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Möbius invariant energy of tori of revolution

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Abstract. We show that among tori of revolution the Clifford torus gives the minimum value of the Möbius invariant surface energy defined by Auckly and Sadun.

1. Introduction and main result

Recently Fernando C. Marques and André Neves (Marques & Neves) proved the Willmore conjecture, namely, they showed the following. Let Σ be an immersed torus in $\mathbb{R}^3$. Let κ_1 and κ_2 be principal curvatures. The Willmore functional is given by

$$\mathcal{W}(\Sigma) = \int_\Sigma \left(\frac{\kappa_1+\kappa_2}{2}\right)^2 d\Sigma = \int_\Sigma \left(\frac{\kappa_1-\kappa_2}{2}\right)^2 d\Sigma,$$

where the second equation is the consequence of the Gauss-Bonnet theorem. It is known to be invariant under Möbius transformations of $\mathbb{R}^3$. Then the Willmore conjecture, now the theorem of Marques and Neves, asserts that $\mathcal{W}(\Sigma) \geq 2\pi^2$ and that the equality holds if and only if Σ is a torus of revolution whose generating circle has radius 1 and center at distance $\sqrt{2}$ from the axis of revolution up to a Möbius transformation, in other words, if and only if Σ is the image of a stereographic projection of the Clifford torus

$$\left\{(z_1,z_2)\in\mathbb{C}\times\mathbb{C}\,\middle|\,|z_1|=|z_2|=1/\sqrt{2}\right\}\subset S^3=\left\{(z_1,z_2)\in\mathbb{C}\times\mathbb{C}\,\middle|\,|z_1|^2+|z_2|^2=1\right\}.$$

In this paper, we give another characterization of the Clifford torus using the surface energy introduced by David Auckly and Lorenzo Sadun ((1)), which is also invariant under Möbius transformations. To be precise, we have not yet succeeded in proving that the Clifford torus gives the minimum energy among all the immersed tori[1]. We only show that it gives the minimum energy among one-parameter family of tori of revolution. Since the energy we use is conformally invariant, it follows that the Clifford torus gives the minimum energy among Dupin cyclides. As the surface energy that we use in this paper is generelization of knot energy, we start with the review of it.

Energy of knots was introduced in (5) motivated to give a functional on the space of knots that can produce a representative configuration of a knot for each knot type as an embedding

[1] To show it, it suffice to show that the Clifford torus gives the minimum energy among all the embedded tori, since the energy blows up if a torus has a double point

that minimizes the energy in the knot type. Let K be a knot and x be a point on it. Define

$$V(x;K) = \lim_{\varepsilon\to 0}\left(\int_{K\setminus B_\varepsilon(x)} \frac{dy}{|x-y|^2} - \frac{2}{\varepsilon}\right), \quad \text{and} \quad E(K) = \int_K V(x;K)dx, \tag{1}$$

where $B_\varepsilon(x)$ is a ball with center x and radius ε. Let us call a process as in the definition of $V(x;K)$ in (1) the *renormalization* in this paper. In general, when we are interested in a diverging integral, we first restrict the integration to the complement of an ε-neighbourhood of the set where the integrand blows up, then expand the result in a Laurent series in ε, and finally take the constant term. In the case of a knot, the integrand of V in (1) blows up at the one-point set $\{x\}$. There are two ways to define an ε-neighbourhood of it, according to the choice of the distance between a pair of points on the knot; either the chord length as in (1) or the arc-length along the knot as in (5). Both types of the renormalization give the same result ((6)).

The energy $E(K)$ in (1) was proved to be invariant under Möbius transformations by Freedman, He, and Wang ((2)), which is the reason why it is sometimes called the *Möbius energy* of knots.

After this energy was found, it has been generalized to functionals that can measure geometric complexity of knots, surfaces, and in general, submanifolds ((1), (3), et al.). Among several ways of generalization to surface energy, in this paper we study the one by Auckly and Sadun that uses a similar renormalization process as in (1).

Let S be an embedded surface in $\mathbb{R}^3$ without boundary and x be a point in S. Define the *renormalized r^{-4}-potential* V and the *renormalized r^{-4}-potential energy* E by

$$\begin{aligned} V(x;S) &= \lim_{\varepsilon\to 0}\left(\int_{S\setminus B_\varepsilon(x)} \frac{d^2y}{|x-y|^4} - \frac{\pi}{\varepsilon^2} + \frac{\pi\Delta(x)}{16}\log\left(\Delta(x)\varepsilon^2\right) + \frac{\pi K(x)}{4}\right), \\ E(S) &= \int_S V(x;S)d^2x, \end{aligned} \tag{2}$$

where d^2y and d^2x mean the volume element of S, $\Delta(x)$ is given by $\Delta(x) = (\kappa_1(x) - \kappa_2(x))^2$, and $K(x)$ is the Gauss curvature; $K = \kappa_1\kappa_2$. This energy $E(S)$ was proved to be invariant under Möbius transformations in (1). It blows up as S degenerates to an immersed surface with double points.

As was pointed out in (1), the choice of the log-term in (2) is not the unique reasonable one. The reason why there is a factor $\Delta(x)$ in the log-term is to make the resulting energy scale invariant, but $c\Delta(x)$ $(c \neq 0)$ also has the same effect. Thus there is ambiguity in the definition of the renormalized potential.

In (1), Auckly and Sadun has computed the energy of spheres and planes and the potentials V of an infinitely long straight cylinder and a surface called dimple. In this article we compute the energies of one-parameter family of tori of revolution.

