

Goal-oriented updating of mechanical models

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Résumé. We introduce a goal-oriented procedure for the updating of mechanical models. It exploits as usual information coming from experimental data, but these are post-processed in a specific way in order to firstly update model parameters which are the most influent for the prediction of a given quantity of interest. The objective is thus to perform a partial model updating that enables to obtain an approximate value of the quantity of interest with sufficient accuracy and minimal model identification effort. The updating method uses the constitutive relation error framework, as well as duality and adjoint techniques, and defines dedicated cost functions. It leads to a convenient strategy that automatically selects the relevant parameter set to be updated. Performances of the approach are analyzed on examples involving linear elasticity and transient thermal models with possible noisy measurements.

1. Introduction

A major concern with mathematical models is their capability to represent a faithful abstraction of the real world. To address this issue and control the error between physical and mathematical models, model validation methods have been used for a long time [13]. In such methods, model parameters are identified or updated in order to minimize the discrepancy between numerical predictions and experimental measurements. The process leads to inverse problems which are usually ill-posed and require special care and regularization techniques in order to ensure solvability [3].

We focus here on the updating of Computational Mechanics models, in which a major component is the constitutive equation that describes the local behavior of the material [6]. We consider that only few localized measurements are available. Several procedures exist in this framework to identify parameters, both in a deterministic setting (minimization of cost functions associated with regularization techniques [3]) and in a stochastic setting (Bayesian inference [14]). The concept of Constitutive Relation Error (CRE) defines another model updating we use here. First introduced for dynamics models [10, 2, 7], this method was latter successfully used in many calibration applications with defects [5], uncertain measurements and behaviors [12], or corrupted measurements [1]. Recent applications of the method dealt with the updating of models used in bolted assemblies [8], or association with PGD reduced models to deal with real-time calibration of machining models [4]. The use of the CRE presents various advantages, such as the capacity to localize structural defects spatially, the robustness with respect to noisy measurements, and good convexity properties. In the CRE framework, reliable theoretical and experimental information (equilibrium, sensor position) are favored compared to other information (material behavior, sensor measures), and the hierarchical updating which is employed (only most erroneous zones



are corrected) directly provides for a regularization process.

In this work, we consider that the prediction target is only the value of a given output of the model (quantity of interest) that implicitly depends on model parameters and is relevant for design (local stress, maximal displacement,...). Therefore, if the quantity of interest is not very sensitive with respect to some parameters, there is probably no need to estimate these parameters with high accuracy. This would lead to a partial calibration of the model so as to ensure the quality of predicted quantities of interest with a minimal calibration effort. We thus define a goal-oriented version of updating methods performed using the CRE, focusing on the sensitivity of the considered quantity of interest with respect to parameters and measurements. We define new dedicated cost functions that lead to a convenient goal-oriented updating process, selecting automatically the relevant model parameters that need to be updated for the prediction of the quantity of interest. For the sake of simplicity, we present the new method and tools in the framework of linear elasticity models, whereas numerical experiments will also involve time-dependent problems.

2. Classical updating method based on the constitutive relation error

We consider a linear elastic body whose undeformed configuration occupies domain $\Omega \subset \mathbb{R}^d$, with boundary $\partial\Omega$. It is subjected to a given displacement field \mathbf{u}_d on $\partial_1\Omega \subset \partial\Omega$, whereas a body force field \mathbf{d} and a traction force field \mathbf{F}_d are imposed in Ω and on $\partial_2\Omega$, respectively, with $\partial_1\Omega \cup \partial_2\Omega = \partial\Omega$ and $\partial_1\Omega \cap \partial_2\Omega = \emptyset$. The corresponding model is governed by compatibility, equilibrium and constitutive equations which respectively read :

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}) \quad \text{in } \Omega \quad ; \quad \mathbf{u} = \mathbf{u}_d \quad \text{on } \partial_1\Omega \quad (1)$$

$$\text{div } \sigma + \mathbf{d} = \mathbf{0} \quad \text{in } \Omega \quad ; \quad \sigma \mathbf{n} = \mathbf{F}_d \quad \text{on } \partial_2\Omega \quad (2)$$

$$\sigma = \mathbf{K}\varepsilon(\mathbf{u}) \quad \text{in } \Omega \quad (3)$$

\mathbf{u} is the displacement vector, ε the linearized strain tensor, σ the Cauchy stress tensor, and \mathbf{n} the outward unit normal vector. \mathbf{K} denotes the Hooke tensor, possibly heterogeneous, and we suppose it is described by a set \mathbf{p} of parameters. We introduce classical sets $\mathcal{U} = \{\mathbf{v} \in [H^1(\Omega)]^d, \mathbf{v} \text{ verifies (1)}\}$ and $\mathcal{S} = \{\pi \in [L^2(\Omega)]^{d(d+1)/2}, \pi = \pi^T, \pi \text{ verifies (2)}\}$. Equations (1–3) constitute a direct problem that can be recast in the following weak form :

