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Generalized helicity and Beltrami fields

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Abstract. We propose Lorentz-covariant generalizations of the magnetic helicity and Beltrami equation. The gauge invariance, variational principle, conserved current, energy-momentum tensor and choice of boundary conditions elucidate the subject. In particular, we prove that any extremal of the Maxwell action functional $\frac{1}{4}\int_\Omega F_{\mu\nu}F^{\mu\nu}\, d^4x$ subject to the local constraint $\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 0$ satisfies the covariant Beltrami equation.

1. Introduction

The introduction of the concept of magnetic helicity (Woltier, 1958; Moffatt, 1969, 1985) revolutionized our understanding of plasma physics phenomena, from dynamos (Moffatt, 1978) to the solar wind, to the operation of controlled fusion devices (Marsh, 1996; Bellan, 2000, 2006), and it plays a central role when applied to a variety of concepts such as the Beltrami equation and force-free fields in the form of Taylor states (Taylor, 1974, 1986). Helicity was introduced in a three-dimensional context, but we find it useful to express it covariantly (Carter, 1978; Bekenstein, 1987). We propose and explore a covariantization of the Beltrami equation, and in particular study its relation to helicity conservation. For instance, new terms can arise that vanish in the non-relativistic case, but which can contribute when we have a multi-component helicity system where the components are moving with relativistic velocities with respect to each other. This could be the case for helicity in relativistic plasmas (Lichnerowicz, 1967) ejected from astrophysical objects colliding with another plasma clouds (Goedbloed *et al.*, 2010), but it can also apply in particle physics to the hadronization process where relativistic flux tubes interact (Casher *et al.*, 1979; Neuberger, 1979; Casher *et al.*, 1980; Buniy & Kephart, 2003, 2005; Buniy *et al.*, 2014) or to the early universe as it cools through various epochs.

2. Non-covariant case

We first briefly review the three-dimensional Beltrami equation and helicity.

Consider a region $\Omega \subset \mathbb{R}^3$ and let (x, ∇) be the Cartesian coordinates and the derivative operator in Ω. A vector field B in Ω is called a Beltrami vector field if it satisfies

$$B \times (\nabla \times B) = 0, \tag{1}$$

where $\times$ is the vector product in $\mathbb{R}^3$. Eq. (1) implies that the vectors $\nabla \times B$ and B are parallel, which leads to the Beltrami equation

$$\nabla \times B = \lambda B, \tag{2}$$

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where λ is a scalar function in Ω.

Eq. (2) shows that Beltrami fields are eigenfields of the curl operator. Eigenfields of the curl operator can be related to more familiar functions by using the identity

$$\nabla \times \nabla \times B = \nabla(\nabla \cdot B) - \Delta B, \tag{3}$$

where $\cdot$ is the scalar product and Δ is the Laplace operator in $\mathbb{R}^3$. It follows that the square of the curl operator, when restricted to the space of divergence-free vector fields, is the negative of the Laplace operator $-\Delta$. Thus, in some sense, the curl operator is the square root of the operator $-\Delta$ (which itself is a positive operator).

The restriction to the space of divergence-free vector fields is not accidental, but is required by physical considerations of B being a magnetic field. In such a case, the divergence-free condition $\nabla \cdot B = 0$ for B in (2) implies

$$B \cdot \nabla\lambda = 0, \tag{4}$$

so that λ is constant along any field line of B. Eq. (4) is the consistency condition for (2). According to the Beltrami equation (2), the Maxwell current $J = \nabla \times B$ is parallel to the magnetic field, $J = \lambda B$. Note that the current conservation $\nabla \cdot J = 0$ also implies (4).

To learn more about a Beltrami field B, it is instructive to consider a vector potential A such that $B = \nabla \times A$. Since a vector potential is defined only up to the gradient of an arbitrary function, it will be important to ensure gauge invariance of various physical quantities under a gauge transformation

$$A \mapsto A + \nabla g, \tag{5}$$

where g is an arbitrary real-valued function in Ω. The two simplest such gauge invariant quantities are the energy W and helicity H of the field B in the region Ω,

$$W = \int_\Omega \tfrac{1}{2}\|B\|^2\, d^3x, \tag{6}$$

$$H = \int_\Omega A \cdot B\, d^3x, \tag{7}$$

where $\| \; \|$ is the scalar norm in $\mathbb{R}^3$.

