)| |\gamma'(u+v)| |\gamma'(u+w)|}{R^{(p,q)}(\gamma(u), \gamma(u+v), \gamma(u+w))} dw dv du, \quad p, q > 0. \quad (3')$$

Of course,  $\mathcal{M}_p = 2^p \text{intM}^{(p,p)}$ . As before,  $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ . The elements of this family are knot energies for certain ranges of parameters only. More precisely, they are self-avoiding if and only if  $p \geq \frac{2}{3}q + 1$ . On the other hand, they are non-singular iff  $p < q + \frac{2}{3}$ .

In contrast to classical energies  $\mathcal{M}_p$ , the integrand of the generalized energies lacks an appealing geometric interpretation. But we will see that they have nicer analytic properties, basically due to the fact that their first variation leads to non-degenerate elliptic operator if  $q = 2$ .

### *Tangent-point energies*

Another variant of these three-point circle-based functionals, already appearing as  $U_{p,2}[\mathcal{C}]$  in the article by Gonzalez and Maddocks [1999], are the *tangent-point energies*

$$\mathcal{E}_q(\Gamma) := \iint_{[0,L]^2} \frac{ds dt}{r_\Gamma(s,t)^q}, \quad q \geq 2, \quad (4)$$

where  $r_\Gamma(s,t) := \lim_{u \rightarrow t} R(\gamma(s), \gamma(t), \gamma(u))$  denotes the radius of the smallest circle tangent to  $\gamma(t)$  and passing through  $\gamma(s)$ . It amounts to half the ratio of the squared distances  $|\Gamma(s) - \Gamma(t)|^2$  to the distance of the current tangent line  $\ell(t) = \Gamma(t) + \mathbb{R}\Gamma'(t)$  from  $\Gamma(s)$ , i.e.

$$r_\Gamma(s,t) = \frac{|\Gamma(s) - \Gamma(t)|^2}{2 \text{dist}(\Gamma(s), \ell(t))}. \quad (5)$$

The first main result by Strzelecki and von der Mosel on the tangent-point energies defined in (4) is that the image of  $\Gamma$  is a one-dimensional topological manifold if  $\mathcal{E}_q(\Gamma)$ ,  $q \geq 2$ , is finite [Strzelecki & von der Mosel, 2010, Thms. 1.1 and 1.4]. Restricting to  $q > 2$  for the remainder of this paragraph, the manifold is of class  $C^{1,1-2/q}$  [2010, Thm. 1.3]. If the Hausdorff distance of two given curves is bounded by some threshold only depending on the respective tangent-point energy values, they are ambient isotopic [2010, Thm. 1.2]. Furthermore,  $\mathcal{E}_q$  is in fact a knot energy [2010, Prop. 5.1], which improves an earlier result requiring higher regularity by Sullivan [2002], who used this functional to approach *ropelength*. Furthermore, there are minimizers within every knot class [Strzelecki *et al.*, 2012, Cor. 2.3].

The above-mentioned result can be refined as follows, providing an explicit characterization of finite-energy curves: the image of  $\Gamma$  is an embedded manifold of class  $W^{2-1/q,q} \subset C^{1,1-2/q}$  if and only if  $\mathcal{E}_q(\Gamma)$  is finite [Blatt, 2011]. Results for higher-dimensional analoga to  $\mathcal{E}_q$  can be found in [Strzelecki & von der Mosel, 2011b; Blatt, 2011; Kolasinski *et al.*, 2012].

Starting point for our investigation in [Blatt & Reiter, 2012] is again decoupling of powers in the nominator and denominator of  $r_\Gamma$ . Using  $\text{dist}(\gamma(u+w), \ell(u)) = \left| P_{\gamma'(u)}^\perp(\gamma(u+w) - \gamma(u)) \right|$ , where

$$P_{\gamma'(u)} a := \left\langle a, \frac{\gamma'(u)}{|\gamma'(u)|} \right\rangle \frac{\gamma'(u)}{|\gamma'(u)|}, \quad P_{\gamma'(u)}^\perp a := a - P_{\gamma'(u)} a \quad \text{for } a \in \mathbb{R}^n \quad (6)$$

denote the projection onto the tangential and normal part along  $\gamma$  respectively, we define

$$\tilde{r}_\Gamma^{(p,q)}(t,s) := \frac{|\Gamma(s) - \Gamma(t)|^p}{\text{dist}(\ell(t), \Gamma(s))^q} \quad \text{for } p, q \geq 1 \quad (5')$$

which leads to the *generalized tangent-point energies*

$$\text{TP}^{(p,q)}(\gamma) = \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\left| P_{\gamma'(u)}^\perp(\gamma(u+w) - \gamma(u)) \right|^q}{|\gamma(u+w) - \gamma(u)|^p} |\gamma'(u+w)| |\gamma'(u)| \, dw \, du \quad (4')$$

where  $p \in [q+2, 2q+1)$ . Note that for  $\gamma \in C^{1,1}$ , by

$$P_{\gamma'(u)}^\perp(\gamma(u+w) - \gamma(u)) = P_{\gamma'(u)}^\perp(\gamma(u+w) - \gamma(u) - w\gamma'(u)),$$

the integrand in (4') behaves like  $\mathcal{O}(|w|^{2q-p})$  as  $w \rightarrow 0$ .