$$\text{Find } \mathbf{u}(\mathbf{p}) \in \mathcal{U} \text{ such that } \mathcal{A}(\mathbf{p}, \mathbf{u}, \mathbf{v}) = a(\mathbf{p}, \mathbf{u}, \mathbf{v}) - l(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{U}_0 \quad (4)$$

where \mathcal{U}_0 denotes the vectorial space associated with \mathcal{U} .

The constitutive relation error (CRE) is a concept with strong mechanical content [11]. It defines an energy measure, denoted \mathcal{E} , of the distance between a given stress field π and another stress field obtained from a given displacement field \mathbf{v} using (3) :

$$\mathcal{E}^2(\mathbf{p}, \mathbf{v}, \pi) = \frac{1}{2} \int_{\Omega} [\pi - \mathbf{K}(\mathbf{p})\varepsilon(\mathbf{v})] : \mathbf{K}^{-1}(\mathbf{p}) [\pi - \mathbf{K}(\mathbf{p})\varepsilon(\mathbf{v})] d\Omega \quad (5)$$

Its usefulness is explained by the fact that the solution (\mathbf{u}, σ) of the well-posed direct problem (1–3) is characterized by $(\mathbf{u}, \sigma) = \text{argmin}_{(\mathbf{v}, \pi) \in \mathcal{U} \times \mathcal{S}} \mathcal{E}(\mathbf{p}, \mathbf{v}, \pi)$. In recent model calibration formulations using the CRE, the cost (misfit) function \mathcal{F} is constructed from \mathcal{E} ; boundary conditions coming from experimental data are relaxed and imposed by penalization in a new

definition of the constitutive relation error called *modified constitutive relation error* [10, 2, 7] and denoted \mathcal{E}_m :

$$\mathcal{E}_m^2(\mathbf{p}, \mathbf{v}, \pi) = \mathcal{E}^2(\mathbf{p}, \mathbf{v}, \pi) + \frac{1}{2} \frac{r}{1-r} \|\mathbf{s}[\mathbf{v}] - \mathbf{s}_{obs}\|_{L^2}^2 \quad (6)$$

The two terms composing \mathcal{E}_m can be respectively seen as measures of modeling error and measurement error. r is a scalar parameter that enables to modulate the influence of these terms. The philosophy of the modified constitutive relation error is thus to favor reliable theoretical and experimental information (equilibrium, sensor location) compared to other information (constitutive relation, sensor measurement). The inverse problem is defined as :

$$\mathbf{p}_{sol} = \underset{\mathbf{p} \in \mathcal{P}}{\operatorname{argmin}} \mathcal{F}(\mathbf{p}) \quad ; \quad \mathcal{F}(\mathbf{p}) = \min_{(\mathbf{v}, \pi) \in \mathcal{U} \times \mathcal{S}} \mathcal{E}_m^2(\mathbf{p}, \mathbf{v}, \pi) \quad (7)$$

It leads to an iterative method, each iteration consisting of two minimizations steps :

- the first minimization step involved in (7), i.e. the computation of $\mathcal{F}(\mathbf{p})$ is called the *localization step*. This is a constrained minimization problem solved by means of a Lagrangian. Splitting the modeling error term of $\mathcal{F}(\mathbf{p})$ into contributions of each element p_i of \mathbf{p} enables to localize the set of parameters that contribute the most to the error. Contributions of the measurement error enable to detect erroneous sensors ;
- the second minimization step involved in (7), i.e. the computation of \mathbf{p}_{sol} is called the *correction step*. In practice, $\mathcal{F}(\mathbf{p})$ is minimized with respect to parameters p_i chosen in the previous step. This is a nonlinear process that uses an optimization algorithm, such as the gradient method with optimal path.

The iterative process with localization and correction steps is in practice stopped when the cost function $\mathcal{F}(\mathbf{p})$ reaches a given tolerance value, without waiting for convergence. Furthermore, only parameters which are selected in the localization step are updated during the correction step (hierarchical updating). This naturally regularizes the inverse formulation.

