

# On combinatorial properties of a higher asymptotic ergodic invariant of magnetic lines

P.M.Akhmet'ev<sup>1</sup>

<sup>1</sup> IZMIRAN,

Troitsk, Moscow region, Russia

E-mail: pmakhmet@izmiran.ru

**Abstract.** We investigate combinatorial properties of a higher invariant of magnetic lines, which is defined in the paper Akhmet'ev-1 (2013). Assume that a 3-component link  $\mathbf{L}$  is modeled by a magnetic field  $\mathbf{B}$ , which is represented by 3 closed magnetic lines. Main Theorem relates the integral invariant  $M(\mathbf{B})$  and a combinatorial invariant  $\tilde{M}(\mathbf{L})$ , defined from the Conway polynomial. As a corollary of Main Theorem, asymptotic properties for combinatorial links are proposed. The combinatorial invariant  $\tilde{M}$  satisfies these asymptotic properties.

## 1. Introduction

Magnetic fields are investigated by the helicity integral  $\chi_{\mathbf{B}}$ , for details see the paper (Moffatt, 2001). The helicity integral is related with the Gauss integral for the linking number of two closed curves. The dimension of the helicity integral is  $Gs^2 \cdot sm^4$ . The idea of topological investigation of turbulent magnetic fields is assumed step-by-step generalization, using higher invariants of links instead of the linking number. Such an approach is possible because of a theorem by V.I.Arnol'd, see Arnol'd V.I. & Khesin B.A. (1998) Ch. 3, paragraph 4. This theorem provides the formula of the helicity integral as the asymptotic ergodic Hopf invariant. In the papers Akhmet'ev (2012), Akhmet'ev-2 (2013) the theorem by V.I.Arnol'd is generalized.

Briefly, the idea of the considered approach proposed by V.I.Arnol'd is following. Assume, for simplicity, that a magnetic field  $\mathbf{B}$  is structured into a finite set  $U = \cup_i U_i$  of magnetic tubes; definition of magnetic tubes is proposed in Berger M.A. & Field G.B. (1984). The case when a magnetic field  $\mathbf{B}$  is inside the only magnetic tube  $U = U_0$  is possible. Take the set of all magnetic lines of  $\mathbf{B}$  (generally speaking, non-closed) and take all collections of  $k$  magnetic lines (non-trivial examples for  $k \geq 4$  are unknown). Take an appropriate Vassiliev's invariant  $I$  of  $k$ -component links and apply this invariant formally for a  $k$ -collection of non-closed magnetic lines of the fixed length  $l$ . Take the asymptotic limit  $I(+\infty) = \lim_{l \rightarrow +\infty} I(l)$  of the integrals, if this limit exists. The average value of  $I(+\infty)$  over all  $k$ -collections of magnetic line is called an asymptotic ergodic invariant of  $\mathbf{B}$ .

In the paper Akhmet'ev (2012) a modification of the helicity invariant for  $k = 2$  is introduced. This new invariant is called the quadratic helicity and is denoted by  $\chi_{\mathbf{B}}^{[2]}$ . The quadratic helicity is not a higher invariant: this invariant is derived from the square of the linking number, the dimension of the quadratic helicity invariant is  $Gs^4 sm^4$ .

A geometrical meaning of  $\chi_{\mathbf{B}}^{[2]}$  is following. The asymptotic linking number of all pairs of magnetic lines determines the (absolutely) integrable function  $\chi : U \times U \rightarrow \mathbb{R}$  on the 2-point



configurations space (recall that an integrable function could be infinite-valued on a set of the zero-measure). This function is frozen-in for the ideal magnetic field and is called the helicity density. The helicity integral  $\chi_{\mathbf{B}}$  is the mean value of the helicity density  $\chi$  on  $U \times U$ . Moreover, the square  $\chi^2$  of the helicity density is an integrable function. The quadratic helicity  $\chi_{\mathbf{B}}^{[2]}$  is the dispersion of  $\chi$ . In the prove that  $\chi_{\mathbf{B}}^{[2]}$  is invariant for ideal magnetic fields one replaces short paths of end-points of open long magnetic lines by pairs "‘monopole–antimonopole’" as in Akhmet’ev (2012), Section 2.2.

Because the quadratic helicity is defined as an ergodic invariant, this invariant is continuously depended of dissipations of  $\mathbf{B}$ . A priori this is not evident, because a  $C^\infty$ –small perturbation of  $\mathbf{B}$  changes totally a structure of magnetic lines. In the paper Akhmet’ev (2012), Theorem 2, shows that the quadratic helicity depends even with a Lipschitz coefficient (but not smoothly!) of the magnetic field  $\mathbf{B}$ . The quadratic helicity could be explicitly estimated from the observed magnetic field  $\mathbf{B}$  as in the paper Akhmet’ev (2012), inequality (5).