Convergence of the integrals in (6) and (7) imposes certain restrictions on A and B. We are concerned here with the case of a non-compact Ω and restrictions derived from the required behavior of A and B for $\|x\| \to \infty$. In such a case, the integral in (7) converges if

$$A = O(\|x\|^p), \;\; \|x\| \to \infty, \;\; p < -1. \tag{8}$$

This leads to $B = O(\|x\|^{p-1})$, $\|x\| \to \infty$, which means that allowed field configurations do not include magnetic monopoles. It follows that the integral in (6) also converges for such fields.

The gauge invariance of the energy is obvious, while the corresponding gauge transformation of the helicity is

$$H \mapsto H + \int_{\partial\Omega} g(n \cdot B) d^2\sigma, \tag{9}$$

where $\partial\Omega$ is the boundary of Ω, n is the unit normal vector to $\partial\Omega$, and $d^2\sigma$ is the area differential on $\partial\Omega$. Since we require gauge invariance of H, we set the boundary condition

$$(n \cdot B)|_{\partial\Omega} = 0, \tag{10}$$

which means that the field lines do not cross the boundary. For a non-compact Ω, the asymptotic behavior $B = O(\|x\|^{p-1})$, $\|x\| \to \infty$ ensures the gauge invariance of H as well since the boundary integral in (9) vanishes.

The helicity is often conserved in physical systems involving magnetic fields, and this restricts their dynamics. For example, suppose that B is a field in Ω satisfying the boundary condition (10) which minimizes its energy W and conserves its helicity H. The resulting variational problem is equivalent to finding B minimizing the functional

$$W - \tfrac{1}{2}\lambda H = \int_{\Omega} L\, d^3x \tag{11}$$

with the Lagrangian

$$L = \tfrac{1}{2}\|B\|^2 - \tfrac{1}{2}\lambda A \cdot B. \tag{12}$$

(We have chosen the form of the Lagrange multiplier λ which leads to the conventional form of the Beltrami equation.) For an infinitesimal variation of the vector potential δA, we find

$$\delta L = (\nabla \times B - \lambda B) \cdot \delta A + \nabla \cdot \big[(-B + \tfrac{1}{2}\lambda A) \times \delta A\big], \tag{13}$$

which leads to

$$\delta(W - \tfrac{1}{2}\lambda H) = \int_{\Omega} (\nabla \times B - \lambda B) \cdot \delta A\, d^3x + \int_{\partial\Omega} n \cdot \big[(-B + \tfrac{1}{2}\lambda A) \times \delta A\big]\, d^2\sigma. \tag{14}$$

We eliminate the boundary term in (14) by setting the boundary condition

$$\delta A|_{\partial\Omega} = 0. \tag{15}$$

The variation (14) vanishes for any δA satisfying (15) if and only if $\nabla \times B = \lambda B$. Thus, a Beltrami field with $\lambda = \text{const}$ is a stationary point of the energy functional W subject to the condition $H = \text{const}$. It can be further proved that such a field is a local minimum of W with constant H.

Another aspect of the helicity relates to the conserved Noether current. Gauge transformations are the symmetry operations of the theory defined by the Lagrangian L. The proof of the invariance of the theory requires showing (without using the equation of motion) that L is changed only by the divergence term. (For the following derivation we assume λ is constant.) Indeed, for a gauge transformation (5) with $\delta A = \nabla g$, (13) becomes

$$\delta L = \nabla \cdot \Gamma, \tag{16}$$

$$\Gamma = -\tfrac{1}{2}\lambda g B. \tag{17}$$

On the other hand, using the equation of motion we find

$$\delta L = \nabla \cdot \left(\frac{\partial L}{\partial \nabla A_k} \delta A_k\right). \tag{18}$$