3. Goal-oriented updating with the constitutive relation error

In the previous section, the model was updated globally with respect to experiments as a (modified) constitutive relation error, defined on the whole domain Ω and with the whole parameter set \mathbf{p} , was used as a cost function. Furthermore, parameters to be updated first were selected using contributions of this global error. Here, we aim at modifying the formulation to obtain an updating process which is dedicated to the prediction of given outputs of the model. We consider a quantity of interest Q which is supposed to be the goal of the computation. Performing an optimal calibration process with respect to Q means updating relevant parameters only. We introduce in the following cost functions which are dedicated to this purpose.

3.1. Goal-oriented cost function when the quantity of interest is not measured

We assume here that the spatial region in which Q is defined is not a measurement point. Keeping the philosophy and flexibility of the modified CRE, composed of modeling and measurements error terms, we introduce a new cost function, denoted $\mathcal{F}_Q(\mathbf{p})$, associated with the considered quantity of interest. It reads :

$$\mathcal{F}_Q(\mathbf{p}) = \min_{q \in \mathbb{R}} \left[\frac{1}{2} |q - Q_{mod}(\mathbf{p})|^2 + \frac{1}{2} \frac{r}{1-r} |q - Q_{obs}(\mathbf{p})|^2 \right] \quad (8)$$

The modeling error term $\frac{1}{2} |q - Q_{mod}(\mathbf{p})|^2$ involves a value Q_{mod} of the quantity of interest defined from the model only, i.e. by means of state equation (4) :

$$Q_{mod}(\mathbf{p}) = Q(\mathbf{u}_1(\mathbf{p})) \quad ; \quad \mathcal{A}(\mathbf{p}, \mathbf{u}_1, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{U}_0 \quad (9)$$

The measurement error term $\frac{1}{2}|q - Q_{obs}(\mathbf{p})|^2$ involves a value Q_{obs} of the quantity of interest defined from an interpolation of measurements \mathbf{s}_{obs} , i.e. :

$$Q_{obs}(\mathbf{p}) = Q(\mathbf{u}_2(\mathbf{p})) \quad ; \quad (\mathbf{u}_2, \sigma_2) = \underset{(\mathbf{v}, \pi) \in \mathcal{U} \times \mathcal{S}}{\operatorname{argmin}} \mathcal{E}_m^2(\mathbf{p}, \mathbf{v}, \pi) \quad (10)$$

The definition of \mathbf{u}_2 thus involves \mathbf{s}_{obs} and leads to a system $\bar{\mathbb{K}}(\mathbf{p})\mathbf{U}_2 = \bar{\mathbf{F}}$ after discretization. Therefore, the first step in the iterative updating process consists in computing $\mathcal{F}_Q(\mathbf{p})$. As it is a constrained minimization problem with fixed \mathbf{p} , we introduce the semi-discrete Lagrangian :

$$\mathcal{L}_Q(\mathbf{p}, q, \mathbf{u}_1, \mathbf{U}_2, \boldsymbol{\mu}, \boldsymbol{\Lambda}) = \frac{1}{2}|q - Q(\mathbf{u}_1)|^2 + \frac{1}{2} \frac{r}{1-r} |q - Q(\mathbf{U}_2)|^2 - \mathcal{A}(\mathbf{p}, \mathbf{u}_1, \boldsymbol{\mu}) - \boldsymbol{\Lambda}^T (\bar{\mathbb{K}}(\mathbf{p})\mathbf{U}_2 - \bar{\mathbf{F}}) \quad (11)$$

Finding the saddle-point of \mathcal{L}_Q leads to verify, in particular, the following adjoint problems for all $(\delta \mathbf{u}_1, \delta \mathbf{U}_2) \in \mathcal{U}_0 \times \mathcal{U}_0^h$ and with $\beta = r[Q(\mathbf{u}_1) - Q(\mathbf{U}_2)]$:

$$\mathcal{A}(\mathbf{p}, \delta \mathbf{u}_1, \boldsymbol{\mu}) = \beta Q'(\mathbf{u}_1; \delta \mathbf{u}_1) \quad \forall \delta \mathbf{u}_1 \in \mathcal{U}_0 \quad ; \quad \boldsymbol{\Lambda}^T \bar{\mathbb{K}}(\mathbf{p}) \delta \mathbf{U}_2 = -\beta Q'(\mathbf{U}_2; \delta \mathbf{U}_2) \quad \forall \delta \mathbf{U}_2 \in \mathcal{U}_0^h \quad (12)$$