Considering only continuously differentiable curves, our method of proof [Blatt & Reiter, 2012]—based on techniques developed in [Blatt, 2011]—works entirely without using the techniques by von der Mosel and Strzelecki [2010] which are quite involved and technical. In fact, we developed a completely independent and fast approach towards existence and regularity of minimizers completely avoiding decay estimates of Jones' beta numbers that play a fundamental rôle in [Strzelecki & von der Mosel, 2010].

In fact, there are many desirable properties of a knot energy one could ask for. Does it penalize pulling-tight (which does not lead to a self-intersection) of small knotted arcs? Are there only finitely many distinct knot types under each energy level? Does the knot energy detect the unknot? Is the round circle the unique minimizer among all curves? A recent account that comments on these questions for several energy functionals including those presented in this text, providing proofs and references, is given by Strzelecki, Szumańska and von der Mosel [2012].

In the following we will concentrate on the analysis of stationary points. Of course, by identifying energy spaces as in the following section we obtain *a priori* estimates which are useful for proving the above-mentioned properties.

## 2. How fractional Sobolev spaces come into play

The first important step in the investigation of these functionals is to identify the energy spaces, i.e. the subset of embedded  $C^{0,1}$ -curves on which the respective energy takes finite values. In order to avoid strange parametrizations, we restrict ourselves to arc-length parametrizations.

As to O'Hara's energies, the integrand then reads (for  $p = 1$ )

$$\frac{1}{|\gamma(u+w) - \gamma(u)|^\alpha} - \frac{1}{|w|^\alpha} = \left| \frac{\gamma(u+w) - \gamma(u)}{w} \right|^{-\alpha} \frac{\left(1 - \left| \frac{\gamma(u+w) - \gamma(u)}{w} \right|^\alpha\right)}{|w|^\alpha}. \quad (7)$$

The main idea is now that, using arc-length parametrization and  $|a - b|^2 = 2 - 2\langle a, b \rangle$  for  $|a| = |b| = 1$  as well as  $1 - x^\alpha \approx 1 - x^2$  for  $0 < c \leq x \leq 1$ ,

$$\begin{aligned} 1 - \left| \frac{\gamma(u+w) - \gamma(u)}{w} \right|^\alpha &\approx 1 - \left| \frac{\gamma(u+w) - \gamma(u)}{w} \right|^2 \\ &= 1 - \iint_{[0,1]^2} \langle \gamma'(u + \vartheta_1 w), \gamma'(u + \vartheta_2 w) \rangle \, d\vartheta_1 \, d\vartheta_2 \\ &= \frac{1}{2} \iint_{[0,1]^2} |\gamma'(u + \vartheta_1 w) - \gamma'(u + \vartheta_2 w)|^2 \, d\vartheta_1 \, d\vartheta_2. \end{aligned}$$

Furthermore,  $(u, w) \mapsto \left| \frac{\gamma(u+w) - \gamma(u)}{w} \right|$  is continuous, taking values in some compactum in  $(0, 1]$  on  $\mathbb{R}/\mathbb{Z} \times [-\frac{1}{2}, \frac{1}{2}]$ . By means of the functions  $g_\alpha : x \mapsto x^{-\alpha}$  and  $h_\alpha : x \mapsto \frac{1-x^\alpha}{1-x^2}$ , which are analytic and non-negative on  $(0, 1]$  provided  $\alpha \geq 2$ , we may express the right-hand side of (7) by

$$\frac{1}{2} g_\alpha \left( \left| \frac{\gamma(u+w) - \gamma(u)}{w} \right| \right) h_\alpha \left( \left| \frac{\gamma(u+w) - \gamma(u)}{w} \right| \right) \iint_{[0,1]^2} \frac{|\gamma'(u + \vartheta_1 w) - \gamma'(u + \vartheta_2 w)|^2}{|w|^\alpha} \, d\vartheta_1 \, d\vartheta_2.$$

