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Higher order topological invariants from the Chern-Simons action

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Abstract. It is well known that for a field theory with Chern-Simons action, expectation values of Wilson line operators are topological invariants. The standard result is expressed in terms of the Gaussian linkings of closed curves defining the operators. We show how a judicious choice of Wilson lines leads to higher order topological linkings.

1. Introduction

Our understanding of knots and links in topological field theory began with the work of Polyakov (Polyakov, 1988) who showed that the expectation value of a Wilson line in Chern-Simons theory gives the Gaussian linking of the components of a link. This work was soon followed by Witten's (Witten, 1989) extensive exploration of expectation values of Wilson lines in non-abelian Chern-Simons theory which he showed is a natural framework for understanding Jones polynomials from knot theory. Perturbative expansion of the expectation values leads to higher order linking and the corresponding link polynomials (Guadagnini *et al.*, 1989). Our approach is somewhat different. Wilson loops in the abelian Chern-Simons theory also implicitly contains the requisite apparatus for the study of higher order linking, and we show here that this can be facilitated by a set of gauges (Buniy & Kephart, 2008a,b, 2006) tailored to the higher order linking problem. We begin with a discussion of intersection number and its relation to Gaussian linking. Next we review the minimal required background from Poincaré duality and de Rham theorem needed to carry out our calculations. We then introduce a special set of gauge potentials that allow us to arrive at our main result—higher order linking invariants at all orders. We conclude with detailed computations for several examples where linkings of various orders appear.

2. Intersection and linking numbers

Suppose C and C' are disjoint oriented closed curves in $\mathbb{R}^3$, and S and S' are surfaces such that $\partial S = C$ and $\partial S' = C'$. By smooth deformations of C which leave it in $\mathbb{R}^3\backslash C'$, we can change the number of points of intersection of C and S'. To find a quantity which is invariant under such deformations, we note that additional points that appear due to the deformations come in pairs, and the intersections of C and S' have opposite orientations for points in each pair (and likewise for disappearing pairs). If C and S' intersect transversely at a point P, we define the intersection index $I(C, S', P)$ to be equal 1 or -1 depending on the relative orientation of C and

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S' at P. We also define the intersection number $I(C, S')$ as the sum of the intersection indices over all points of intersection,

$$I(C, S') = \sum_{P \in C \cap S'} I(C, S', P). \tag{1}$$

It is clear that $I(C, S')$ is invariant under the above deformations as contributions due to additional pairs of points cancel.

One can also deform S' into $\tilde{S}'$ and notice that $I(C, S') = I(C, \tilde{S}')$, where $\partial S' = \partial \tilde{S}' = C'$. This means that the intersection number depends only on C and C'; for this reason we call it the linking number, $L(C, C') = I(C, S')$. Examination of relevant orientations gives $L(C, C') = L(C', C)$. The method of Green functions leads to an explicit Gauss expression for the linking number of the curves in terms of their parametrizations,

$$L(C, C') = (4\pi)^{-1} \int_{C \times C'} \sum_{a,b,c} \varepsilon_{abc} \frac{(x - x')^a}{\|x - x'\|^3} dx^b \wedge dx'^c. \tag{2}$$

3. Duality

From Poincaré duality and de Rham theorem (Griffiths & Harris, 1978; Hatcher, 2002), for a closed curve C in three dimensions, there exists a closed 2-form F such that for any 1-form B we have

$$\int_C B = \int_{\mathbb{R}^3} B \wedge F. \tag{3}$$

We call (C, F) a dual set. Since B is arbitrary, it is clear that F has support only on C, i.e. $\operatorname{supp} F = C$. Since $\mathbb{R}^3$ is simply connected, there exists a 1-form A such that $F = dA$. The Stokes theorem then gives [1]