Theorem: *Let T_R be a torus of revolution whose generating circle has radius* 1 *and center at distance R $(R > 1)$ from the axis of revolution. Then the renormalized potential energy is given by*

$$E(T_R) = \frac{\pi^3}{2\sqrt{R^2-1}}\left(R^2\left(3\log 2 - 1\right) + 2 - \frac{2}{R^2}\right).$$

Corollary: *Among tori of revolution, a stereographic projection of the Clifford torus gives the minimum energy.*

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Proof of Corollary: Since

$$\frac{d}{dR}E(T_R)=\frac{\pi^3(R^2-2)\left((R^2-2)^2+3\,R^4\log\left(4/e\right)\right)}{4R^3\left(R^2-1\right)^{3/2}},$$

the energy takes the minimum value $\pi^3(6\log 2-1)/2$ when $R=\sqrt{2}$. $\square$

Problem: (1) Does $T_{\sqrt{2}}$ give the minimum energy among all the embedded tori in $\mathbb{R}^3$, hence among all the immersed tori in $\mathbb{R}^3$?

(2) When we change the power of $|x-y|$ in the denominator in (2) from 4 to any number λ, we obtain a new potential energy $E_{r^{-\lambda}}$ after suitable renormalization. It is no longer scale invariant when $\lambda\neq 4$. What is $R=R(\lambda)$ that makes T_R give the minimum energy $E_{r^{-\lambda}}$ after rescaling to have area 1?

2. Computation of the energy of a torus

Proof of Theorem: Let T be a torus of revolution parametrized by

$$p(u,v)=((R+\cos u)\cos v,(R+\cos u)\sin v,\sin u).$$

Let $x=p(\alpha,0)=(R+\cos\alpha,0,\sin\alpha)$ be a point on T. First we fix α and compute the renormalized r^{-4}-potential of T at x. This is the main part of the paper. Some of the complicated computation to obtain expansion in seiries in ε has been done with the help of Maple.

Let Dist $=$ Dist(u,v) be the distance between x and a point $y=p(u,v)$:

$$\begin{aligned}\mathrm{Dist}^2&=|x-y|^2=|p(u,v)-p(\alpha,0)|^2\\&=((R+\cos\alpha)-(R+\cos u)\cos v)^2+(R+\cos u)^2\sin^2 v+(\sin\alpha-\sin u)^2\\&=2R^2+2+2R\,(\cos\alpha+\cos u)-2\sin\alpha\sin u-2\,(R+\cos\alpha)\,(R+\cos u)\cos v.\end{aligned}$$

To compute the integral $\int_{T\setminus B_\varepsilon(x)}|x-y|^{-4}d^2y$, we will divide the domain of the integration into four parts and use several kinds of changes of variables. We may assume without loss of generality that $u\in[\alpha-\pi,\alpha+\pi]$ and $v\in[-\pi,\pi]$. Put

$$\theta=\frac{\pi}{2}-\frac{u-\alpha}{2},\ t=2\cos\theta=2\sin\frac{u-\alpha}{2}.$$

Then, as $(u-\alpha)/2\in[-\pi/2,\pi/2]$ we have $\theta\in[0,\pi]$ and $\cos(u-\alpha)/2\geq 0$. Therefore,

$$\cos u=\frac{2-t^2}{2}\cos\alpha-\frac{t\sqrt{4-t^2}}{2}\sin\alpha,\quad \sin u=\frac{t\sqrt{4-t^2}}{2}\cos\alpha+\frac{2-t^2}{2}\sin\alpha,$$

which implies that the distance can be expressed as

$$\begin{aligned}\mathrm{Dist}^2&=t^2+\left[4(R+\cos\alpha)^2-2(R+\cos\alpha)\cos\alpha\cdot t^2-2(R+\cos\alpha)\sin\alpha\cdot t\sqrt{4-t^2}\right]\sin^2\frac{v}{2}\\&=4\left[\cos^2\theta+(R+\cos\alpha)\left(R+\cos\alpha-2\cos\alpha\cos^2\theta-2\sin\alpha\sin\theta\cos\theta\right)\sin^2\frac{v}{2}\right]\\&=4\left[\cos^2\theta+(R+\cos\alpha)\,(R-\cos(\alpha-2\theta))\sin^2\frac{v}{2}\right].\end{aligned}$$

Define $b = b(\theta)$ and $c = c(t)$ by

$$\begin{aligned} c(t) &= 2(R+\cos\alpha)\left[2(R+\cos\alpha) - \cos\alpha\cdot t^2 - \sin\alpha\cdot t\sqrt{4-t^2}\,\right] \\ b &= \frac{c}{4} = (R+\cos\alpha)\,(R-\cos(\alpha-2\theta))\,. \end{aligned} \tag{3}$$

Then

$$\mathrm{Dist}^2 = t^2 + c(t)\sin^2\frac{v}{2} = 4\left(\cos^2\theta + b\sin^2\frac{v}{2}\right).$$