In a second step, the gradient of $\mathcal{F}_Q^h(\mathbf{p})$ is computed using the adjoint state method and from fields $(q, \mathbf{u}_1, \mathbf{U}_2, \boldsymbol{\mu}, \boldsymbol{\Lambda})$ obtained in the previous step :

$$\mathbf{d}_{\mathbf{p}} \mathcal{F}_Q^h(\mathbf{p}; \delta \mathbf{p}) = \mathcal{L}'_{Q, \mathbf{p}}(\mathbf{p}, q, \mathbf{u}_1, \mathbf{U}_2, \boldsymbol{\mu}, \boldsymbol{\Lambda}; \delta \mathbf{p}) = -\mathbf{a}'_{\mathbf{p}}(\mathbf{p}, \mathbf{u}_1, \boldsymbol{\mu}; \delta \mathbf{p}) - \boldsymbol{\Lambda}^T \mathbf{d}_{\mathbf{p}} \bar{\mathbb{K}}(\mathbf{p}; \delta \mathbf{p}) \mathbf{U}_2 \quad (13)$$

Elements of \mathbf{p} associated with high components for the gradient of $\mathcal{F}_Q^h(\mathbf{p})$ are then selected to drive the nonlinear hierarchical minimization.

3.2. Goal-oriented cost function when the quantity of interest is measured

In this specific case, there is no interpolation of the data and therefore the regularization term involved in the previous cost function is missing. We propose a new cost function based on a local version of the modified CRE, i.e. a measure at point (or in the area) where the quantity of interest is defined. We denote by Q^* the dual quantity of Q (in the energy sense), and we write the local constitutive relation linking Q^* and Q under the form $Q^* = k(\mathbf{p})Q$. The new cost function thus reads :

$$\mathcal{F}_Q(\mathbf{p}) = \min_{(\mathbf{v}, \pi) \in \mathcal{U} \times \mathcal{S}} \mathcal{E}_{m, loc}^2(\mathbf{p}, \mathbf{v}, \pi) \quad (14)$$

with

$$\mathcal{E}_{m, loc}^2(\mathbf{p}, \mathbf{v}, \pi) = \frac{1}{2}|Q^*(\pi) - k(\mathbf{p})Q(\mathbf{v})|^2 + \frac{1}{2} \frac{r}{1-r} |Q(\mathbf{v}) - Q_{obs}|^2 \quad (15)$$

The two terms in $\mathcal{E}_{m, loc}^2$ correspond to local modeling and measurement errors, respectively. Following a procedure similar to the one detailed in Section ??, we write the discrete version of (15) :

$$\mathcal{E}_{m, loc}^{h2}(\mathbf{p}, \mathbf{U}, \mathbf{V}) = \frac{1}{2}|Q^*(\mathbf{V}) - k(\mathbf{p})Q(\mathbf{U})|^2 + \frac{1}{2} \frac{r}{1-r} |Q(\mathbf{U}) - Q_{obs}|^2 \quad (16)$$

with $\mathbb{K}(\mathbf{p})\mathbf{V} = \mathbf{F}$ (equilibrium), and the computation of $\mathcal{F}_Q^h(\mathbf{p})$ is performed by searching the saddle-point of the following lagrangian $\mathcal{L}_Q(\mathbf{p}, \mathbf{U}, \mathbf{V}, \boldsymbol{\Lambda}) = \mathcal{E}_{m, loc}^{h2}(\mathbf{p}, \mathbf{U}, \mathbf{V}) - \boldsymbol{\Lambda}^T (\mathbb{K}(\mathbf{p})\mathbf{V} - \mathbf{F})$. This leads to solving, in particular, the following adjoint problem for all $\delta \mathbf{V} \in \mathcal{S}_0^h$:

$$Q_u^*(\mathbf{V}; \delta \mathbf{V}) (Q^*(\mathbf{V}) - k(\mathbf{p})Q(\mathbf{U})) - \boldsymbol{\Lambda}^T \mathbb{K}(\mathbf{p}) \delta \mathbf{V} = 0 \quad (17)$$

The evaluation of the gradient of $\mathcal{F}_Q^h(\mathbf{p})$ is then direct with the adjoint state method :

$$\mathbf{d}_{\mathbf{p}} \mathcal{F}_Q^h(\mathbf{p}; \delta \mathbf{p}) = \mathcal{L}'_{\mathbf{p}}(\mathbf{p}, \mathbf{U}, \mathbf{V}, \boldsymbol{\Lambda}; \delta \mathbf{p}) \quad (18)$$

with $(\mathbf{U}, \mathbf{V}, \boldsymbol{\Lambda})$ solution of (17). Here again, elements of \mathbf{p} associated with high components of the gradient are selected in order to minimize $\mathcal{F}_Q^h(\mathbf{p})$ with respect to these parameters only.