For arbitrary  $p > 0$  this leads us to

$$\left( \frac{1}{|\gamma(u+w) - \gamma(u)|^\alpha} - \frac{1}{|w|^\alpha} \right)^p \approx \frac{|\gamma'(u + \vartheta_1 w) - \gamma'(u + \vartheta_2 w)|^{2p}}{|w|^{\alpha p}}.$$

Integrating over  $\mathbb{R}/\mathbb{Z} \times [-\frac{1}{2}, \frac{1}{2}]$  this resembles (the  $2p$ -th power of) the *Sobolev-Slobodeckii seminorm*

$$[f]_{W^{s,\varrho}} := \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f(u+w) - f(u)|^\varrho}{|w|^{1+\varrho s}} \, dw \, du \right)^{1/\varrho} \quad (8)$$

where  $f = \gamma'$ ,  $s = \frac{\alpha p - 1}{2p}$ , and  $\varrho = 2p$ . Full details of this argument can be found in [Blatt, 2012a]. In other words, an embedded arc-length parametrized curve  $\gamma$  belongs to  $W^{1+\alpha/2-1/(2p), 2p}$  if and only if its energy  $E^{\alpha,p}(\gamma)$  is finite. Here,  $W^{k+s,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) := \{f \in W^{k,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \mid \|f\|_{W^{k+s,\varrho}} < \infty\}$  is equipped with the norm  $\|f\|_{W^{k+s,\varrho}} := \|f\|_{W^{k,\varrho}} + [f^{(k)}]_{W^{s,\varrho}}$  for  $k \in \mathbb{N} \cup \{0\}$ ,  $s \in (0, 1)$ , and  $\varrho \in [1, \infty)$ , whereas  $W^{k,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  denotes the usual Sobolev space. One of the most important features is the (compact) embedding

$$W^{k+s,\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \hookrightarrow C^{k,s-1/\varrho}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \quad (9)$$

in the *sub-critical\** range  $s \in (\varrho^{-1}, 1)$ .

\* The range of parameters that lead to a *sub-critical* Euler-Lagrange equation is referred to as *sub-critical* in this text. It is also characterized by the existence of the embedding (9).—In contrast, the corresponding range is called *super-critical* in [Strzelecki et al., 2012] as it lies above the respective critical value for which the corresponding energy is scale-invariant.



For the tangent-point energies, we may proceed in a similar way, making use of  $\gamma'' \perp \gamma'$ . The integrand reads

$$\begin{aligned} \frac{\left| P_{\gamma'(u)}^\perp (\gamma(u+w) - \gamma(u)) \right|^q}{|\gamma(u+w) - \gamma(u)|^p} &\approx \frac{\left| P_{\gamma'(u)}^\perp \int_0^1 \gamma'(u + \vartheta w) d\vartheta \right|^q}{|w|^{p-q}} \\ &= \frac{\left| \int_0^1 P_{\gamma'(u)}^\perp (\gamma'(u + \vartheta w) - \gamma'(u)) d\vartheta \right|^q}{|w|^{p-q}} \approx \int_0^1 \frac{|\gamma'(u + \vartheta w) - \gamma'(u)|^q}{|w|^{p-q}} d\vartheta, \end{aligned}$$

so  $\text{TP}^{(p,q)}(\gamma) < \infty \iff \gamma \in W^{(p-1)/q,q}$  for arc-length parametrized  $\gamma$ . Of course, this is merely a hand-waving argument that illustrates the idea, see [Blatt, 2011; Blatt & Reiter, 2012] for details.

The situation is a little bit more delicate for the integral Menger curvature as we have to deal with three integrations, cf. [Blatt, 2013; Blatt & Reiter, 2013b]. We skip this part for sake of simplicity.

### 3. Towards regularity theory

The next step is to compute the derivative. Formally this is a straight-forward calculation, but one has to take into account that this involves interchanging differentiation and integration. Especially in the case of O'Hara's energies this is a quite subtle process which can be tackled by a careful analysis of approximating energy functionals, see [Blatt & Reiter, 2013a]. In this particular situation, the first variation turns out to be a principal value integral of type  $\lim_{\varepsilon \searrow 0} \iint_{|w| \geq \varepsilon} \dots$ .

The situation is quite different for the integral Menger curvature which has been pointed out by Hermes [2012]. Here the integrand of the first variation is  $L^1$ . The same applies to the tangent-point energies, cf. [Blatt & Reiter, 2012]. This might in fact be seen as an advantage of Menger curvature based knot energies over O'Hara's energies.