$$\int_S dB = \int_{\mathbb{R}^3} dB \wedge A. \tag{4}$$

This means that there is a particular solution A such that

$$A(x) = \int_{y \in S} \delta(x - y) \sum_a dx^a * dy^a. \tag{5}$$

This is a singular gauge with $\operatorname{supp} A = S$. We can similarly introduce a dual set (C', F') such that $F' = dA'$ and for any 1-form B' we have

$$\int_{C'} B' = \int_{\mathbb{R}^3} B' \wedge F'. \tag{6}$$

Taking $B = A'$, $B' = A$ and using the Stokes theorem again, we find

$$\int_C A' = \int_{C'} A = \int_{\mathbb{R}^3} A \wedge dA' = L(C, C'). \tag{7}$$

Since $L(C, C')$ is a topological invariant, smooth deformations of C which leave it in $\mathbb{R}^3 \backslash C'$ should not change its value. This is possible if and only if $dA'|_{\mathbb{R}^3 \backslash C'} = 0$. If A' is exact, then $L(C, C') = 0$, and so a nontrivial case is when A' is a closed 1-form which is not exact.

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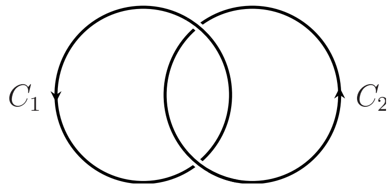

Figure 1. The Hopf link.

As an example with the first order linking, consider the Hopf link (Rolfsen, 1990; Kauffman, 2001) in Fig. 1. We use the singular gauge (5) to compute

$$L(C_1, C_2) = \int_{\mathbb{R}^3} dA_1 \wedge A_2 = 1. \tag{8}$$

This follows since we can choose $C_1 \cap S_2 = P$, where P is a point. The first order non-self linking is $\tilde{L}_1(C, C) = 2$. For $C = \cup_{1 \le i \le N} C_i$, we find

$$L_1(C, C) = \sum_{i,j} L(C_i, C_j), \tag{9}$$

which is in agreement with results in Refs. (Polyakov, 1988; Witten, 1989).

4. Second order fields

Suppose $\{C_i\}_{1 \le i \le N}$ are disjoint closed curves in $\mathbb{R}^3$ and let $F_i = dA_i$ be dual to C_i. We now construct a dual set (C_{ij}, F_{ij}) of the second order which satisfies

$$\int_{C_{ij}} B = \int_{\mathbb{R}^3} B \wedge F_{ij} \tag{10}$$

for any 1-form B and $i \neq j$. Since B is arbitrary, it follows that $\operatorname{supp} F_{ij} = C_{ij}$. The most general 2-form which can be expressed in terms of A_i and A_j is

$$F_{ij} = f_i dA_j - f_j dA_i + g A_i \wedge A_j, \tag{11}$$

where f_i, f_j and g are arbitrary functions. In order for F_{ij} to be closed, the functions have to satisfy certain conditions. A requirement $dF_{ij}|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} = 0$ gives $dg|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} = 0$ and without loss of generality we set $g = 1$. Requirements $dF_{ij}|_{C_i} = 0$ and $dF_{ij}|_{C_j} = 0$ then give

$$(df_j - A_j)|_{C_i} = 0, \tag{12}$$

$$(df_i - A_i)|_{C_j} = 0. \tag{13}$$

Integrating these conditions, we find a constraint $L(C_i, C_j) = 0$. This means that the second order field associated with a pair of closed curves can be defined only if the curves are unlinked.

Since $dF_{ij} = 0$, there exists a 1-form A_{ij} such that $F_{ij} = dA_{ij}$. We seek a solution in the form

$$A_{ij} = \tfrac{1}{2}(\gamma_i A_j - \gamma_j A_i). \tag{14}$$

[1] We assume that values of appropriate quantities vanish at infinity, so there is no contribution from boundary terms. Boundary terms also vanish if we use $\mathbb{S}^3$ instead of $\mathbb{R}^3$.