In the paper Arnol’d V.I. (2000) the following Problem 1984-12 is formulated: "To transform asymptotic ergodic definition of the Hopf invariant of divergence-free vector fields to a theory by S.P.Novikov, which generalized Whitehead product in homotopy group of spheres". As it is well-known, by Whitehead product in 3D is assumed the Massey integrals for Milnor’s invariants of links. The Massey integrals is applied in the framework of MHD problems by Monastyrsky, M.V. & Retakh, V.S. (1986), for a detailed exposition and references see, f.ex. Mayer Ch. (2003). The Massey integrals are defined with restrictions. New aspects of the problem, related with configuration spaces, recently, is proposed in the paper De Turck, D. *et al.* (2011), Cohen, F.R. *et al.* (2012).

To apply the Massey integrals for magnetic lines the following problem arises. Massey integrals are not totally–defined: an integral of an order  $k$  is well defined, only if all integrals of the orders  $\leq k - 1$  are vanished. In particular, to define the simplest integral  $\mu_{1,2,3}$  for 3-component link one should assume that all pairwise linking numbers between components are vanished (3 conditions). In the paper Akmetiev P. & Ruzmaikin A (1955) is observed that for the Sato-Levine invariant (this invariant for 2-component links is defined as the  $\mu_{1,1,2,2}$  integral) the only linking number between components has to be vanished. This assumption is restrictive for applications. The Massey integrals are not asymptotic ergodic invariants.

In the paper Akhmetiev (2005) an invariant  $M$  for magnetic fields, which is structured into 3 disjoint magnetic tubes in introduced. This invariant is written as a totally well-defined Massey’s integral. This is a higher integral:  $M$  is not a function of the pairwise integral linking coefficients of the magnetic tubes. In the paper Akhmet’ev-1 (2013) the integral  $M$  is well-defined for magnetic lines using a integral over a finite-type configuration space. In the paper Akhmet’ev-2 (2013) (Section 6, Theorem 14) is particularly proved the following conjecture, which is a positive solution of the Problem 1984-12 by V.I.Arnol’d.

*Conjecture 1* The invariant  $M$  is well-defined as a higher asymptotic ergodic invariant of magnetic fields.

The most difficult part of the proof of the Conjecture is to rewrite peripheral terms of the integral (9) – (14) in an ergodic stile, analogously to the invariant  $\chi_{\mathbf{B}}^{[2]}$ , Akhmet’ev (2012).

The goal of the paper is to express the integral  $M$  in terms of Conway polynomial of links. Additionally, a combinatorial property of the invariant  $M$  namely, the formula (23), is deduced from Conjecture 1. This combinatorial formula (23) is explicitly proved in Akhmet’ev (2011) (Section 6 Theorem 8).

## 2. The integral $M$

Let us recall the integral formula from Akhmetiev (2005) in the form Akhmet'ev-1 (2013). Assume that a magnetic field  $\mathbf{B}$  is localized into 3 disjoint magnetic tubes

$$\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3,$$

each magnetic tube is presented by a thin solid torus  $U_i \subset \mathbb{R}^3$ , in this torus a prescribed coordinate system  $U_i = D^2 \times S^1$  is well-defined, where  $S^1 \subset \mathbb{C}$  is the standard circle. The integral magnetic flow of  $\mathbf{B}_i$  is well-defined through a cross-section  $D^2 \times pt \subset D^2 \times S^1$  of the magnetic tube  $U_i$ .

Denote by  $\mathbf{A}_i$  the vector-potentials of  $\mathbf{B}_i$ ,  $i = 1, 2, 3$  in the corresponding magnetic tube:

$$\text{rot} \mathbf{A}_i = \mathbf{B}_i.$$

Denote by  $(1, 2), (2, 3), (3, 1)$  the integral linking coefficients, which are given by the integrals:

$$(i, j) = \int_{U_i} (\mathbf{A}_j, \mathbf{B}_i) dU_i = \int_{U_j} (\mathbf{A}_i, \mathbf{B}_j) dU_j, \quad i, j = 1, 2, 3, \quad i \neq j.$$

The restriction of the potential  $(1, 3)(2, 3)\mathbf{A}_2 - (3, 2)(1, 2)\mathbf{A}_3$  to the magnetic tube  $U_1$  coincides with the gradient of a function, which is well-defined up to an additive constant. This function is denoted by  $\phi_1 : U_1 \rightarrow \mathbb{R}$ , two analogous functions are denoted by  $\phi_2 : U_2 \rightarrow \mathbb{R}$ ,  $\phi_3 : U_3 \rightarrow \mathbb{R}$ . Those functions are called scalar potentials.

We get  $\varphi_{2,1} = (2, 1)\varphi_1^0 + \varphi_{j,i}^{var} + C_{2,1}$ , where  $\varphi_1^0$  is a linear multivalued function in  $U_1 = D^2 \times S^1$  with respect to the second factor. The functions  $\varphi_{2,1}^{var}$ ,  $\varphi_{3,1}^{var}$  in  $U_1$  satisfy the equation  $\int_{U_1} \varphi_{2,1}^{var} dU_1 = 0$ ;  $\int_{U_1} \varphi_{3,1}^{var} dU_1 = 0$ .