Equating (16) and (18), we arrive at the conserved Noether current ($\nabla \cdot j = 0$),

$$j = \frac{\partial L}{\partial \nabla A_k} \delta A_k - \Gamma, \tag{19}$$

$$j = (-B + \tfrac{1}{2}\lambda A) \times \nabla g + \tfrac{1}{2}\lambda g B. \tag{20}$$

Since the gauge function g is arbitrary, we can define another conserved Noether current k by

$$gk = j + \nabla \times \big[(-B + \tfrac{1}{2}\lambda A)g\big]. \tag{21}$$

and find

$$k = -\nabla \times B + \lambda B. \tag{22}$$

Now the Beltrami equation (2) gives $k = 0$, so that the Noether current associated with the helicity, λB, equals the Maxwell current $J = \nabla \times B$. As expected, there is only one independent conserved current in the problem.

Computing the Noether energy-momentum tensor

$$\begin{aligned}\theta^i{}_j &= \frac{\partial L}{\partial(\nabla_i A_k)}\nabla_j A_k - \delta^i{}_j L \\ &= (B_l - \tfrac{1}{2}\lambda A_l)\varepsilon^{lik}\nabla_j A_k - \delta^i{}_j\big(\tfrac{1}{2}\|B\|^2 - \tfrac{1}{2}\lambda A\cdot B\big),\end{aligned} \tag{23}$$

we find that its divergence

$$\nabla_i \theta^i{}_j = (-\nabla \times B + \lambda B)^k \nabla_j A_k \tag{24}$$

vanishes for any solution of the Beltrami equation (2), which implies the conservation equation $\nabla_i \theta^i{}_j = 0$.

We now derive the lower bound for the energy in terms of helicity and constant λ (Arnold & Khesin, 1978). We first integrate the Beltrami equation (2) once to obtain

$$\nabla \times A = \lambda A + \nabla\varphi, \tag{25}$$

where φ is an arbitrary scalar function in Ω. We can now use the gauge transformation (5) with $g = -\lambda^{-1}\varphi$ to replace (25) with

$$\nabla \times A = \lambda A, \tag{26}$$

which is of the same form as (2). Hence in this gauge $B = \lambda A$, which gives

$$W = \tfrac{1}{2}\lambda H. \tag{27}$$

Although (26) is not gauge invariant, its consequence, (27), is gauge invariant. We conclude that the minimal value of the variational functional $W - \frac{1}{2}\lambda H$ equals zero for any solution of the Beltrami equation with constant λ.

The field satisfying $B = \lambda A$ saturates the lower bound for the energy in terms of helicity (Arnold & Khesin, 1978). To derive this, we consider a non-local operator curl^{-1} acting on the space of divergence-free vector fields. We use the Schwarz inequality

$$\left|\int_\Omega B\cdot \mathrm{curl}^{-1} B\, d^3x\right| \le \left[\int_\Omega \|B\|^2\, d^3x\right]^{1/2}\left[\int_\Omega \|\mathrm{curl}^{-1} B\|^2\, d^3x\right]^{1/2} \tag{28}$$

and the Poincaré inequality

$$\int_\Omega \|\mathrm{curl}^{-1} B\|^2\, d^3x \le C^{-2}\int_\Omega \|B\|^2\, d^3x, \tag{29}$$

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where $C > 0$ is a certain constant depending on Ω. Combination of the two inequalities gives $W \geq \frac{1}{2} C\,|H|$. Finally, using the Rayleigh min-max theorem

$$B \cdot \operatorname{curl}^{-1} B \leq |\mu|_{\max} \|B\|^2, \tag{30}$$

where

$$|\mu|_{\max} = \max_a |\mu_a|, \tag{31}$$

$$\operatorname{curl}^{-1} B_a = \mu_a B_a, \tag{32}$$

we see that we can use $C = 2\,|\mu|_{\max}^{-1}$ and find

$$W \geq \tfrac{1}{2}\,|\mu|_{\max}^{-1}\,|H|\,. \tag{33}$$

It is clear that the field satisfying $B = \lambda A$ saturates the bound (33) since in this case we have $|\mu|_{\max}^{-1} = |\lambda|$ and $W = \frac{1}{2}\,|\lambda|\,|H|$.