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From a requirement $F_{ij} = dA_{ij}$, the unknown functions γ_i and γ_j are found to satisfy

$$(d\gamma_i - A_i)|_{\mathbb{R}^3\backslash(C_i \cup C_j)} = 0, \tag{15}$$

$$d\gamma_i|_{C_i} = 0, \tag{16}$$

$$(d\gamma_i - 2A_i)|_{C_j} = 0, \tag{17}$$

$$(\gamma_i - 2f_i)|_{C_j} = 0 \tag{18}$$

and the same expressions with i and j interchanged. This implies $\gamma_i|_{\mathbb{R}^3\backslash(\cup_k C_k)} = \int_{\Gamma_i} A_i$, where Γ_i is a curve in $\mathbb{R}^3\backslash(\cup_k C_k)$; this means γ_i is a nonlocal quantity. If $S_i \cap C_j = \varnothing$, then $d\gamma_i|_{C_j} = 0$, and so $\gamma_i|_{C_j}$ is a constant. If $S_i \cap C_j \neq \varnothing$, then $S_i \cap S_j = \cup_m \big(S_i \cap S_j\big)_{(m)}$, where m labels disjoint segments of the intersection; for an example, see Fig. 2. Let $C_{j(m)}$ be the segment

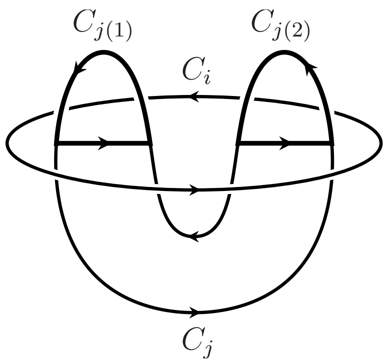

Figure 2. An example of the construction of the second order curve C_{ij}. Two components of C_{ij} are drawn with thick lines. S_i is a disk bounded by C_i, and the thick line segments in this disk are two disjoint segments of $S_i \cap S_j$.

of C_j which closes the curve $\big(S_i \cap S_j\big)_{(m)}$ and agrees with its orientation; this closed curve is $C_{ij(m)} = C_{j(m)} \cup \big(S_i \cap S_j\big)_{(m)}$. We define $C'_j = \cup_m C_{j(m)}$ and its complement in C_j is C''_j. It follows from the above relations that $\gamma_i|_{C'_j}$ and $\gamma_i|_{C''_j}$ are constants such that $\gamma_i|_{C'_j} - \gamma_i|_{C''_j} = 2$. Without loss of generality, we set $\gamma_i|_{C'_j} = 2$ and $\gamma_i|_{C''_j} = 0$. Using the definition of F_{ij}, the duality condition now becomes

$$\int_{C_{ij}} B = -\int_{C'_i} B + \int_{C'_j} B + \int_{S_i \cap S_j} B. \tag{19}$$

This implies $C_{ij} = C_i'^{-1} \cup C'_j \cup \big(S_i \cap S_j\big)$, which agrees with $\operatorname{supp} F_{ij} = C_{ij}$. See Fig. 2 for an example of the above construction. The Stokes theorem gives

$$\int_{S_{ij}} dA = \int_M dA \wedge A_{ij}, \tag{20}$$

where a surface S_{ij} is such that $\partial S_{ij} = C_{ij}$. This means $\operatorname{supp} A_{ij} = S_{ij}$.

The above construction requires $L(C_i, C_j) = 0$. Since the curves C_i and C_j are unlinked, the surfaces S_i and S_j can be chosen to be disjoint, in which case $dA_{ij}|_{\mathbb{R}^3\backslash(C_i \cup C_j)} = 0$. One can similarly proceed to construct higher order fields. For example, if $L(C_{ij}, C_k) = 0$ for all (i, j, k), then we can define the third order 1-forms A_{ijk}. In particular, they satisfy

$$dA_{ijk}|_{\mathbb{R}^3\backslash(C_i \cup C_j \cup C_k)} = (A_i \wedge A_{jk} + A_{ij} \wedge A_k)|_{\mathbb{R}^3\backslash(C_i \cup C_j \cup C_k)} \tag{21}$$

for $i \neq j \neq k$.

