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Magnetic field generated by current filaments

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Abstract. We investigate the magnetic field generated by two straight current filaments using the analogy between steady MHD and Euler flows. Using the Biot-Savart law, we present a dynamical system describing the extension of magnetic lines around the current filaments. It is demonstrated that, if two current filaments are non-parallel, a magnetic line starting near one current goes to infinity by the drifting effect of the other.

1. Introduction

It is well-known that a practically important analogy resides between steady Euler and MHD flows. If we denote incompressible velocity and vorticity as $\mathbf{u}$ and $\boldsymbol{\omega}$ for steady Euler flows, and a magnetic field (or the magnetic flux density) and an electric current distribution as $\mathbf{B}$ and $\mathbf{j}$ for steady MHD flows, the following analogy for the two sets holds exactly :[1]

$$\nabla \times \mathbf{u} = \boldsymbol{\omega} \qquad \nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\nabla \times \mathbf{B} = \mathbf{j} \qquad \nabla \cdot \mathbf{B} = 0 \tag{2}$$

From the Biot-Savart law, which is equivalent to take the inverse of rotation, we have

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \mathrm{d}V', \tag{3}$$

for $\mathbf{u}(\mathbf{x})$, and

$$\mathbf{B}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\mathbf{j}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \mathrm{d}V', \tag{4}$$

for $\mathbf{B}(\mathbf{x})$.

We should note that this analogy breaks if we deal with a time-dependent problem, because corresponding dynamics, the Euler equation for $\mathbf{u}$ and the magnetic induction equations for $\mathbf{B}$, involve $\mathbf{u}$ and $\mathbf{B}$ in a different manner. As long as we stay in steady problems, however, we can extract useful information from one side and apply to the other. The content of this article make full use of this analogy.

Using the Biot-Savart law (3), we have investigated the particle motion around a vortex soliton (Kimura & Koikari (2004)). It is found that particles near the loop part of a vortex soliton are trapped and move inside a knotted torus around the loop of the vortex soliton.

[1] We have omitted the permeability index μ_0 which would appear when S. I. units are used.

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To understand this torus formation, a simple toy model, which replaced a vortex soliton with a couple of vortex filaments tangent to the soliton filament was introduced and named the chopsticks model. A variety of particle motions including chaotic ones are found inside the torus, and the discovery of such convoluted particle motions even for a simple setting of vortex filaments is the direct motivation to study possible complication of the magnetic field structure around a simple configuration of electric current filaments. For similar motivations, reader is referred to, for example, Gascon & Peralta-Salas (2005) and Aguirre *et al.* (2008) and the references therein.

2. Formulation

Let us first calculate the magnetic field generated by a single electric current filament of a (steady) current J which passes $\mathbf{x}_0$ with the unit tangent vector $\mathbf{t}$ (Figure 1). A direct integration of (4) results

$$\mathbf{B}(\mathbf{x}) = \frac{J}{2\pi}\frac{\mathbf{t}\times(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{t}\times(\mathbf{x}-\mathbf{x}_0)|^2}\,. \tag{5}$$

Now a couple of current filaments, J_1 and J_2, are placed as in Figure 2. We may assume that J_1 is on the z-axis, and J_2 is on a plane parallel to the yz-plane with a distance of d, and is tilted by the angle θ from the z direction. For the formula (5), we have $\mathbf{t} = (0, 0, 1)$, $\mathbf{x}_0 = (0, 0, 0)$ for J_1 and $\mathbf{t} = (0, \sin\theta, \cos\theta)$, $\mathbf{x}_0 = (d, 0, 0)$ for J_2. The total magnetic field is a superposition of the contributions from each current filament. To see how a given magnetic line extends in the 3D space, we start at the initial point and connect segments of the magnetic vector field "nicely". For the present case, it is equivalent to integrate the following dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{j_1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + \frac{j_2}{(z\sin\theta - y\cos\theta)^2 + (x-d)^2}\begin{pmatrix} z\sin\theta - y\cos\theta \\ (x-d)\cos\theta \\ -(x-d)\sin\theta \end{pmatrix}, \tag{6}$$

where we have replaced $J_1/2\pi \to j_1$ and $J_2/2\pi \to j_2$.