3. Covariant case

The proceeding non-covariant analysis is sufficient for the description of magnetic fields in nonrelativistic plasmas. Generalizations to the electric case have been carried out (Ranada, 1989) and applied (Irvine & Bouwmeester, 2008; Arrayas & Trueba, 2012), but relativistic plasmas require a full covariant analysis. In particular, this applies to Beltrami fields and helicity.

To derive the covariant forms of equations obtained in the preceding section (Buniy & Kephart, 2014), we consider Lorentzian $(\mathbb{R}^{1,3}, \Omega, x, \nabla)$, where $\mathbb{R}^{1,3}$ has a constant pseudo-Riemannian metric with signature $(1,3)$ and $\Omega = [t_1, t_2] \times \Omega'$, $\Omega' \subset \mathbb{R}^3$. The magnetic field B is now a part of the gauge field strength tensor $F_{\mu\nu}$. Using

$$B_i = \tfrac{1}{2}\varepsilon_{ijk} F^{jk}, \tag{34}$$

we first write (1), (2), (4) in the form

$$F_i{}^j \nabla^k F_{jk} = 0, \tag{35}$$

$$\nabla^j F_{ij} = \tfrac{1}{2}\lambda \varepsilon_{ijk} F^{jk}, \tag{36}$$

$$\varepsilon^{ijk} F_{jk} \nabla_i \lambda = 0, \tag{37}$$

then setting

$$E_i = F_{i0}, \tag{38}$$

$$\varepsilon_{0ijk} = \varepsilon_{ijk}, \tag{39}$$

$$\lambda_0 = \lambda, \tag{40}$$

we arrive at the covariant form of (35), (36), (37),

$$F_\alpha{}^\mu \nabla^\nu F_{\mu\nu} = 0, \tag{41}$$

$$\nabla^\nu F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} \lambda^\nu F^{\alpha\beta}, \tag{42}$$

$$\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \nabla_\mu \lambda_\nu = 0. \tag{43}$$

These covariant expressions require that we identify non-covariant λ with the time component of a 4-vector λ_μ. Note that the left-hand side of (43) vanishes identically if we set

$$\nabla_\mu \lambda_\nu - \nabla_\nu \lambda_\mu = 0, \tag{44}$$

The requirement (44) will appear later in the variational formulation of the problem.

Similarly to (2) implying (4) and (4) not implying (2) for $\nabla \cdot B = 0$, we have (42) implying (43) and (43) not implying (42). However, although (1) and (2) are equivalent, their covariant counterparts (41) and (42) are not equivalent; in fact, none of the two implies the other.

In terms of the E and B fields, the time and space components of (41) become

$$E \cdot \nabla_0 E - E \cdot (\nabla \times B) = 0, \tag{45}$$
$$E(\nabla \cdot E) + B \times (\nabla_0 E) - B \times (\nabla \times B) = 0, \tag{46}$$

the time and space components of (42) become

$$-\nabla \cdot E = \lambda \cdot B, \tag{47}$$
$$-\nabla_0 E + \nabla \times B = \lambda_0 B + \lambda \times E, \tag{48}$$

and (43) becomes

$$-B \cdot \nabla \lambda_0 + B \cdot \nabla_0 \lambda + E \times (\nabla \times \lambda) = 0. \tag{49}$$

Note that the left-hand side of (49) vanishes identically if we set $\nabla \lambda_0 - \nabla_0 \lambda = 0$ and $\nabla \times \lambda = 0$, which combine to give (44).