If all linkings of order q, where $1 \leq q \leq p-1$, vanish, then we can similarly construct dual sets (C_{I_p}, dA_{I_p}), where $I_p = (i_1, i_2, \ldots, i_p)$. The quantities dA_{I_p} are related to what is known in algebraic topology as the Massey products of cohomology groups (Massey, 1959, 1968; Kraines, 1966; O'Neill, 1979; Fenn, 1983); see also Refs. (Monastyrsky & Retakh, 1986; Berger, 1990).

As an example with second order linking, consider the Borromean rings (Rolfsen, 1990; Kauffman, 2001) in Fig. 3. We use the singular gauge to compute

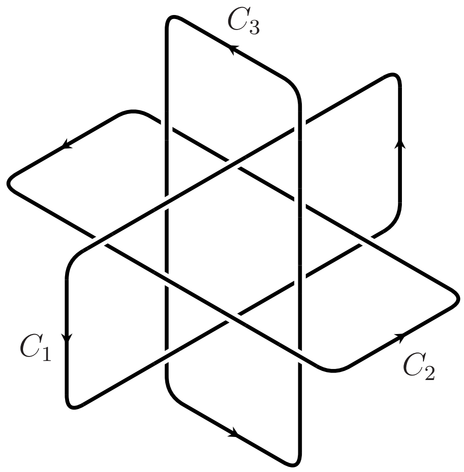

Figure 3. The Borromean rings.

$$L(C_i, C_j) = \int_{\mathbb{R}^3} dA_i \wedge A_j = 0 \tag{22}$$

for $i \neq j$. This follows since we can choose $C_i \cap S_j = \varnothing$ for $i \neq j$. We take $C = C_1 \cup C_2 \cup C_3$ and find that the first order non-self linking vanishes,

$$\tilde{L}_1(C, C) = 2L(C_1, C_2) + 2L(C_2, C_3) + 2L(C_3, C_1) = 0. \tag{23}$$

We define the second order curves $\{C_{ij}\}$ for $i \neq j$. Again we use the singular gauge to compute

$$L(C_{ij}, C_k) = \int_{\mathbb{R}^3} A_i \wedge A_j \wedge A_k - \tfrac{1}{2} \int_{C_i} \gamma_j A_k + \tfrac{1}{2} \int_{C_j} \gamma_i A_k = \varepsilon_{ijk}, \tag{24}$$

where (i, j, k) is a permutation of $(1, 2, 3)$. The integral over $\mathbb{R}^3$ equals ε_{ijk} since we can choose $S_1 \cap S_2 \cap S_3 = P$, where P is a point. The integrals over C_i and C_j vanish since we can choose S_k such that $C_i \cap S_k = \varnothing$ and $C_j \cap S_k = \varnothing$. Similarly,

$$\tilde{L}(C_{ij}, C_j) = \int_{\mathbb{R}^3} A_i \wedge A_j \wedge A_j - \tfrac{1}{2} \int_{C_i} \gamma_j A_j + \tfrac{1}{2} \int_{C_j} \gamma_i A_j = 0 \tag{25}$$

for any $i \neq j$. We take $C = C_1 \cup C_2 \cup C_3 \cup C_{12} \cup C_{23} \cup C_{31}$ and find that the second order non-self linking of C is

$$\tilde{L}_2(C, C) = 2L(C_{12}, C_3) + 2L(C_{23}, C_1) + 2L(C_{31}, C_2) = 6. \tag{26}$$

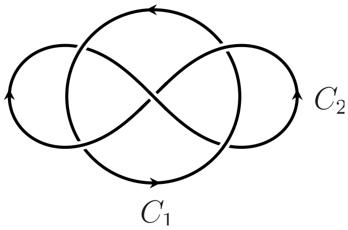

Figure 4. The Whitehead link.