Put

$$V(\varepsilon,x) = \iint_{\mathrm{Dist}\ge\varepsilon} \frac{d^2y}{|x-y|^2}.$$

Since the area element of T is given by $d^2y = (R+\cos u)\,dudv$, $V(\varepsilon,x)$ is given by

$$\iint_{\mathrm{Dist}\ge\varepsilon} \frac{(R+\cos u)}{\left(2R^2+2+2R\,(\cos\alpha+\cos u) - 2\sin\alpha\sin u - 2\,(R+\cos\alpha)\,(R+\cos u)\cos v\right)^2}\,dudv$$

$$= \iint_{\mathrm{Dist}\ge\varepsilon} \frac{R + \left(\frac{2-t^2}{2}\cos\alpha - \frac{t\sqrt{4-t^2}}{2}\sin\alpha\right)}{\left(t^2 + c(t)\sin^2\frac{v}{2}\right)^2}\,\frac{2dt}{\sqrt{4-t^2}}\,dv \tag{4}$$

$$= \frac{1}{8}\iint_{\mathrm{Dist}\ge\varepsilon} \frac{(R-\cos(\alpha-2\theta))}{\left(\cos^2\theta + b\sin^2\frac{v}{2}\right)^2}\,d\theta dv. \tag{5}$$

The integration by dv an be executed as

$$\begin{aligned} &\int \frac{dv}{(A^2+B\sin^2\frac{v}{2})^2} \\ &= \frac{2A^2+B}{A^3(A^2+B)^{\frac{3}{2}}}\tan^{-1}\left(\frac{\sqrt{A^2+B}}{A}\tan\frac{v}{2}\right) + \frac{B\tan\frac{v}{2}}{A^2(A^2+B)\left((A^2+B)\tan^2\frac{v}{2}+A^2\right)}\,. \end{aligned} \tag{6}$$

We devide the domain of the integration $T\setminus B_\varepsilon(x)$ into eight parts as is illustrated in Figure 1. Using symmetry, we have only to compute integrals over the following four regions

$$\begin{aligned} &\left\{p(u,v)\,|\,2\sin^{-1}(\varepsilon/2) \le u-\alpha \le \pi, 0\le v\le\pi\right\}, \\ &\left\{p(u,v)\,|\,0 \le u-\alpha \le \sin^{-1}(\varepsilon/2), 0\le v\le\pi, |p(u,v)-x|\ge\varepsilon\right\}, \\ &\left\{p(u,v)\,|\,-\pi \le u-\alpha \le -2\sin^{-1}(\varepsilon/2), 0\le v\le\pi\right\}, \\ &\left\{p(u,v)\,|\,-2\sin^{-1}(\varepsilon/2) \le u-\alpha \le 0, 0\le v\le\pi, |p(u,v)-x|\ge\varepsilon\right\}, \end{aligned}$$

which we denote by I_1, I_2, I_3, and I_4 respectively. We remark that the regions given by the first and the third lines do not have intersection with $B_\varepsilon(x)$.

Let us first compute I_2 and I_4 using variables (t,v). Since $0 \le u-\alpha \le 2\sin^{-1}(\varepsilon/2)$ we have

$$0 \le t = 2\sin((u-\alpha)/2) \le \varepsilon,$$

and the condition $|p(u,v)-x|\ge\varepsilon$ implies

$$\sin^2\frac{v}{2} = \frac{\mathrm{Dist}^2 - t^2}{c(t)} \ge \frac{\varepsilon^2-t^2}{c(t)},\ \text{i.e., } v \ge 2\sin^{-1}\left(\sqrt{\frac{\varepsilon^2-t^2}{c(t)}}\right).$$

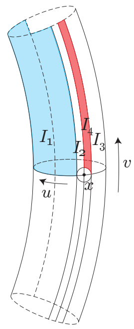

Figure 1. Parts of the domains of integration of I_1 (blue or light grey) and I_4 (red or dark grey) near the point x

Therefore, I_2 is given by

$$I_2 = \int_0^{\varepsilon} \int_{2\sin^{-1}\left(\sqrt{\frac{\varepsilon^2-t^2}{c(t)}}\right)}^{\pi} \frac{2R + \cos\alpha\cdot(2-t^2) - \sin\alpha\cdot t\sqrt{4-t^2}}{(t^2 + c(t)\sin^2\frac{v}{2})^2} dv \frac{dt}{\sqrt{4-t^2}}.$$

By the formula (6) we get

$$\begin{aligned} I_2 = &\int_0^{\varepsilon} \left\{ \frac{2R + \cos\alpha\cdot(2-t^2) - \sin\alpha\cdot t\sqrt{4-t^2}}{\sqrt{4-t^2}} \right. \\ &\times \left[\frac{2t^2 + c(t)}{t^3(t^2+c(t))^{\frac{3}{2}}} \tan^{-1}\left(\frac{\sqrt{t^2+c(t)}}{t} \tan\frac{v}{2} \right) \right. \\ &\left.\left. + \frac{c(t)\tan\frac{v}{2}}{t^2(t^2+c(t))\left((t^2+c(t))\tan^2\frac{v}{2} + t^2\right)} \right]_{v=2\sin^{-1}\left(\sqrt{\frac{\varepsilon^2-t^2}{c(t)}}\right)}^{\pi} \right\} dt \\ = &\int_0^{\varepsilon} \frac{2R + \cos\alpha\cdot(2-t^2) - \sin\alpha\cdot t\sqrt{4-t^2}}{\sqrt{4-t^2}} \\ &\times \left\{ \frac{2t^2 + c(t)}{t^3(t^2+c(t))^{\frac{3}{2}}} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{\varepsilon^2-t^2}}{t} \sqrt{\frac{c(t)+t^2}{c(t)+t^2-\varepsilon^2}} \right) \right) - \frac{\sqrt{\varepsilon^2-t^2}\sqrt{c(t)+t^2-\varepsilon^2}}{\varepsilon^2 t^2(t^2+c(t))} \right\} dt \end{aligned}$$