A consistency condition is required for compatibility of (41) and (42) for arbitrary λ. Indeed, combining these equations, we find

$$F^{\gamma\mu} \varepsilon_{\mu\nu\alpha\beta} \lambda^\nu F^{\alpha\beta} = 0. \tag{50}$$

Considering the values $\gamma = 0$ and $\gamma = i$ in (50), yields $\lambda^0 E \cdot B = 0$ and $\lambda^i E \cdot B = 0$, respectively. This requires the same consistency condition $E \cdot B = 0$ for each values of γ, which we write in the covariant form

$$\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = 0. \tag{51}$$

We could have arrived at the consistency condition $E \cdot B = 0$ also by noting that it is an appropriate covariant form of the three-dimensional constraint $E = 0$.

The covariant analogue of the energy W is the negative of the Maxwell action

$$\begin{aligned} W &= \int_\Omega \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \, d^4x, \\ &= \int_\Omega \left(-\tfrac{1}{2}\|E\|^2 + \tfrac{1}{2}\|B\|^2\right) d^4x, \end{aligned} \tag{52}$$

(We have introduced the sign difference in the definition of W so that the non-covariant W is a limiting case of the covariant W.) As a covariant form of the helicity H, we propose

$$\begin{aligned} H(f) &= -\int_\Omega \tfrac{1}{2} \varepsilon^{\mu\nu\alpha\beta} (\nabla_\mu f) A_\nu F_{\alpha\beta} \, d^4x \\ &= \int_\Omega \tfrac{1}{2} \big[(\nabla_0 f)(A \cdot B) - A_0 (B \cdot \nabla f) - \nabla f \cdot (A \times E)\big] d^4x, \end{aligned} \tag{53}$$

where f is an arbitrary scalar function in Ω. (It will become clear in what follows why in (53) we use $\lambda_\mu = \nabla_\mu f$ instead of a general λ_μ.)

For a non-compact Ω, the integral in (53) converges if

$$A = O(\|x\|^p), \ \|x\| \to \infty, \ p < -1 - \tfrac{1}{2} q, \tag{54}$$

where we assumed

$$f = O(\|x\|^q), \ \ \|x\| \to \infty \tag{55}$$

for a certain q. Since $F = O(\|x\|^{p-1})$, $\|x\| \to \infty$, the integral in (52) now converges for $p < -1$.

Our definition (53) is motivated by the following limiting case of covariant helicity $H(f)$. Suppose $\Omega = [t_1, t_2] \times \Omega'$, where $\Omega' \subset \mathbb{R}^3$, and f is a function of $x^0 = t$ only. It follows that

$$H(f) = \int_{t_1}^{t_2} H'(t)(\partial f/\partial t)dt, \tag{56}$$

where $H'(t)$ is the non-covariant helicity of the vector potential $A_i(t, x)$. In particular, for the conserved non-covariant helicity H', we find

$$H(f) = [f(t_2) - f(t_1)]\, H', \tag{57}$$

More generally, for an arbitrary f, (53) implies

$$H(f) = \tilde{H}(f) - \int_{\partial\Omega} \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} f A_\nu F_{\alpha\beta}\, d^3\sigma_\mu, \tag{58}$$

$$\tilde{H}(f) = \int_{\Omega} \tfrac{1}{4}\varepsilon^{\mu\nu\alpha\beta} f F_{\mu\nu} F_{\alpha\beta}\, d^4x, \tag{59}$$

which means that $H(f)$ is a boundary term when consistency condition (51) is satisfied.

Under a gauge transformation

$$A_\mu \mapsto A_\mu + \nabla_\mu g, \tag{60}$$

where g is an arbitrary real-valued function in Ω, the gauge invariance of W is obvious, while the corresponding gauge transformation of the helicity is

$$H(f) \mapsto H(f) + \int_{\partial\Omega} \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(\nabla_\nu f) g F_{\alpha\beta}\, d^3\sigma_\mu. \tag{61}$$

Since we require gauge invariance of $H(f)$, we set

$$\left[\varepsilon^{\mu\nu\alpha\beta} n_\mu (\nabla_\nu f) F_{\alpha\beta}\right]_{\partial\Omega} = 0, \tag{62}$$

where n is the 4-vector normal to $\partial\Omega$. Using now (42) with $\lambda_\mu = \nabla_\mu f$, we find

$$(n^\mu \nabla^\nu F_{\mu\nu})_{\partial\Omega} = 0, \tag{63}$$

which is a covariant version of the boundary condition (10). For a non-compact Ω, the asymptotic behavior $F = O(\|x\|^{p-1})$, $\|x\| \to \infty$ ensures the gauge invariance of $H(f)$ as well, since the boundary integral in (61) vanishes.