As an example with the third order linking, consider the Whitehead link (Rolfsen, 1990; Kauffman, 2001; Monastyrsky & Retakh, 1986) in Fig. 4. We use the singular gauge to compute

$$L(C_1, C_2) = \int_{\mathbb{R}^3} dA_1 \wedge A_2 = 0. \tag{27}$$

This follows since we can choose $C_1 \cap S_2 = P_1 \cup P_2$, where P_1 and P_2 are points at which the contributions of $dA_1 \wedge A_2$ to the integral cancel due to the opposite orientations. We take $C = C_1 \cup C_2$ and find that the first order non-self linking vanishes, $\tilde{L}_1(C, C) = 0$.

We define the second order curve C_{12} and use the singular gauge to compute

$$L(C_{12}, C_k) = \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_k - \tfrac{1}{2} \int_{C_1} \gamma_2 A_k + \tfrac{1}{2} \int_{C_2} \gamma_1 A_k = 0 \tag{28}$$

for $k = 1$ or $k = 2$. We take $C = C_1 \cup C_2 \cup C_{12}$ and find that the second order non-self linking vanishes, $\tilde{L}_2(C, C) = 0$.

We use the singular gauge to compute the third order self linking,

$$L(C_{12}, C_{12}) = \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_{12} = 2. \tag{29}$$

This follows since S_{12} is the union of disks $S_{12(1)}$ and $S_{12(2)}$ such that $S_{12(1)} \cap S_{12(2)} = P$, where P is a point, and we can choose S_1 and S_2 such that $S_1 \cap S_2 \cap S_{12} = P$.

We define the third order curves C_{121} and C_{212}. We use the singular gauge to compute

$$L(C_{121}, C_2) = \int_{\mathbb{R}^3} (A_1 \wedge A_{21} + A_{12} \wedge A_1) \wedge A_2 = 2 \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_{12} = 4 \tag{30}$$

and similarly $L(C_{212}, C_1) = 4$. We take $C = C_1 \cup C_2 \cup C_{12} \cup C_{121} \cup C_{212}$ and find that the third order non-self linking of C is

$$\tilde{L}_3(C, C) = 2L(C_{121}, C_2) + 2L(C_{212}, C_1) = 16. \tag{31}$$

5. Path integral

We want to construct a field theory for which expectation values of observables are topological invariants of various orders. For this we need to specify the action and the observables. We choose the Chern-Simons action,

$$S(B) = \int_{\mathbb{R}^3} B \wedge dB, \tag{32}$$

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since it has all the necessary topological properties (Witten, 1989). First, it does not depend on the choice of metric. Second, it can be related to linking numbers in the following way. Suppose Γ_α is a closed curve on which dB takes a constant value dB_α. (In order for Γ_α to be closed, we may need to identify points at infinity by considering $\mathbb{S}^3$ instead of $\mathbb{R}^3$. We also avoid field configurations with sources, such as monopoles.) Since a union of curves $\Gamma = \cup_\alpha \Gamma_\alpha$ densely fills $\mathbb{R}^3$, an arbitrary 1-form B can be written as $B = \sum_\alpha B_\alpha$. The action becomes

$$S(B) = \sum_{\alpha,\beta} L(\Gamma_\alpha, \Gamma_\beta), \tag{33}$$

and so (apart from the choice of measure for Γ) we interpret it as the self-linking $L(\Gamma,\Gamma)$ of the set of closed field lines of dB.