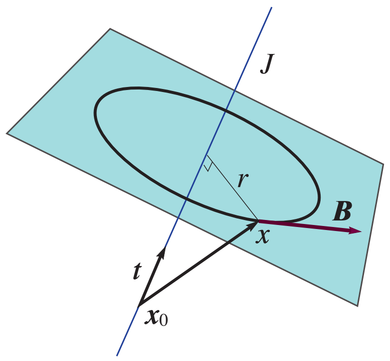

Figure 1. A Single electric current filament J and the magnetic field B generated by J.

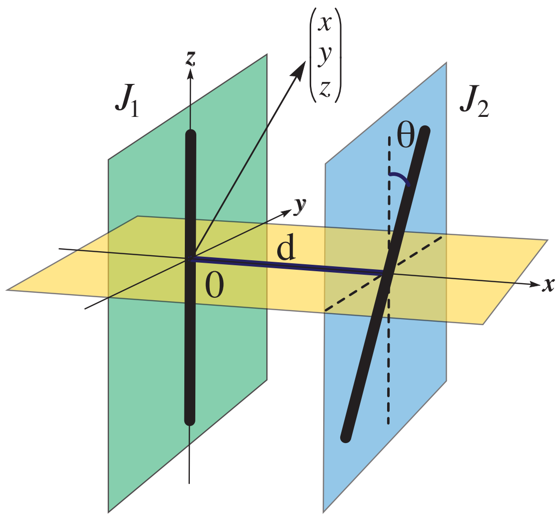

Figure 2. The configuration of a couple of electric current filaments J_1 and J_2.

We should note that the right hand side of (6) can be written with a vector potential as follows,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \nabla\times\left[\frac{-j_1}{2}\log(x^2+y^2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{-j_2}{2}\log\left\{(z\sin\theta - y\cos\theta)^2 + (x-d)^2\right\}\begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}\right]. \tag{7}$$

3. Comparison with the vortex chopsticks model

With the vortex chopsticks model, we obtained the following expression corresponding to (7),

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \nabla\times\left[-\frac{\Gamma_1}{2\pi}\log|\mathbf{t}_1\times(\mathbf{x}-\mathbf{x}_1)|\ \mathbf{t}_1 - \frac{\Gamma_2}{2\pi}\log|\mathbf{t}_2\times(\mathbf{x}-\mathbf{x}_2)|\ \mathbf{t}_2\right] \\ &\quad +\nabla\times\left[\frac{1}{2}(\nu^2+\tau^2)G(x^2+y^2)\hat{z}\right] + \nabla\times\nabla\times\left[\tau G(x^2+y^2)\hat{z}\right], \end{aligned}\tag{8}$$

where Γ_1, Γ_2 are the circulation of vortex filaments. In the above equation, the first two terms on the right hand side correspond directly to the right hand side of (7), but there are additional terms. These terms attribute to the steady rotational and translational motion of a vortex soliton, which can be parametrized by a half of the maximum curvature ν, a constant torsion τ and the local induction constant G of a vortex soliton(Hasimoto (1972), Kida (1981)). In Kimura & Koikari (2004), a wide variety of combinations of these parameters are examined in integrating (8) to calculate the volume of the torus as a fluid mass transported by the vortex chopsticks. While examining the parameter dependence on the volume of the torus, we noticed a tendency that the volume decrease drastically with decreasing value of the torsion parameter τ, and in the limit when $\tau \to 0$, we predict that the volume of torus also goes to 0. On the other hand, as a finite volume remains even for zero ν (with nonzero τ), we may assume that the last term in (8), which corresponds to a uniform translation of the system, is essential for a finite volume torus to exist. So if we use the analogy between a vortex filament and a current filament, we may expect that the translation of the system, which is understood as an anti-drifting motion, is also essential for the existence of closed magnetic lines or their stability, if any. To corroborate this, we will investigate (6) for some special cases.