With respect to the definition above, we get:

$$\phi_1 = (3, 1)\varphi_{2,1}^{var} - (1, 2)\varphi_{3,1}^{var} + C_1. \quad (1)$$

The functions  $\phi_2, \phi_3$  are defined analogously:

$$\phi_2 = (1, 2)\varphi_{3,2}^{var} - (2, 3)\varphi_{1,2}^{var} + C_2, \quad (2)$$

$$\phi_3 = (2, 3)\varphi_{1,3}^{var} - (3, 3)\varphi_{2,3}^{var} + C_3. \quad (3)$$

The normalized constant  $C_1$  is well-defined by the equation:

$$2C_1 = \int_{U_1} \varphi_{2,1}^{var} (\mathbf{grad} \varphi_{3,1}^{var}, \mathbf{B}_1) - (\mathbf{grad} \varphi_{2,1}^{var}, \mathbf{B}_1) \varphi_{3,1}^{var} dU_1 + \frac{2}{3} \int_{\mathbb{R}^3} \langle \mathbf{A}_1(x), \mathbf{A}_2(x), \mathbf{A}_3(x) \rangle d\mathbb{R}^3, \quad (4)$$

$$2C_2 = \int_{U_2} \varphi_{3,2}^{var} (\mathbf{grad} \varphi_{1,2}^{var}, \mathbf{B}_2) - (\mathbf{grad} \varphi_{3,2}^{var}, \mathbf{B}_2) \varphi_{1,2}^{var} dU_2 + \frac{2}{3} \int_{\mathbb{R}^3} \langle \mathbf{A}_1(x), \mathbf{A}_2(x), \mathbf{A}_3(x) \rangle d\mathbb{R}^3, \quad (5)$$

$$2C_3 = \int_{U_3} \varphi_{1,3}^{var} (\mathbf{grad} \varphi_{2,3}^{var}, \mathbf{B}_3) - (\mathbf{grad} \varphi_{1,3}^{var}, \mathbf{B}_3) \varphi_{2,3}^{var} dU_3 + \frac{2}{3} \int_{\mathbb{R}^3} \langle \mathbf{A}_1(x), \mathbf{A}_2(x), \mathbf{A}_3(x) \rangle d\mathbb{R}^3. \quad (6)$$

In the construction it is sufficient to assume that the constants  $C_1, C_2, C_3$  are well-defined using the only equation: the sum of the the equations (4), (5), (6).

Using the functions  $\phi_i$  by the equations (1) – (3) define a divergence-free vector function  $\mathbf{F}$  by the formula:

$$\mathbf{F} = (1, 3)(2, 3)\mathbf{A}_1 \times \mathbf{A}_2 + (2, 1)(3, 1)\mathbf{A}_2 \times \mathbf{A}_3 + (3, 2)(1, 2)\mathbf{A}_3 \times \mathbf{A}_1 - \phi_1\mathbf{B}_1(2, 3) - \phi_2\mathbf{B}_2(3, 1) - \phi_3\mathbf{B}_3(1, 2). \quad (7)$$

Let us define the invariant  $M$  as the the helicity integral of the vector-function (7) by the formula:

$$M(\mathbf{B}) = \int_{\mathbb{R}^3} (\mathbf{G}, \mathbf{F}) d\mathbb{R}^3 + \dots, \quad (8)$$

where  $\mathbf{G}$  is the vector-potential of  $\mathbf{F}$ , and  $\dots$  in the formula (8) means the sum of the following 6 terms:

$$d_{1,1} = -(2, 3)^2 \int (\mathbf{A}_1, \mathbf{B}_1) \phi_1^2 dU_1, \quad (9)$$

$$d_{2,2} = -(3, 1)^2 \int (\mathbf{A}_2, \mathbf{B}_2) \phi_2^2 dU_2, \quad (10)$$

$$d_{3,3} = -(1, 2)^2 \int (\mathbf{A}_3, \mathbf{B}_3) \phi_3^2 dU_3, \quad (11)$$

$$d_{1,3} = (2, 3)(1, 2) \int (\mathbf{A}_3, \mathbf{B}_1) \phi_1^2 dU_1, \quad (12)$$

$$d_{2,1} = (3, 1)(2, 3) \int (\mathbf{A}_1, \mathbf{B}_2) \phi_2^2 dU_2, \quad (13)$$

$$d_{3,2} = (1, 2)(3, 1) \int (\mathbf{A}_2, \mathbf{B}_3) \phi_3^2 dU_3. \quad (14)$$

In the paper Akhmetiev (2005) (the considered form of the integral is presented in Akhmet'ev-2 (2013), Theorem 2) the following result is proved:

### 2.1. Theorem

The expression (8), in which the scalar potentials, given by the formulas (1) - (3) with the gauge conditions (4) – (6), is the invariant (dimension  $Gs^{12}sm^6$ ) of volume-preserved diffeomorphisms of the space. This invariant is not a function of pairwise integral integer coefficients of magnetic tubes.