In terms of the E and B fields, the boundary condition (63) becomes

$$[-n_0(\nabla f \cdot B) + (\nabla_0 f)(n \cdot B) + n \cdot (\nabla f \times E)]_{\partial\Omega} = 0. \tag{64}$$

In particular, for time-like and space-like hypersurface $\partial\Omega$ we have

$$(\nabla f \cdot B)_{\partial\Omega} = 0 \text{ for time-like } \partial\Omega, \tag{65}$$

$$[(\nabla_0 f)(n \cdot B) + n \cdot (\nabla f \times E)]_{\partial\Omega} = 0 \text{ for space-like } \partial\Omega. \tag{66}$$

We further emphasize the choice of the definition (53) by the following theorem.

Theorem 1. *Any extremal of the action functional* $W = \int_\Omega \frac{1}{4}F_{\mu\nu}F^{\mu\nu}d^4x$ *subject to the constraint* $\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = 0$ *and the boundary condition* $\delta A|_{\partial\Omega} = 0$ *satisfies the covariant Beltrami equation* $\nabla^\nu F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\lambda^\nu F^{\alpha\beta}$ *for* $\lambda_\mu = \nabla_\mu f$*, where* $f = f(x)$ *is an arbitrary function.*

Proof. Any extremal of (52) subject to the local constraint (51) must be an extremal of the functional $W - \frac{1}{2}\tilde{H}(f)$, where $f(x)$ is a space-time dependent Lagrange multiplier (Elsgolc, 1961; Akhiezer, 1962). For an arbitrary variation of the gauge potential δA, we find

$$\begin{aligned}\delta(W - \tfrac{1}{2}\tilde{H}(f)) &= \int_\Omega \left[-(\nabla_\mu F^{\mu\nu}) + \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(\nabla_\mu f)F_{\alpha\beta}\right]\delta A_\nu \, d^4x \\ &\quad + \int_{\partial\Omega}(F^{\mu\nu} - \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}fF_{\alpha\beta})\delta A_\nu \, d^3\sigma_\mu. \end{aligned} \tag{67}$$

Using the boundary condition

$$\delta A|_{\partial\Omega} = 0, \tag{68}$$

we arrive at (42) with $\lambda_\mu = \nabla_\mu f$, which proves the theorem. □

Note that $\lambda_\mu = \nabla_\mu f$ derived in the proof implies (44), which we have already seen as a sufficient condition for (43) to be satisfied identically.

We now consider the covariant version of the conserved Noether current (Jackiw, 1985). Equations (11), (12), (16), (19), (20), (22) become

$$W - \tfrac{1}{2}\tilde{H}(f) = \int_\Omega L(f)\, d^4x, \tag{69}$$

$$L(f) = \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{8}\varepsilon^{\mu\nu\alpha\beta}fF_{\mu\nu}F_{\alpha\beta}, \tag{70}$$

$$\delta L(f) = 0, \tag{71}$$

$$j^\mu(f) = \frac{\partial L(f)}{\partial \nabla_\mu A_\nu}\delta A_\nu, \tag{72}$$

$$j^\mu(f) = \left(F^{\mu\nu} - \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}fF_{\alpha\beta}\right)\nabla_\nu g, \tag{73}$$

$$k^\mu(f) = -\nabla_\nu F^{\mu\nu} + \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(\nabla_\nu f)F_{\alpha\beta}. \tag{74}$$

Now using the Beltrami equation (42), we find $k^\mu(f) = 0$, so that the Noether current associated with the helicity, $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}(\nabla_\nu f)F_{\alpha\beta}$, equals the negative of the Maxwell current $J^\mu = -\nabla_\nu F^{\mu\nu}$. As expected, there is only one independent conserved current in the problem.