Putting $\tau = \sin^{-1}(t/\varepsilon)$, i.e., $t = \varepsilon\sin\tau$, we have

$$\begin{aligned} I_2 = &\int_0^{\frac{\pi}{2}} \frac{2R + \cos\alpha\cdot(2-\varepsilon^2\sin^2\tau) - \varepsilon\sin\alpha\sin\tau\sqrt{4-\varepsilon^2\sin^2\tau}}{\sqrt{4-\varepsilon^2\sin^2\tau}} \\ &\times \left\{ \frac{\cos\tau\left(2\varepsilon^2\sin^2\tau + c(\varepsilon\sin\tau)\right)}{\varepsilon^2\sin^3\tau(\varepsilon^2\sin^2\tau + c(\varepsilon\sin\tau))^{\frac{3}{2}}} \left(\frac{\pi}{2} - \tan^{-1}\left(\cot\tau\sqrt{\frac{c(\varepsilon\sin\tau)+\varepsilon^2\sin^2\tau}{c(\varepsilon\sin\tau)+\varepsilon^2\sin^2\tau-\varepsilon^2}} \right) \right) \right. \\ &\left. - \frac{\cot^2\tau\sqrt{c(\varepsilon\sin\tau)+\varepsilon^2\sin^2\tau-\varepsilon^2}}{\varepsilon^2(\varepsilon^2\sin^2\tau + c(\varepsilon\sin\tau))} \right\} d\tau. \end{aligned}$$

Expanding the integrand in a series in ε with the help of Maple we obtain

$$\begin{aligned}I_2 = &\frac{1}{2\varepsilon^2}\int_0^{\frac{\pi}{2}}\left(\tau\frac{\cos\tau}{\sin^3\tau}-\frac{\cos^2\tau}{\sin^2\tau}\right)d\tau - \frac{\sin\alpha}{4(R+\cos\alpha)\varepsilon}\int_0^{\frac{\pi}{2}}\left(\tau\frac{\cos\tau}{\sin^2\tau}-\frac{\cos^2\tau}{\sin\tau}\right)d\tau\\ &+\frac{R^2}{16(R+\cos\alpha)^2}\int_0^{\frac{\pi}{2}}\left(\tau\frac{\cos\tau}{\sin\tau}-\cos^2\tau\right)d\tau+\frac{1}{8(R+\cos\alpha)^2}\int_0^{\frac{\pi}{2}}\cos^2\tau d\tau + O(\varepsilon).\end{aligned} \tag{7}$$

Direct integration shows

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\left(\tau\frac{\cos\tau}{\sin^3\tau}-\frac{\cos^2\tau}{\sin^2\tau}\right)d\tau = \frac{1}{2}\left[\frac{\tau+\sin\tau\cos\tau-2\tau\cos^2\tau}{\sin^2\tau}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4},\\ &\int_0^{\frac{\pi}{2}}\left(\tau\frac{\cos\tau}{\sin^2\tau}-\frac{\cos^2\tau}{\sin\tau}\right)d\tau = \left[-\cos\tau-\frac{\tau}{\sin\tau}\right]_0^{\frac{\pi}{2}} = \frac{4-\pi}{2}.\end{aligned} \tag{8}$$

On the other hand, $\int_0^{\frac{\pi}{2}}\left(\tau\cot\tau-\cos^2\tau\right)d\tau$ can be computed as follows.

Put

$$I = \int_0^{\frac{\pi}{2}}\log(\sin\tau)d\tau.$$

Then

$$I = \int_0^{\frac{\pi}{2}}\log\left(\sin\left(\frac{\pi}{2}-\tau'\right)\right)d\tau' = \int_0^{\frac{\pi}{2}}\log(\cos\tau)d\tau.$$

Adding the two above, we get

$$\begin{aligned}2I &= \int_0^{\frac{\pi}{2}}\log(\sin\tau\cos\tau)d\tau\\ &= \int_0^{\frac{\pi}{2}}\log(\sin 2\tau)d\tau - \frac{\pi}{2}\log 2\\ &= \frac{1}{2}\int_0^{\pi}\log(\sin\tau')d\tau' - \frac{\pi}{2}\log 2\\ &= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\log(\sin\tau)d\tau + \int_{\frac{\pi}{2}}^{\pi}\log(\sin\tau)d\tau\right) - \frac{\pi}{2}\log 2\\ &= I - \frac{\pi}{2}\log 2,\end{aligned}$$

which implies

$$I = \int_0^{\frac{\pi}{2}}\log(\sin\tau)d\tau = -\frac{\pi}{2}\log 2.$$

Now we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\left(\tau\frac{\cos\tau}{\sin\tau}-\cos^2\tau\right)d\tau &= \Big[\tau\log(\sin\tau)\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\log(\sin\tau)d\tau - \frac{\pi}{4}\\ &= \frac{\pi}{4}(2\log 2 - 1).\end{aligned} \tag{9}$$

Substituting (8) and (9) to (7) we obtain

$$I_2 = \frac{\pi}{8\varepsilon^2} + \frac{(\pi-4)\sin\alpha}{8(R+\cos\alpha)\varepsilon} + \frac{\pi R^2}{64(R+\cos\alpha)^2}(2\log 2 - 1) + \frac{\pi}{32(R+\cos\alpha)^2} + O(\varepsilon).$$