### 3. A combinatorial formula for the integral $M$

Let us consider a Conway polynomial for  $m$ -component link  $\mathbf{L}$ :

$$\nabla_{\mathbf{L}}(z) = z^{m-1}(c_0 + c_1 z^2 + \cdots + c_n z^{2n}). \quad (15)$$

We shall consider invariants of links, which are expressed from the first two coefficients  $c_0, c_1$  of this polynomial (the Conway polynomial is applied to the link  $\mathbf{L}$  itself and to the all proper sublinks of  $\mathbf{L}$ ).

Let us start with the case  $m = 2$  of 2-component links  $\mathbf{L} = L_1 \cup L_2$ . Recall results from the papers Melikhov (2003), Matveev S. & Polyak M. (2009), and Nikkuni, R. (2008). The simplest invariant of 2-component links, which is not expressed from pairwise linking coefficients of components, is called the generalized Sato-Levine invariant:

$$\beta(\mathbf{L}) = c_1(\mathbf{L}) - c_0(\mathbf{L})(c_1(L_1) + c_1(L_2)). \quad (16)$$

In this formula  $c_0(\mathbf{L})$  coincides with linking coefficient  $lk(L_1, L_2)$ ,  $c_1(L_1)$ ;  $c_1(L_2)$  are called Casson's invariants of the corresponding (knotted) component, Denote  $lk(L_1, L_2) = c_0(\mathbf{L})$  by  $(1, 2)$  for short.

Let us considered the case  $m = 3$  of 3-component links  $\mathbf{L} = L_1 \cup L_2 \cup L_3$ . Define the Melikhov's invariant  $\gamma(\mathbf{L})$ , as in the papers Melikhov (2003), Matveev S. & Polyak M. (2009) by the following formula:

$$\begin{aligned} \gamma(\mathbf{L}) = c_1(\mathbf{L}) - & \\ & ((1, 2)(2, 3) + (2, 3)(3, 1) + (3, 1)(1, 2))(c_1(L_1) + c_1(L_2) + c_1(L_3)) \\ & - ((3, 1) + (2, 3))(c_1(L_1 \cup L_2) - (1, 2)(c_1(L_1) + c_1(L_2))) \\ & - ((1, 2) + (3, 1))(c_1(L_2 \cup L_3) - (2, 3)(c_1(L_2) + c_1(L_3))) \\ & - ((2, 3) + (1, 2))(c_1(L_3 \cup L_1) - (3, 1)(c_1(L_3) + c_1(L_1))), \end{aligned} \quad (17)$$

where  $(i, j)$  is the linking number of the pair of components  $L_i, L_j$ ,  $i, j = 1, 2, 3$ ,  $i \neq j$ , of the link  $\mathbf{L}$ .

Recall the formula by S.A.Melikhov of jumps of the invariant  $\gamma$  for homotopy of 3-component links with self-intersection and with no intersection between different components. Let  $\mathbf{L}_{sing;3}$  be a singular 3-component link, the components  $L_1, L_2$  are regular, the component  $L_{sing,3}$  has the only self-intersection point. Denote by  $\mathbf{L}_{+,3}, \mathbf{L}_{-,3}$  two 3-component links, the first 2 components of this links coincide with  $L_1, L_2$ , the third component  $L_{3,+}$  of the link  $\mathbf{L}_{+,3}$  is defined by means of the prescribed resolution of the self-intersection of the component  $L_{sing,3}$ , the third component  $L_{3,-}$  of the link  $\mathbf{L}_{-,3}$  is defined by means of the opposite resolution of the self-intersection of the component of the link  $L_{sing,3}$ . (The sign convention of resolutions of singular link is standard: a sign of the resolution is determined by the orientation of the basis, which is formed by the (non-ordered) pair of the tangent vectors and the vector from the (non-ordered) resolution points of the singular point.)

Let us define 4-component link  $\mathbf{L}_{s;3}$  with the components  $(L_1, L_2, L_{3+}, L_{3-})$ . The first two components of the links  $\mathbf{L}_{s;3}, \mathbf{L}_{sing;3}$  coincide, the components  $L_{3+}, L_{3-}$  of the link  $\mathbf{L}_{s;3}$  are obtained by the orientation-preserved smoothing of the singular component  $L_{sing,3}$  of the link  $\mathbf{L}_{sing;3}$ .

Denote by  $(1, 3^+)$   $(1, 3^-)$  the linking coefficients of the component  $L_1$  with the components  $L_{3+}, L_{3-}$  correspondingly. Define by  $(2, 3^+)$   $(2, 3^-)$  the linking coefficients of the component  $L_2$  with the components  $L_{3+}, L_{3-}$  correspondingly. Analogical denotations are well-defined after the replacement  $1 \rightarrow 2$  of the numbers of the components.

### 3.1. Lemma

The invariant  $\gamma(\mathbf{L})$  of 3-component links satisfies the equations, which are obtained from the following equation by cyclic permutation of indexes:

$$\gamma(\mathbf{L}_{+;1}) - \gamma(\mathbf{L}_{-;1}) = (2, 3)((2, 1^+)(3, 1^-) + (2, 1^-)(3, 1^+)), \quad (18)$$

*Proof of Lemma 3.1* Lemma 3.1 is proved in Melikhov (2003), p.11.