The Noether energy-momentum tensor is

$$\begin{aligned}\theta^\mu{}_\nu(f) &= \frac{\partial L(f)}{\partial(\nabla_\mu A_\sigma)}\nabla_\nu A_\sigma - \delta^\mu{}_\nu L(f), \\ &= (F^{\mu\sigma} - \tfrac{1}{2}f\varepsilon^{\mu\sigma\alpha\beta}F_{\alpha\beta})\nabla_\nu A_\sigma - \delta^\mu{}_\nu(\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \tfrac{1}{8}\varepsilon^{\alpha\beta\gamma\delta}fF_{\alpha\beta}F_{\gamma\delta}) \end{aligned} \tag{75}$$

and the corresponding energy-momentum 4-vector is

$$P_\nu(f) = \int_{\Omega'} \theta^0{}_\nu(f)\, d^3x, \tag{76}$$

$$P_0(f) = -\int_{\Omega'}\left(\tfrac{1}{2}\|E\|^2 + \tfrac{1}{2}\|B\|^2\right) d^3x + \int_{\partial\Omega'} n\cdot(E + fB)A_0\, d^2\sigma, \tag{77}$$

$$P_i(f) = -\int_{\Omega'} (E \times B)_i \, d^3x + \int_{\partial\Omega'} n \cdot (E + fB) A_i \, d^2\sigma, \tag{78}$$

where we assumed $\Omega = [t_1, t_2] \times \Omega'$, $\Omega' \subset \mathbb{R}^3$. We set the boundary condition

$$n \cdot (E + fB)|_{\partial\Omega'} = 0 \tag{79}$$

and obtain the relation $P_\nu(f) = P_\nu(0)$ which is consitent with $L(f) - L(0)$ being a topological term. Also note that although $\theta^\mu{}_\nu(f)$ is not gauge invariant, the resulting $P_\nu(f)$ is.

To prove conservation of $\theta^\mu{}_\nu(f)$, we need to use the Beltrami equation. Indeed, in the expression

$$\nabla_\mu \theta^\mu{}_\nu(f) = \big((\nabla_\mu F^{\mu\sigma}) - \tfrac{1}{2}\varepsilon^{\mu\sigma\alpha\beta}(\nabla_\mu f)F_{\alpha\beta}\big)\nabla_\nu A_\sigma + \tfrac{1}{8}\varepsilon^{\alpha\beta\gamma\delta}(\nabla_\nu f)F_{\alpha\beta}F_{\gamma\delta}, \tag{80}$$

the first term on the right-hand side vanishes for any solution of (42) and the second term vanishes due to the constraint (51). Since (51) follows from (42), we conclude that the conservation equation $\nabla_\mu \theta^\mu{}_\nu(f) = 0$ holds for any solution of the Beltrami equation.

It is straightforward to carry out this analysis for non-abelian fields. See (Buniy & Kephart, 2014), where our results are also conveniently expressed in terms of differential forms.

4. Conclusions

We have generalized the magnetic helicity and Beltrami equation to the relativistic form. In the process, we discussed various interconnected features associated with this generalization. In particular, we found that the helicity is related to the Chern-Simons action and can also be viewed as a constraint requiring the vanishing of a generalized instanton term.

Besides its theoretical appeal, the covariant formulation of the magnetic helicity and Beltrami equation has an experimental advantage as well. It turns out that, for an ideal nonrelativistic plasma, charges flow until the electric field is completely shorted out. In the relativistic case, even for an ideal plasma, however, the current flow may not be able to keep up, and so the electric fields do not necessarily always vanish. Some possible applications of our results for the relativistic generalization of the Beltrami equation may be found for dynamos inside millisecond pulsars, pulsar and quasar atmospheres, collisions of plasma shock waves with other shocks or gas clouds and nuclear fusion via laser confinement.

Explicit solutions of the covariant Beltrami equation are of particular interest for applications, and we will address these elsewhere.

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