For simplicity, if we assume that the initial point (x, y, z) is much closer to J_1 than J_2 in Figure 2, *i.e.* $\sqrt{x^2+y^2}/d \ll 1$, then (6) is approximated as

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{j_1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} + \frac{j_2/d}{((z/d)\sin\theta)^2+1}\begin{pmatrix} (z/d)\sin\theta \\ -\cos\theta \\ \sin\theta \end{pmatrix}. \tag{9}$$

From the above equation, we see that a trajectory is near a circle around J_1 which is perturbed by the effect of J_2. We are concerned with how the effect of perturbation remains if d becomes larger. To see this we check the development of the z-component of the above equation. The z-component is

$$\frac{\mathrm{dz}}{\mathrm{dt}} = \frac{(j_2/d)\sin\theta}{(z^2/d^2)\sin^2\theta + 1},$$

and if we denote $a = \sin^2\theta/d^2$ and $b = (j_2/d)\sin\theta$, and assume the initial condition $z(0) = 0$, we have an initial value problem of

$$dz/dt = b/(az^2+1), \quad z(0) = 0, \tag{10}$$

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which can be integrated easily to give

$$\frac{a}{3}z^3 + z = bt \qquad \rightarrow \qquad z \sim \left(\frac{3b}{a}t\right)^{1/3} = \left(\frac{3\,d\,j_2}{\sin\theta}\,t\right)^{1/3}. \tag{11}$$

The speed depends on the parameters, but $z(t)$ goes to plus or minus infinity as $t \to \infty$ for any values of θ except 0 and π. From (9) the equation for the square distance of trajectory from J_1, $r^2(t) = x(t)^2 + y(t)^2$, is given by

$$\frac{\mathrm{d}r^2}{dt} = \frac{2j_2\,d}{(z^2/d^2)\sin^2\theta + 1}\left((xz/d)\sin\theta - y\cos\theta\right). \tag{12}$$

Using (11), the right hand side of this equation is evaluated as $\sim t^{-1/3}$ for $t \to \infty$ which provides the asymptotic behavior $r^2 \sim t^{2/3}$.

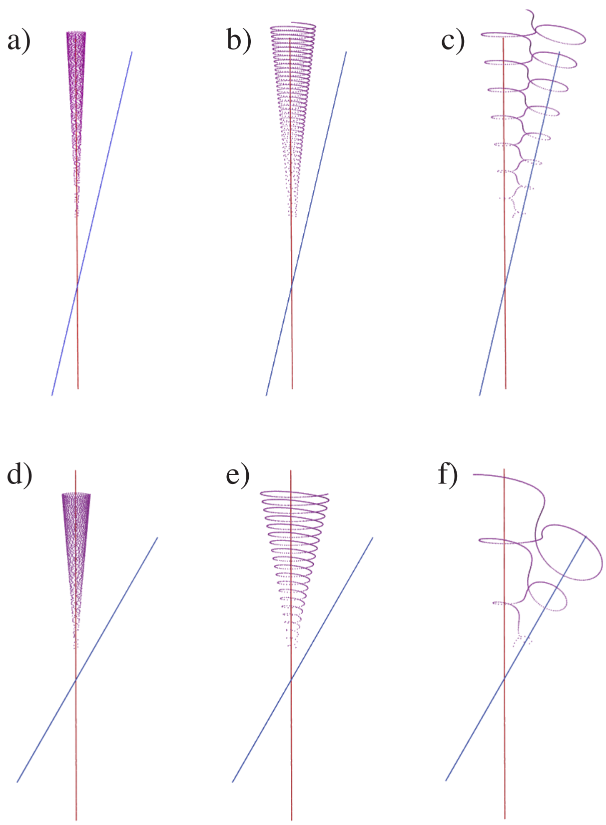

Figure 3. Simulation results of (6). Parameters are set $j_1 = j_2 = 1$ and $d = 1$, and $\theta = \pi/6$ for $a), b), c)$ and $\theta = \pi/3$ for $d), e), f)$. The initial conditions are $(0.1, 0.333, 0)$ for $a), d)$, $(0.3, 0.333, 0)$ for $b), e)$, $(0.5, 0.333, 0)$ for $c), f)$.