Let  $\mathbf{L} = (L_1 \cup L_2 \cup L_3)$  be an arbitrary 3-component link. Define the invariant  $\tilde{M}(\mathbf{L})$  by the following formula:

$$\tilde{M}(\mathbf{L}) = (1, 2)(2, 3)(3, 1)\gamma(\mathbf{L}) - \quad (19)$$

$$((1, 2)^2(1, 3)^2\beta(L_2 \cup L_3) + (2, 3)^2(2, 1)^2\beta(L_3 \cup L_1) + (2, 3)^2(2, 1)^2\beta(L_3 \cup L_1)).$$

The invariant  $\tilde{M}$  is of the order 7 in the sense of V.A.Vassiliev.

### 3.2. Theorem

For homotopies of links with one self-intersection point on the corresponding component the invariant  $\tilde{M}$  satisfies crossing change formulas, which is obtained from the following formula by cyclic permutation of indexes:

$$\begin{aligned} \tilde{M}(\mathbf{L}_{+;1}) - \tilde{M}(\mathbf{L}_{-;1}) &= (1, 2)(2, 3)^2(3, 1)((2, 1^+)(3, 1^-) + (2, 1^-)(3, 1^+)) \\ &\quad - (2, 3)^2(1, 2)^2(3, 1^+)(3, 1^-) - (3, 1)^2(3, 2)^2(2, 1^+)(2, 1^-), \end{aligned} \quad (20)$$

### Proof of Theorem 3.2

The crossing change formula for  $\beta$  is well known, see f.ex Matveev S. & Polyak M. (2009). Theorem follows from Lemma 3.1.

### 3.3. Main Theorem

Let  $\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3$  a magnetic fields in 3 disjoint magnetic tubes  $U_1 \cup U_2 \cup U_3$  with the central lines  $\mathbf{L} = L_1 \cup L_2 \cup L_3$ , and with the unit integral magnetic flows trough the cross-sections of the each magnetic tube. (Let us call in this case that  $\mathbf{B}$  is a model of  $\mathbf{L}$ .) The integral invariant (8) is expressed from the combinatorial invariant (19) by the following equality:

$$M(\mathbf{B}) = \tilde{M}(\mathbf{L}) + P((1, 2), (2, 3), (3, 1)), \quad (21)$$

where  $P$  is a polynomial of the pairwise linking numbers  $(1, 2)$ ,  $(2, 3)$ ,  $(3, 1)$  of the components of  $\mathbf{L}$ ,  $\deg(P) \leq 14$ .

*Proof of Main Theorem*

Let  $\mathbf{L}$  be an arbitrary 3-component link. In the case  $(1, 2) = (2, 3) = (3, 1) = 0$  we get  $M(\mathbf{B}) = \tilde{M}(\mathbf{L}) = 0$ .

Assume  $(1, 2)^2 + (2, 3)^2 + (3, 1)^2 \neq 0$ . Let  $k \in \mathbb{Z}$  be an arbitrary integer. Let us define the  $k$ -modification  $\Psi_k : \mathbf{L} \mapsto \Psi_k(\mathbf{L})$  of  $\mathbf{L}$ ,  $k \in \mathbb{Z}$  (the corresponding modification of the magnetic tubes denote by  $\Psi_k : \mathbf{B} \mapsto \Psi_k(\mathbf{B})$ ). Represents the link  $\mathbf{L}$  in its isotopy class, such that there exists a ball  $D^3 \subset \mathbb{R}^3$ , which intersects  $\mathbf{L}$  by 3 parallel segments, which belong to the different components. Each commutator is decomposed into 2 intersections of  $L_1$  with  $L_2$  with opposite signs. The commutator of the component  $L_1$  with the component  $L_2, L_3$  is called the "delta move", the delta move was introduced by Murakami and Nakanishi, see De Turck, D. *et al.* (2011) for a picture and a more detailed explanation of the commutator.

Apply  $k$ -times commutator,  $k \in \mathbb{Z}$ , of the segment on the component  $L_1$  with the segments on the components  $L_2, L_3$  inside  $D^3$ . Evidently, the links  $\Psi_{k_1} \circ \Psi_{k_2}(\mathbf{L})$  and  $\Psi_{k_1+k_2}(\mathbf{L})$  are isotopic.

*3.4. Lemma*

The integral invariant  $M$ , given by the formula (8), satisfies the crossing change formulas, as the formula (20), when the corresponding magnetic tube is changed by a homotopy with a self-intersection.

*Proof of Lemma 3.4*

It is sufficient to prove the lemma for a homotopy with the self-intersection of the tube  $U_1$  (of the component  $L_1$ ). The idea of the proof is in the paper by Akhmet'ev P.M. & Kunakovskaya O.V. (2009), where the invariant  $\beta$  is investigated.