For the majority of parameters, the Euler-Lagrange operators seem to be somehow “degenerate”. Therefore our regularity result applies to the non-degenerate sub-critical case, namely  $E^{\alpha,1}$ ,  $\alpha \in (2,3)$ , for O'Hara's energies,  $\text{intM}^{(p,2)}$ ,  $p \in (\frac{7}{3}, \frac{8}{3})$ , for the integral Menger curvature, and  $\text{TP}^{(p,2)}$ ,  $p \in (4,5)$ , for the tangent-point energies. As these energies are not scaling invariant, we have to fix length.

**THEOREM** (Stationary points are smooth [Blatt & Reiter, 2013a, 2012, 2013b]). *Any stationary point of  $E^{\alpha,1}$ ,  $\text{intM}^{(p,2)}$ , and  $\text{TP}^{(p,2)}$  in the non-degenerate sub-critical case with respect to fixed length, being injective and parametrized by arc-length, is  $C^\infty$ -smooth.*

Surprisingly, up to some technical difficulties which have to be dealt with separately, the respective proofs roughly follow the same lines—yet another indication that the three families of energies are not too different from the perspective of an analyst.

Let us briefly outline the common structure of the proofs with some details for the case of O'Hara's energies. First, one has to identify the highest-order term in the Euler-Lagrange equation and show that it is in fact an elliptic operator. Now the remainder consists of lower-order terms that can be brought into a common form which allows to deal with them simultaneously.

For O'Hara's knot energies  $E^{\alpha,1}$ ,  $\alpha \in (2,3)$ , these two steps show that a stationary curve  $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  of finite energy parametrized by arc-length satisfies the Euler-Lagrange

equation

$$Q^{(\alpha)}(\gamma; h) = R^{(\alpha)}(\gamma; h) \quad (10)$$

for all  $h \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$  where  $Q^{(\alpha)}$  is a symmetric bilinear operator given by

$$Q^{(\alpha)}(\gamma; h) = \sum_{k \in \mathbb{Z}} q_k |k|^{1+\alpha} \hat{\gamma}(k) \overline{\hat{h}(k)} \quad (11)$$

for some sequence  $(q_k)_{k \in \mathbb{Z}} \in \ell^\infty(\mathbb{R})$  with  $\lim_{|k| \rightarrow \infty} q_k \rightarrow q_\infty > 0$ . Here  $\hat{f}(k)$  denotes the  $k$ -th Fourier coefficient of  $f$ . Note that

$$f \in W^{s,2} \iff \sum_{k \in \mathbb{Z}} |k|^{2s} |\hat{f}_k|^2 < \infty. \quad (12)$$

If now  $\gamma \in W^{\frac{\alpha+1}{2}+\sigma,2}$  for some  $\sigma \geq 0$ , the remainder term  $R^{(\alpha)}(\gamma; \cdot)$  defines a bounded (conjugate-) linear operator on  $W^{\frac{3}{2}-\tilde{\sigma}}$  for all  $\tilde{\sigma} < \sigma$ ; so

$$R^{(\alpha)}(\gamma; h) = \sum_{k \in \mathbb{Z}} r_k |k|^{\frac{3}{2}-\tilde{\sigma}} \overline{\hat{h}(k)}$$

for some sequence  $(r_k)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{C})$ . Plugging this into (10), using (11), and testing with basis functions  $h(u) = e^{2\pi i k u}$ , we arrive at

$$q_k |k|^{1+\alpha} \hat{\gamma}(k) = r_k |k|^{\frac{3}{2}-\tilde{\sigma}} \quad \text{for all } k \in \mathbb{Z},$$

thus

$$\left( q_k |k|^{(1+\alpha)-\frac{3}{2}+\tilde{\sigma}} \hat{\gamma}(k) \right)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{C}).$$

Since  $q_k \rightarrow q_\infty > 0$  as  $|k| \rightarrow \infty$ , Equation (12) implies

$$\gamma \in W^{1+\alpha-\frac{3}{2}+\tilde{\sigma},2} = W^{\frac{\alpha+1}{2}+\sigma+(\frac{\alpha-2}{2}+\tilde{\sigma}-\sigma),2}$$

for all  $\tilde{\sigma} < \sigma$ . As  $\alpha > 2$ , we can finally iterate the argument to show that critical points are smooth and thus prove the Theorem.

Although our method fundamentally relies on the sub-critical range on which we can make use of the embedding (9), the statement of the theorem above should also hold for the non-degenerate *critical case*. For the Möbius-energy  $E^{2,1}$ , we could establish a corresponding result [Blatt *et al.*, 2012] that—using sophisticated methods from harmonic analysis—improves previous results by Freedman, He, Wang [1994] and He [2000]. In light of the technical difficulties arising here we expect these situation for  $\text{intM}^{(7/3,2)}$  and  $\text{TP}^{(4,2)}$  to be much more involved.

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