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The integral I_4 can be obtained from I_2 by changing α to $-\alpha$:

$$\begin{aligned} I_4 &= \int_{-\varepsilon}^{0} \int_{2\sin^{-1}\left(\sqrt{\frac{\varepsilon^2-t^2}{c(t)}}\right)}^{\pi} \frac{2R+\cos\alpha\cdot(2-t^2)-\sin\alpha\cdot t\sqrt{4-t^2}}{(t^2+c(t)\sin^2\frac{v}{2})^2} \frac{dt}{\sqrt{4-t^2}}\,dv \\ &= \frac{\pi}{8\varepsilon^2} - \frac{(\pi-4)\sin\alpha}{8(R+\cos\alpha)\varepsilon} + \frac{\pi R^2}{64(R+\cos\alpha)^2}(2\log 2-1) + \frac{\pi}{32(R+\cos\alpha)^2} + O(\varepsilon). \end{aligned}$$

Let us next compute I_1 and I_3. Since $u-\alpha \in [2\sin^{-1}(\varepsilon/2), \pi]$ $\theta = \pi/2-(u-\alpha)/2 \in [0, \cos^{-1}(\varepsilon/2)]$. The integral I_1 is given by

$$I_1 = \frac{1}{8}\int_0^{\cos^{-1}\frac{\varepsilon}{2}}\int_0^{\pi} \frac{R-\cos(\alpha-2\theta)}{(\cos^2\theta+b\sin^2\frac{v}{2})^2}\,dv d\theta,$$

where $b=(R+\cos\alpha)\,(R-\cos(\alpha-2\theta))$ as was given by (3). Put $a=\cos\theta$. Then (6) shows

$$\int_0^{\pi} \frac{dv}{(a^2+b\sin^2\frac{v}{2})^2} = \frac{2a^2+b}{a^3(a^2+b)^{\frac{3}{2}}}\cdot\frac{\pi}{2}\,.$$

Since

$$R-\cos(\alpha-2\theta) = R+\cos\alpha-2\cos\alpha\cos^2\theta-2\sin\alpha\sin\theta\cos\theta,$$

we have

$$\begin{aligned} I_1 &= \frac{\pi}{16}\int_0^{\cos^{-1}\frac{\varepsilon}{2}} (R-\cos(\alpha-2\theta))\,\frac{2a^2+b}{a^3(a^2+b)^{\frac{3}{2}}}d\theta \\ &= \frac{\pi}{16}\int_0^{\cos^{-1}\frac{\varepsilon}{2}} \left[(R+\cos\alpha)-2\cos\alpha\cos^2\theta-2\sin\alpha\sin\theta\cos\theta\right]\frac{(a^2+b)+a^2}{a^3(a^2+b)^{\frac{3}{2}}}\,d\theta \\ &= \frac{\pi(R+\cos\alpha)}{16}I_{11} - \frac{\pi\cos\alpha}{8}I_{12} - \frac{\pi\sin\alpha}{8}I_{13} + \frac{\pi(R+\cos\alpha)}{16}I_{14} - \frac{\pi\cos\alpha}{8}I_{15} - \frac{\pi\sin\alpha}{8}I_{16}, \end{aligned}$$

where

$$\begin{aligned} I_{11} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{d\theta}{a^3\sqrt{a^2+b}}, \quad I_{12} = \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{\cos^2\theta\,d\theta}{a^3\sqrt{a^2+b}}, \quad I_{13} = \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{\sin\theta\cos\theta\,d\theta}{a^3\sqrt{a^2+b}}, \\ I_{14} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{d\theta}{a(a^2+b)^{\frac{3}{2}}}, \quad I_{15} = \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{\cos^2\theta\,d\theta}{a(a^2+b)^{\frac{3}{2}}}, \quad I_{16} = \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{\sin\theta\cos\theta\,d\theta}{a(a^2+b)^{\frac{3}{2}}}. \end{aligned}$$

These integrals can be computed using

$$a^2+b = \cos^2\theta\left(R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2\right), \quad \tan\left(\cos^{-1}\left(\frac{\varepsilon}{2}\right)\right) = \frac{\sqrt{4-\varepsilon^2}}{\varepsilon}.$$

Journal of Physics: Conference Series **544** (2014) 012019 doi:10.1088/1742-6596/544/1/012019

We have used Maple to expand them in series in ε (the last step of each computation of I_{1j}).

$$\begin{aligned}
I_{15} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{1}{\cos^2\theta\left(R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2\right)^{\frac{3}{2}}}\,d\theta \\
&= \left[\frac{(R+\cos\alpha)\tan\theta-\sin\alpha}{R^2(R+\cos\alpha)\sqrt{R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2}}\right]_0^{\cos^{-1}\frac{\varepsilon}{2}} \\
&= \frac{(R+\cos\alpha)\sqrt{4-\varepsilon^2}-(\sin\alpha)\varepsilon}{R^2(R+\cos\alpha)\sqrt{R^2\varepsilon^2+((R+\cos\alpha)\sqrt{4-\varepsilon^2}-(\sin\alpha)\varepsilon)^2}} + \frac{\sin\alpha}{R^2(R+\cos\alpha)\sqrt{R^2+\sin^2\alpha}} \\
&= \frac{1}{R^2(R+\cos\alpha)} + \frac{\sin\alpha}{R^2(R+\cos\alpha)\sqrt{R^2+\sin^2\alpha}} + O(\varepsilon)\,,
\end{aligned}$$