Denote by  $V, W$  the two short segments of the magnetic tube  $U_1$  in a small ball  $D^3 \subset \mathbb{R}^3$ , inside this ball the crossing of the homotopy is. Without loss of a generality we may assume that the scalar potential in  $U_1$  satisfies the equation  $\phi_1|_V = 0$ . Then for the restriction of  $\phi_1$  to the second segment of the magnetic tube we have:  $\phi_1|_W = (1, 2)(3, 1^+) - (1, 3)(2, 1^+)$  (obviously,  $(1, 2)(3, 1^-) - (1, 3)(2, 1^-) = -(1, 2)(3, 1^+) + (1, 3)(2, 1^+)$ ). The gauge (4)-(6) remains fixed by the homotopy, the only term (9) gives a contribution to the crossing change formula by

$$\delta(M) = (2, 3)^2[(1, 2)(3, 1^+) - (1, 3)(2, 1^+)]^2.$$

From the formula (20) (in the case  $(2, 3) \neq 0$ ) we get:

$$(2, 3)^{-2}\delta(\tilde{M}) = (1, 2)(3, 1)[(2, 1^+)(3, 1^-) + (2, 1^-)(3, 1^+)] - (1, 2)^2(3, 1^+)(3, 1^-) - (3, 1)^2(2, 1^+)(2, 1^-).$$

Using the identities  $(3, 1^+) + (3, 1^-) = (3, 1)$ ,  $(2, 1^+) + (2, 1^-) = (1, 2)$ , we get the proof of Lemma 3.4.

From Lemma 3.4 the difference  $M(\mathbf{B}) - \tilde{M}(\mathbf{L})$  is fixed by a singular homotopy with no intersections between different magnetic tubes. By Akhmet'ev-2 (2013), Lemma 11, the invariant  $M$  is derived from a finite-type configuration space. Let us prove that the function  $F(k) : k \mapsto M(\Psi_k(\mathbf{B}))$  is polynomial.

Denote the  $m$ -difference  $\nabla^m F(k)$ ,  $k \in \mathbb{Z}$ , by the inductive formula:

$$\nabla^m F(k) = \nabla(\nabla^{m-1} F(k)) = \nabla^{m-1} F(k) - \nabla^{m-1} F(k-1), \quad i \in \mathbb{Z}.$$

The  $m$ -difference  $\nabla^m F(k)$  is represented as following. Define the collection  $\aleph = \aleph(\varepsilon)$  of  $2^m$  links, which is defined as the modifications of the magnetic tube  $\Psi_k(\mathbf{B})$  by a sequence of  $m$  copies of the delta-move, the delta-move with a number  $i$ ,  $1 \leq i \leq m$ , in the sequence is included, or excluded, correspondingly.

Let us consider an algebraic sum of values of  $M(\alpha)$  over a collection  $\aleph$ ,  $\alpha \in \aleph$ . The algebraic sum is defined as following:  $\sum_{\alpha \in \aleph} \pm M(\alpha)$ , with sign  $+1$  ( $-1$ ), if the corresponding collection  $\alpha$  consists of an even (odd) number of delta-moves, correspondingly. Obviously,  $\nabla^m F(k) = \sum_{\alpha \in \aleph} \pm M(\alpha)$ .

Analogously to Akhmet'ev-2 (2013), Theorem 6, there exists a positive  $m$ , such that for arbitrary  $\mathbf{B}$ ,  $k$  the following identity is satisfied:

$$\nabla^m F(k) = 0.$$

From this equitation we derive that  $F(k)$  is a polynomial of a degree  $\leq m$ .

The invariant  $\tilde{M}$  is a finite type invariant in the sense of V.A.Vassiliev, from the definition we get  $\tilde{F}(k) : k \mapsto \tilde{M}(\Psi_k(\mathbf{L}))$  is polynomial. By the Gusarov's Theorem, see Baader, S. & Marché, J. (2008), Matveev S. & Polyak M. (2009) for details and references,  $\tilde{M}$  is the algebraic sum of representations of a suitable collection of Gauss diagrams in the diagram of  $\Psi_k(\mathbf{L})$ . Therefore, for a sufficiently great  $m$  (where  $m$  is less or equals to the order of  $M$ , which is the maximal numbers of arrows of the Gauss diagrams) we get

$$\nabla^m \tilde{F}(k) = 0.$$

Therefore  $F(k) - \tilde{F}(k) : k \mapsto M(\Psi_k(\mathbf{B})) - \tilde{M}(\Psi_k(\mathbf{L}))$  is a polynomial. Assume that  $k \equiv \gcd\{(1, 2), (2, 3), (3, 1)\}$ . Then the link  $\Psi_k(\mathbf{L})$  is related with  $\mathbf{L}$  by a homotopy with self-intersection of components and with no intersection between different components, see f.ex. Cohen, F.R. *et al.* (2012). Therefore for a such  $k$  we get  $M(\Psi_k(\mathbf{B})) - \tilde{M}(\Psi_k(\mathbf{L})) = 0$ , and therefore  $M = \tilde{M} + C$ , where  $C$  is a polynomial, which depends only of the pairwise linking numbers. The degree of  $C$  satisfy the equality:  $\deg(C) \leq 14$ , because  $\tilde{M}$  is a finite-type invariant in the sense of V.A.Vassiliev of the degree 7, and the assumption  $\deg(C) > 14$  contradicts to the Property (22) below.