If $C = \cup_\alpha C_\alpha$ is a union of disjoint closed curves, then an integral $\int_C B$ is invariant with respect to deformations of C if and only if $dB = 0$. Since $dB = 0$ is the classical equation of motion for $S(B)$, the integral is a topological observable (at least in the semiclassical limit). Proceeding as above, we find

$$\int_C B = \sum_{\alpha,\beta} L(\Gamma_\alpha, C_\beta), \tag{34}$$

which we interpret as the linking $L(\Gamma, C)$ of the sets of closed field lines of dB with C. Since the measure in the path integral is $\exp(iS(B))$, it is convenient to consider as an observable a Wilson loop operator

$$W(C,B) = \exp\Big(i\int_C B\Big). \tag{35}$$

We thus need to compute the expectation value

$$Z(C) = \int DB\, \exp(iS(B)) W(C,B). \tag{36}$$

Using duality, this becomes

$$Z(C) = \int DB\, \exp\Big(i\int_{\mathbb{R}^3} B\wedge d(B+A)\Big), \tag{37}$$

where dA is dual to C. The change of the variable $B = B' - \frac{1}{2}A$ gives four terms in the exponent. One term corresponds to the path integral of the measure without a Wilson loop, $Z(\varnothing)$, the two mixed terms combine into a boundary term, and the forth is the linking invariant. Ignoring the boundary term, we find

$$Z(C) = Z(\varnothing)\exp\big(-\tfrac{1}{4}i(L(C,C))\big). \tag{38}$$

A product of Wilson loop operators can be written as a single Wilson loop operator for the union of the corresponding loops,

$$\prod_p \prod_{I_p} W(C_{I_p}, B) = W(C,B), \tag{39}$$

where $C = \cup_p \cup_{I_p} C_{I_p}$ and C_{I_p} is a curve of order p. It follows that

$$L(C,C) = \sum_{p,q}\sum_{I_p,J_q} L(C_{I_p}, C_{J_q}). \tag{40}$$

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Here $L(C_{I_p}, C_{J_q})$ is the first order linking of curves C_{I_p} and C_{J_q}. This quantity can also be interpreted as a linking of order $p+q-1$ of the curves $\{C_i\}_{1\leq i\leq N}$. In general, $L(C,C)$ is a sum of linkings of various orders,

$$L(C,C) = \sum_r L_r(C,C), \tag{41}$$

where the expressions for several lowest order terms are

$$L_1(C,C) = \sum_{i,j} L(C_i, C_j), \tag{42}$$

$$L_2(C,C) = \sum_{i,j,k} L(C_{ij}, C_k), \tag{43}$$

$$L_3(C,C) = L_{3,1}(C,C) + L_{3,2}(C,C), \tag{44}$$

$$L_{3,1}(C,C) = \sum_{i,j,k,l} L(C_{ijk}, C_l), \tag{45}$$

$$L_{3,2}(C,C) = \sum_{i,j,k,l} L(C_{ij}, C_{kl}). \tag{46}$$

6. Conclusions

In Chern-Simons theory the first order (Gaussian) linking of two curves C_1 and C_2 associated with the 1-forms A_1 and A_2 can be ascribed to the topological properties of the expectation value of the associated Wilson lines. Here we have generalized this idea to the case of higher order linking. First, if C_1 and C_2 are unlinked, then we can define a new second order 1-form A_{12} via Eq. (14). Unlike C_1 and C_2, the associated curve C_{12} is not fixed in space, but is however sufficient for our topological needs, which are to investigate its linking with other curves, since this property is invariant. If, for instance, we have a third curve C_3 unlinked with both C_1 and C_2, but linked with C_{12}, then we find nonzero second order linking, i.e., a nonvanishing second order topological invariant. The Borromean rings is the simplest topological configuration for which this invariant is nonzero. The argument generalizes to higher order invariants. The Whitehead link is the simplest example with a nonvanishing third order topological invariant. For curves $\{C_i\}_{1\leq i\leq N}$, with no linking of orders $1, 2, \ldots, p-1$ and nonzero linking of order p, the general results are obtained from the path integral of the expectation value of a product of Wilson loop operators which depend on curves of order p in the Chern-Simons gauge theory. One could apply these results to the investigation of higher order link polynomials.

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