$$\begin{aligned}
I_{16} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{\sin\theta}{\cos^3\theta\left(R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2\right)^{\frac{3}{2}}}\,d\theta \\
&= \left[\frac{\sin\alpha\,((R+\cos\alpha)\tan\theta-\sin\alpha)-R^2}{R^2(R+\cos\alpha)^2\sqrt{R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2}}\right]_0^{\cos^{-1}\frac{\varepsilon}{2}} \\
&= \frac{\sin\alpha\left((R+\cos\alpha)\sqrt{4-\varepsilon^2}-(\sin\alpha)\varepsilon\right)-R^2\varepsilon}{R^2(R+\cos\alpha)^2\sqrt{R^2\varepsilon^2+\left((R+\cos\alpha)\sqrt{4-\varepsilon^2}-(\sin\alpha)\varepsilon\right)^2}} + \frac{\sqrt{R^2+\sin^2\alpha}}{R^2(R+\cos\alpha)^2} \\
&= \frac{\sin\alpha}{R^2(R+\cos\alpha)^2} + \frac{\sqrt{R^2+\sin^2\alpha}}{R^2(R+\cos\alpha)^2} + O(\varepsilon)\,,
\end{aligned}$$

$$\begin{aligned}
I_{12} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{1}{\cos^2\theta\sqrt{R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2}}\,d\theta \\
&= \left[\frac{1}{R+\cos\alpha}\sinh^{-1}\left(\frac{(R+\cos\alpha)\tan\theta-\sin\alpha}{R}\right)\right]_0^{\cos^{-1}\frac{\varepsilon}{2}} \\
&= \frac{1}{R+\cos\alpha}\left\{\sinh^{-1}\left(\frac{(R+\cos\alpha)\sqrt{4-\varepsilon^2}-(\sin\alpha)\varepsilon}{R\varepsilon}\right)+\sinh^{-1}\left(\frac{\sin\alpha}{R}\right)\right\} \\
&= -\frac{1}{2(R+\cos\alpha)}\log\left(\frac{R^2\varepsilon^2}{16(R+\cos\alpha)^2}\right) + \frac{1}{R+\cos\alpha}\sinh^{-1}\left(\frac{\sin\alpha}{R}\right) + O(\varepsilon),
\end{aligned}$$

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$$
\begin{aligned}
I_{13} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{\sin\theta}{\cos^3\theta\sqrt{R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2}}\,d\theta \\
&= \frac{\sin\alpha}{R+\cos\alpha} I_{12} + \frac{1}{(R+\cos\alpha)^2}\left[\sqrt{R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2}\right]_0^{\cos^{-1}\frac{\varepsilon}{2}} \\
&= \frac{\sin\alpha}{R+\cos\alpha} I_{12} + \frac{\sqrt{R^2\varepsilon^2+\left((R+\cos\alpha)\sqrt{4-\varepsilon^2}-(\sin\alpha)\varepsilon\right)^2}}{(R+\cos\alpha)^2\varepsilon} - \frac{\sqrt{R^2+\sin^2\alpha}}{(R+\cos\alpha)^2} \\
&= \frac{\sin\alpha}{R+\cos\alpha} I_{12} + \left\{\frac{2}{(R+\cos\alpha)\varepsilon} - \frac{\sin\alpha}{(R+\cos\alpha)^2} + O(\varepsilon)\right\} - \frac{\sqrt{R^2+\sin^2\alpha}}{(R+\cos\alpha)^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
I_{14} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{1}{\cos^4\theta\left(R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2\right)^{\frac{3}{2}}}\,d\theta \\
&= \frac{1}{(R+\cos\alpha)^2} I_{12} - \frac{(1-2R\cos\alpha-2\cos^2\alpha)}{(R+\cos\alpha)^2} I_{15} + \frac{2\sin\alpha}{R+\cos\alpha} I_{16} \\
&= \frac{1}{(R+\cos\alpha)^2} I_{12} + \frac{1+2R\cos\alpha}{R^2(R+\cos\alpha)^3} + \frac{(1+2R^2)\sin\alpha+2R^2\sin\alpha\cos\alpha}{R^2(R+\cos\alpha)^3\sqrt{R^2+\sin^2\alpha}} + O(\varepsilon)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{11} &= \int_0^{\cos^{-1}\frac{\varepsilon}{2}} \frac{1}{\cos^4\theta\sqrt{R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2}}\,d\theta \\
&= \frac{R^2-1+4R\cos\alpha+3\cos^2\alpha}{2(R+\cos\alpha)^2} I_{12} + \frac{3\sin\alpha}{2(R+\cos\alpha)} I_{13} \\
&\quad + \left[\frac{\sin\theta\sqrt{R^2+((R+\cos\alpha)\tan\theta-\sin\alpha)^2}}{2(R+\cos\alpha)^2\cos\theta}\right]_0^{\cos^{-1}\frac{\varepsilon}{2}} \\
&= \frac{R^2-1+4R\cos\alpha+3\cos^2\alpha}{2(R+\cos\alpha)^2} I_{12} + \frac{3\sin\alpha}{2(R+\cos\alpha)} I_{13} \\
&\quad + \frac{\sqrt{4-\varepsilon^2}\sqrt{R^2\varepsilon^2+\left((R+\cos\alpha)\sqrt{4-\varepsilon^2}-(\sin\alpha)\varepsilon\right)^2}}{2(R+\cos\alpha)^2\varepsilon^2} \\
&= \frac{R^2-1+4R\cos\alpha+3\cos^2\alpha}{2(R+\cos\alpha)^2} I_{12} + \frac{3\sin\alpha}{2(R+\cos\alpha)} I_{13} \\
&\quad + \left\{\frac{2}{(R+\cos\alpha)\varepsilon^2} - \frac{\sin\alpha}{(R+\cos\alpha)^2\varepsilon} - \frac{R^2+4R\cos\alpha+2\cos^2\alpha}{4(R+\cos\alpha)^3} + O(\varepsilon)\right\}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
I_1 &= \frac{\pi(R+\cos\alpha)}{16} I_{11} - \frac{\pi\cos\alpha}{8} I_{12} - \frac{\pi\sin\alpha}{8} I_{13} + \frac{\pi(R+\cos\alpha)}{16} I_{14} - \frac{\pi\cos\alpha}{8} I_{15} - \frac{\pi\sin\alpha}{8} I_{16} \\
&= \frac{\pi R^2}{32(R+\cos\alpha)} I_{12} + \frac{\pi}{8\varepsilon^2} - \frac{\sin\alpha}{8(R+\cos\alpha)\varepsilon} - \frac{\pi(R^2+4R\cos\alpha+2\cos^2\alpha)}{64(R+\cos\alpha)^2} - \frac{\pi}{16R^2(R+\cos\alpha)^2} \\
&\quad + \frac{\pi\sin^2\alpha}{32(R+\cos\alpha)^2} - \frac{\pi\sin\alpha}{16R^2(R+\cos\alpha)^2\sqrt{R^2+\sin^2\alpha}} + \frac{\pi\sin\alpha\sqrt{R^2+\sin^2\alpha}}{32(R+\cos\alpha)^2} + O(\varepsilon) \\
&= -\frac{\pi R^2}{64(R+\cos\alpha)^2}\log\left(\frac{R^2\varepsilon^2}{16(R+\cos\alpha)^2}\right) + \frac{\pi R^2}{32(R+\cos\alpha)^2}\sinh^{-1}\left(\frac{\sin\alpha}{R}\right) \\
&\quad + \frac{\pi}{8\varepsilon^2} - \frac{\sin\alpha}{8(R+\cos\alpha)\varepsilon} - \frac{\pi(R^2+4R\cos\alpha+2\cos^2\alpha)}{64(R+\cos\alpha)^2} - \frac{\pi}{16R^2(R+\cos\alpha)^2} + \frac{\pi\sin^2\alpha}{32(R+\cos\alpha)^2} \\
&\quad - \frac{\pi\sin\alpha}{16R^2(R+\cos\alpha)^2\sqrt{R^2+\sin^2\alpha}} + \frac{\pi\sin\alpha\sqrt{R^2+\sin^2\alpha}}{32(R+\cos\alpha)^2} + O(\varepsilon).
\end{aligned}$$