Main Theorem is proved.

#### 4. Asymptotic properties for combinatorial links

Let  $\mathbf{K}$  be an arbitrary  $m$ -component framed link with a framed marked component  $(K, \xi)$ ,  $K \subset \mathbf{K}$ . The framing  $\xi$  determines the coordinate system on the boundary of a tubular neighborhood of the component  $K$ , such a coordinate system is represented by a family of parallels and meridians.

For an arbitrary integer  $r \in \mathbb{Z}$  let us define the following two operations:

$$A^r : (\mathbf{K}; K, \xi) \rightarrow \mathbf{L},$$

where  $\mathbf{L}$  is  $m$ -component link, and

$$B^r : (\mathbf{K}; K, \xi) \rightarrow \mathbf{L},$$

where  $\mathbf{L}$  is  $(m + 1)$ -component link.

Denote non-marked components of  $\mathbf{K}$  by  $K_i$ ,  $i = 2, \dots, m$ . By the transformation  $A^r$ , all the non-marked components of the link  $\mathbf{K}$  remain unchanged,  $K_i = L_i$ ,  $i = 2, \dots, m$ , unless the first marked component  $K$ , which is transformed into the standard  $(r, 1)$ -time winding  $L_1$  along  $(K, \xi)$ . The component  $L_1$  passes  $r$  times along the parallel and 1-time along the meridian on the boundary of the thin regular tubular neighborhood of the component  $K$ . In the case  $r = 0$  we get the link  $\mathbf{L}$  with a small non-knotted component  $L_1$ .

Denote non-marked components of  $\mathbf{K}$  by  $K_i$ ,  $i = 3, \dots, m + 1$ . By the transformation  $B^r$ , all the non-marked components of the link  $\mathbf{K}$  remain unchanged,  $K_i = L_i$ ,  $i = 3, \dots, m + 1$ , unless



the first marked component  $K$ , which is transformed into the pair of closed parallel standard  $(r, 1)$ -time windings  $L_1 \cup L_2$  along  $(K, \xi)$ . In the case  $r = 0$  we get the link  $\mathbf{L}$  with two small parallel non-knotted components  $L_1, L_2$ .

Let  $\mathbf{L}$  be a  $m$ -component link with the marked component  $L \subset \mathbf{L}$ , which is represented as  $\mathbf{L} = A^r(\mathbf{K}; K, \xi)$  for some  $r \in \mathbb{Z}$ . let us define the following operations

$$C : \mathbf{L} \rightarrow \mathbf{L}^{tw},$$

where  $\mathbf{L}^{tw}$  is an  $m$ -component link, which is obtained from  $\mathbf{L}$  by an arbitrary homotopy (which is sufficiently small in  $C^1$ -topology), inside a small regular neighborhood  $V_K$  of the marked component  $K \subset \mathbf{K}$ ,  $L \subset V_K$ , with self-intersection points of the component  $L$ . To get a definition of  $C$ , which is not ambiguous, we say that  $\mathbf{L}^{tw}$  is obtained from  $\mathbf{L}$  by the modification, using a prescribed colored  $r$ -component braid. Recall, an arbitrary colored braid is a composition of the elementary braids  $\sigma_{i,i+1}$ ,  $1 \leq i < i+1 \leq r$ , in  $\sigma_{i,i+1}$  the 2-strings with numbers  $i, i+1$  are twisted.

#### 4.1. Definition

Let us say that a finite-type invariant  $I$  for  $(m+1)$ -component links is an asymptotic invariant of the degree  $s$ ,  $s \in \mathbb{N}$ , if the following equations (22) -(24) are satisfied:

$$I(A^r(\mathbf{K}; K, \xi)) = r^s I(\mathbf{K}) + o(r^s), \quad (22)$$

where  $o(r^s)$  is a polynomial of a variable  $r$  of the degree less than  $s$ , coefficients of this polynomial depend only on the isotopy class of the framed link  $(\mathbf{K}; K, \xi)$ .

$$I(B^r((\mathbf{K}; K, \xi))) = r^{2s} I_0 + o(r^{2s}), \quad (23)$$

where  $I_0 \in \mathbb{Z}$  is an integer,  $o(r^{2s})$  is a polynomial of  $r$  of the degree less than  $2s$ , coefficients of this polynomial depend only on the isotopy class of the framed link  $(\mathbf{K}; K, \xi)$ .

$$I(\mathbf{L}) = I(\mathbf{L}^{tw}), \quad (24)$$

where  $\mathbf{L}^{tw} = C(\mathbf{L})$ .

#### 4.2. Theorem

The combinatorial invariant  $M$ , determined by the right side of the formula (21), is an asymptotic invariant in the sense of Definition 4.1 for  $m+1 = 3$ ,  $s = 4$ .

#### Proof of Theorem 4.2

Let us replace the combinatorial invariant  $M$  by the integral invariant  $M$  using Theorem 3.3, and prove the analogous equations for the integral invariant  $M$ .