Figure 3 shows some plots of the trajectories (magnetic lines) obtained by integrating (6). The parameters are set $j_1 = j_2 = 1$ and $d = 1$, and $\theta = \pi/6$ for $a), b), c)$ and $\theta = \pi/3$ for $d), e), f)$. All the initial conditions are set on the $z = 0$ plane with $y = 0.0333$, and the x component is 0.1 for $a), d)$, 0.3 for $b), e)$ and 0.5 for $c), f)$. If the initial positions are close to J_1, the trajectories spiral out along J_1 ($a), b), d), e)$). On the other hand, if they are close to

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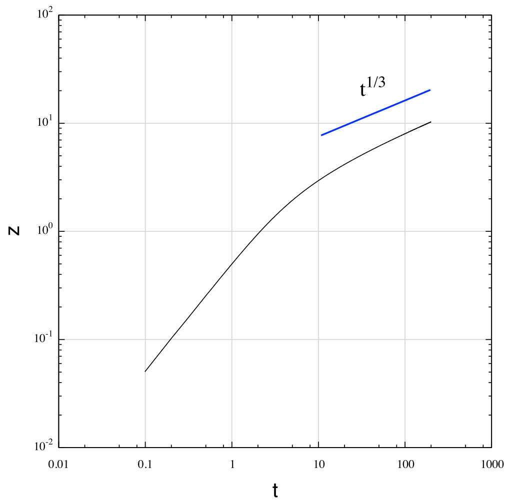

Figure 4. The time development of $z(t)$ for Figure 3a). Asymptotically, it approach the line of $t^{1/3}$.

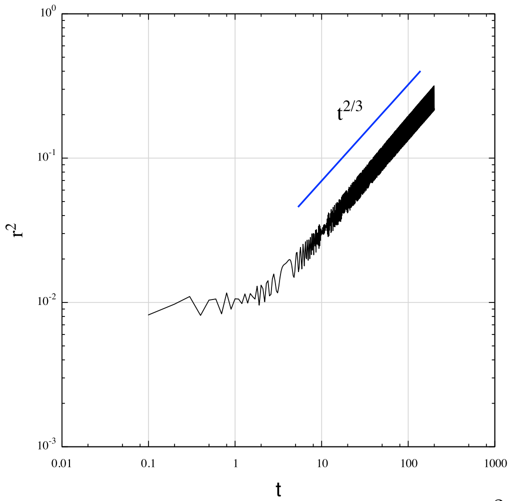

Figure 5. The time development of $r^2 = x(t)^2 + y(t)^2$ for Figure 3a). With oscillations, it approach the line of $t^{2/3}$ asymptotically.

the (approximated) hyperbolic point $(0.5, 0, 0)$, the trajectories meander between J_1 and J_2, but still the z component tends to go to infinity.

In Figure 4 and Figure 5, the time development of $z(t)$ and of $r^2(t)$ are plotted respectively for the case of Figure 3a). As estimated, they show the asymptotic dependence of $t^{1/3}$ and $t^{2/3}$, respectively.

We have demonstrated that a magnetic line which passes a point close to one of the two straight current filaments in a skewed configuration goes to infinity by the drifting effect of the other. Although this observation implies that such a magnetic line can not be closed, it still remains an open problem whether any magnetic lines not close to either current filament could make a closed loop or not. To answer to this question, a thorough investigation of (6) is necessary.

Acknowledgments

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References

Aguirre, J., Giné, J. & Peralta-Salas, D. 2008 Integrability of magnetic field created by current distributions. *Nonlinearity* **21**, 51–69.

Gascon, F. & Peralta-Salas, D. 2005 Some properties of magnetic fields generated by symmetric configurations of wires. *Physica D* **206**, 109–120.

Hasimoto, H. 1972 A soliton on a vortex filament. *Journal of Fluid Mechanics* pp. 477–485.

Kida, K. 1981 A vortex filament moving without change of form. *Journal of Fluid Mechanics* pp. 397–409.

Kimura, Y. & Koikari, S. 2004 Particle transport by a vortex soliton. *Journal of Fluid Mechanics* **510**, 201–218.