The integral I_3 can be obtained from I_1 by changing α to $-\alpha$:

$$\begin{aligned}
I_3 &= \frac{1}{8}\int_0^{\cos^{-1}\frac{\varepsilon}{2}}\int_0^{\frac{\pi}{2}} \frac{R-\cos(-\alpha-2\theta)}{(\cos^2\theta+(R+\cos\alpha)(R-\cos(-\alpha-2\theta))\sin^2\frac{v}{2})^2}\,dv d\theta \\
&= -\frac{\pi R^2}{64(R+\cos\alpha)^2}\log\left(\frac{R^2\varepsilon^2}{16(R+\cos\alpha)^2}\right) - \frac{\pi R^2}{32(R+\cos\alpha)^2}\sinh^{-1}\left(\frac{\sin\alpha}{R}\right) \\
&\quad + \frac{\pi}{8\varepsilon^2} + \frac{\sin\alpha}{8(R+\cos\alpha)\varepsilon} - \frac{\pi(R^2+4R\cos\alpha+2\cos^2\alpha)}{64(R+\cos\alpha)^2} - \frac{\pi}{16R^2(R+\cos\alpha)^2} + \frac{\pi\sin^2\alpha}{32(R+\cos\alpha)^2} \\
&\quad + \frac{\pi\sin\alpha}{16R^2(R+\cos\alpha)^2\sqrt{R^2+\sin^2\alpha}} - \frac{\pi\sin\alpha\sqrt{R^2+\sin^2\alpha}}{32(R+\cos\alpha)^2} + O(\varepsilon).
\end{aligned}$$

By putting all the formulae together we obtain

$$\begin{aligned}
V(\varepsilon,x) &= 2(I_1+I_2+I_3+I_4) \\
&= \frac{\pi}{\varepsilon^2} - \frac{\pi R^2}{16(R+\cos\alpha)^2}\log\left(\frac{R^2\varepsilon^2}{(R+\cos\alpha)^2}\right) + \frac{\pi R^2}{8(R+\cos\alpha)^2}3\log 2 \\
&\quad - \frac{\pi}{8} - \frac{\pi}{4R^2(R+\cos\alpha)^2} + \frac{\pi(1+\sin^2\alpha)}{8(R+\cos\alpha)^2}.
\end{aligned}$$

As the Gauss curvature and $\Delta = (\kappa_1-\kappa_2)^2$ at the point $x = p(\alpha, 0)$ is given by

$$K(x) = \frac{\cos\alpha}{(R+\cos\alpha)}, \quad \Delta(x) = \frac{R^2}{(R+\cos\alpha)^2},$$

the renormalized potential is given by

$$\begin{aligned}
V(x;T) &= \lim_{\varepsilon\to 0}\left(V(\varepsilon,x) - \frac{\pi}{\varepsilon^2} + \frac{\pi\Delta(x)}{16}\log\left(\Delta(x)\varepsilon^2\right) + \frac{\pi K(x)}{4}\right) \\
&= \frac{\pi R^2}{8(R+\cos\alpha)^2}3\log 2 - \frac{\pi}{8} - \frac{\pi}{4R^2(R+\cos\alpha)^2} + \frac{\pi(1+\sin^2\alpha)}{8(R+\cos\alpha)^2} + \frac{\pi\cos\alpha}{4(R+\cos\alpha)}.
\end{aligned}$$