#### 4.3. Lemma

The integral invariant  $M$ , given by (8), depends of the integral flows of the magnetic field through the cross-sections of the magnetic tubes and of a position of the magnetic tubes in the space, and depends no of magnetic fields (with prescribed integral flows) inside the magnetic tubes  $U_i$ ,  $i = 1, 2, 3$ .

*Proof of Lemma 4.3*

Assume that magnetic fields  $\mathbf{B}$ ,  $\mathbf{B}'$  coincide in  $U_2, U_3$ , but, probably, are different in  $U_1$ . Take the gauge with the trivial scalar potential (1) inside  $U_1$ . We may assume that  $\mathbf{A}_2 = \mathbf{A}'_2$ ,  $\mathbf{A}_3 = \mathbf{A}'_3$ , and  $\mathbf{A}_2, \mathbf{A}_3$  are collinear inside  $U_1$ . Moreover,  $\mathbf{A}_1$  and  $\mathbf{A}'_1$  coincide outside of  $U_1$ . In the presented gauge all terms of  $M(\mathbf{B})$ ,  $M(\mathbf{B}')$  coincide correspondingly. Lemma 4.3 is proved.

The equations (24) is a corollary of Lemma 4.3, which is applied for the magnetic fields  $\mathbf{B}$  and  $\mathbf{B}'$ , which are models of  $\mathbf{L}$  and  $C(\mathbf{L})$  correspondingly.

To proof the equation (22) apply the scaling  $\mathbf{B}_1 \mapsto r\mathbf{B}_1$  to the formula  $M(\mathbf{B})$ . By dimension arguments  $r^4 M(\mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3) = M(r\mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3)$ . By Lemma 4.3  $M(r\mathbf{B}_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3) = M(\mathbf{B}'_1 \cup \mathbf{B}_2 \cup \mathbf{B}_3)$ , where  $\mathbf{B}_1$  is a model for the component  $K \subset \mathbf{K}$ ,  $\mathbf{B}'_1$  is a model for the first component of the link  $A^r(\mathbf{K}; K, \xi)$ .

Let us prove the equation (23). Take an arbitrary 2-component link  $\mathbf{K}$ , where  $(K, \xi)$  is a framed component of  $\mathbf{K}$  inside  $U_1$ . Take the magnetic field  $\mathbf{B} = \mathbf{B}_{1,2}(r) \cup \mathbf{B}_3$ , which is a model for the link  $B^r(\mathbf{K}; K, \xi)$ . This magnetic field is defined by the two parallel magnetic lines  $\mathbf{B}_{1,2}(r) = \mathbf{B}_1(r) \cup \mathbf{B}_2(r)$  on the boundary  $\partial U_1$ , the third magnetic line  $\mathbf{B}_3$ , which is a model for  $K_3 \subset \mathbf{K}$ . Denote the magnetic field  $r^{-1}\mathbf{B}_{1,2}(r)$  by  $\mathbf{B}'_{1,2}$  and take the limit  $r \rightarrow +\infty$  of the links  $\mathbf{B}'_{1,2}$ . This limit is an integrable vector field on  $\partial U_1$ , denoted by  $\mathbf{B}_{1,2}^{+\infty}$ . The constant  $I_0$  is defined by  $I_0 = M(\mathbf{B}_{1,2}^{+\infty} \cup \mathbf{B}_3)$ . If  $r_1, r_2$  are sufficiently large,  $\mathbf{B}_{1,2}^{r_1}, \mathbf{B}_{1,2}^{r_2}$  are closed as integrable magnetic fields. The equation (23) is a corollary of Conjecture 1. Theorem 4.2 is proved.

*Remark*

The combinatorial invariant  $M$  is not a function of pairwise linking coefficients. By Conjecture 1, the invariant  $M$  is ergodic. For a higher ergodic invariant by Baader, S. & Marché, J. (2008), Theorem 1, the following inequality is satisfied:

$$2d(M) > (m+1)s,$$

where  $d(M)$  is the order of the combinatorial invariant  $M$  in the sense of V.A.Vassiliev (recall,  $m+1=3$ ,  $s=4$ ). By the formula (21) we get  $d(M) \geq 7$ , because for a combinatorial invariant  $\tilde{M}$ , defined by the formula (19), we get  $d(\tilde{M}) = 7$ .

**5. Conclusion Remarks**

It is well known that Hopf invariants in stable homotopy groups of spheres relates with  $E^1$ -term of the Adams spectral sequence. The helicity integral detects the Hopf bundle  $S^3 \rightarrow S^2$ , which is the main example of a mapping with the Hopf invariant 1. By Browder (1969) the Arf-Kervaire invariants, which are the "quadratic constructions" on the Hopf invariants, are related with  $E^2$ -term of the Adams spectral sequence. Main Theorem of the paper determines a connection of a higher helicity integral  $M$  with the Arf-invariants: the invariants  $\gamma, \beta$  are the integer lifts of the Arf-invariants of 3- and 2-component links. Integer lifts of elements of homotopy groups of the 2-sphere into finite-type invariants of spherical braids is the main subject of the survey by Cohen, F.R. & Wu, J. (2004). From an author's opinion, the problem by V.I.Arnol'd should be investigated using this algebra.

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