Finally, it implies that the renormalized r^{-4}-potential energy of the torus T is given by

$$\begin{aligned}
E(T) &= \int_T V(x;T)\, d^2x \\
&= 2\pi \int_0^{2\pi} \left(\frac{\pi R^2}{8(R+\cos\alpha)^2} 3\log 2 - \frac{\pi}{8} - \frac{\pi}{4R^2(R+\cos\alpha)^2} + \frac{\pi(1+\sin^2\alpha)}{8(R+\cos\alpha)^2} \right. \\
&\qquad \left. + \frac{\pi\cos\alpha}{4(R+\cos\alpha)} \right) (R+\cos\alpha)\, d\alpha \\
&= \frac{\pi^2}{2} \int_0^{\pi} \left(\left(R^2 (3\log 2 - 1) + 2 - \frac{2}{R^2} \right) \frac{1}{R+\cos\alpha} \right) d\alpha \\
&= \frac{\pi^2}{2} \left[\left(R^2 (3\log 2 - 1) + 2 - \frac{2}{R^2} \right) \frac{2}{\sqrt{R^2-1}} \tan^{-1} \left(\sqrt{\frac{R-1}{R+1}} \tan\frac{\alpha}{2} \right) \right]_0^{\pi} \\
&= \frac{\pi^3}{2\sqrt{R^2-1}} \left(R^2 (3\log 2 - 1) + 2 - \frac{2}{R^2} \right).
\end{aligned}$$

□

3. Application as another motivation

First we state our framework ((O'Hara & Solanes)). Let M be an m-dimensional compact orientable submanifold of $\mathbb{R}^n$ and λ be a real number. Put

$$\Delta_\varepsilon = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, |x-y| < \varepsilon\}.$$

It is interesting to see the asymptotics of

$$E_{r^\lambda}(\varepsilon, M) = \iint_{M\times M \setminus \Delta_\varepsilon} |x-y|^\lambda \, d^m x d^m y,$$

where $d^m x$ and $d^m y$ denote the standard Lebesgue measure of M. To be precise, we expand the above in a series in ε and study the coefficients. For example, if K is a knot in $\mathbb{R}^3$ and $\lambda = -2$ then

$$E_{r^{-2}}(\varepsilon, K) = \frac{2L(K)}{\varepsilon} + E(K) + O(\varepsilon),$$

where $L(K)$ is the length of the knot and $E(K)$ denotes the knot energy given by (1), and if S is a closed surface in $\mathbb{R}^3$ and $\lambda = -4$ then

$$E_{r^{-4}}(\varepsilon, S) = \frac{\pi}{\varepsilon^2} A(S) - \frac{\pi}{8} \log\varepsilon \int_S \Delta(x)\, d^2x + E(S) - \frac{\pi}{16} \int_S \Delta(x) \log\Delta(x)\, d^2x - \frac{\pi^2}{2}\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S, and if Ω is a 2-dimensional compact submanifold of $\mathbb{R}^2$ then

$$E_{r^{-4}}(\varepsilon, \Omega) = \frac{\pi}{\varepsilon^2} A(\Omega) - \frac{2}{\varepsilon} L(\partial\Omega) + E_{OS}(\Omega) - \frac{\pi^2}{4}\chi(\Omega) + O(\varepsilon),$$

where $A(\Omega)$ is the area of Ω and E_{OS} is the energy defined in (O'Hara & Solanes). We remark that E_{OS} is also invariant under Möbius transformations. We conjecture that a similar formula holds for compact bodies in $\mathbb{R}^3$.

Let us focus on the constant term of the series of $E_{r^\lambda}(\varepsilon, M)$, which, after some modification if necessary, we call the *renormalized r^λ-potential energy* of M, denoted by $E_{r^\lambda}(M)$.

Now we can define functionals for knots as follows. Let $N_\varepsilon(K)$ be an ε-tubular neighbourhood of K. Expand $E_{r^\lambda}(\partial N_\varepsilon(K))$ and $E_{r^{\lambda'}}(N_\varepsilon(K))$ in series of ε. We conjecture that functionals that can capture global properties of knots appear as a coefficient of ε^2-term of $E_{r^\lambda}(\partial N_\varepsilon(K))$ and as a coefficient of ε^4-term of $E_{r^{\lambda'}}(N_\varepsilon(K))$.

When $K_\circ$ is a round circle with radius 1, our main theorem implies

$$\begin{aligned} E_{r^{-4}}(\partial N_\varepsilon(K_\circ)) &= \frac{\pi^3\varepsilon}{2\sqrt{1-\varepsilon^2}}\left(\frac{3\log 2-1}{\varepsilon^2}+2-2\varepsilon^2\right) \\ &= \frac{\pi^3(3\log 2-1)}{2\varepsilon}+\frac{3\pi^3\,(\log 2+1)}{4}\varepsilon+\frac{\pi^3\,(9\log 2-11)}{16}\varepsilon^3+O(\varepsilon^5). \end{aligned}$$

Therefore, the functional thus obtained from the ε^2-term of $E_{r^{-4}}(\partial N_\varepsilon(K))$ vanishes for round circles.

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