3.3. Remarks on the regularization process

In the goal-oriented updating method involving cost functions \mathcal{F}_Q introduced in Sections 3.1 or 3.2, it is important to notice that an iterative two-step (localization, correction) strategy is conserved. However, the localization step is no more based on a splitting of the modeling error term and comparison between parameter contributions (\mathcal{F}_Q is local in space) as used in Section 2, but on the selection of parameters that bring highest components to the gradient of $\mathcal{F}_Q(\mathbf{p})$. In practice, at each iteration of the method, the parameter (or the set of parameters) that brings the highest gradient component is corrected first. If this correction is not associated with a significant decrease of the cost function value, the parameter (or the set of parameters) that brings the second highest gradient component is corrected, and the process is stopped when a significant decrease of $\mathcal{F}_Q(\mathbf{p})$ is observed. This hierarchical strategy naturally regularizes the goal-oriented inverse formulation.

4. Numerical results

We consider a 2D concrete beam with a steel bar Γ (Figure 1). We divide the steel bar (resp. the lower concrete part Ω_c) into five subdomains Γ_j (resp. Ω_{c_j}), $j \in \{1, \dots, 5\}$ and consider piecewise constant steel bar cross-section and concrete Young modulus. We denote by S_{b_j} the steel bar cross-section in Γ_j , E_{c_j} the concrete Young modulus in Ω_{c_j} , and k_1 (resp. k_2) the rigidity of support 1 (resp. support 2). We take $L = 30 \text{ m}$ and $H = 1 \text{ m}$. To update beam parameters

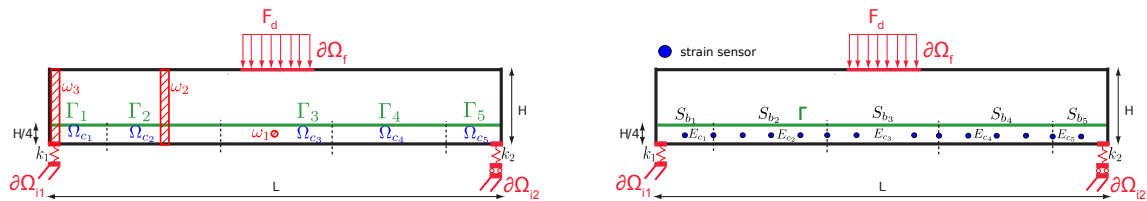


FIGURE 1. 2D concrete beam with a horizontal steel bar, with domains of interest ω_1 , ω_2 and ω_3 (left), and instrumentation with 14 strain sensors (right).

$(\{S_{b_j}\}, \{E_{c_j}\}, k_1, k_2)$, a static loading \mathbf{F}_d is applied and the longitudinal strain component ϵ_{xx} is measured at 14 points (see Figure 1). The sensor outputs are simulated numerically considering the direct model and the following reference beam parameters : $E_{c_1}^{ref} = E_{c_2}^{ref} = E_{c_4}^{ref} = E_{c_5}^{ref} = 40.10^9 \text{ Pa}$, $E_{c_3}^{ref} = 30.10^9 \text{ Pa}$, $S_{b_1}^{ref} = S_{b_2}^{ref} = S_{b_3}^{ref} = S_{b_4}^{ref} = S_{b_5}^{ref} = 0.04 \text{ m}$, $k_1^{ref} = k_2^{ref} = 5.10^7 \text{ N/m}^3$. The only difference between reference and initial models is the value of the concrete Young modulus in subdomain Ω_{c_3} .

The goal-oriented updating procedure is applied to three quantities of interest :

$$Q_1 = \int_{\omega_1} \epsilon_{xx} d\Omega ; \quad Q_2 = \int_{\omega_2} u_y d\Omega ; \quad Q_3 = \int_{\omega_3} u_y d\Omega \quad (19)$$

The associated zones of interest ω_1 , ω_2 and ω_3 are represented in Figure 1.

Regarding the updating strategy, the highest component of the functional gradient is searched at each iteration and the associated parameter is updated. For each iteration of the goal-oriented technique, the updated parameter values and associated predicted values of the quantity of interest are represented in Figures 2 and 3. As regards quantities of interest Q_1 and Q_2 , we observe that two iterations are sufficient to get an error less than 1% on the determination of the concrete Young modulus E_{c_3} : we also remark that only parameter E_{c_3} has been updated using the goal-oriented approach. Quantity Q_3 being not sensitive to E_{c_3} , this parameter is not updated using the goal-oriented approach.

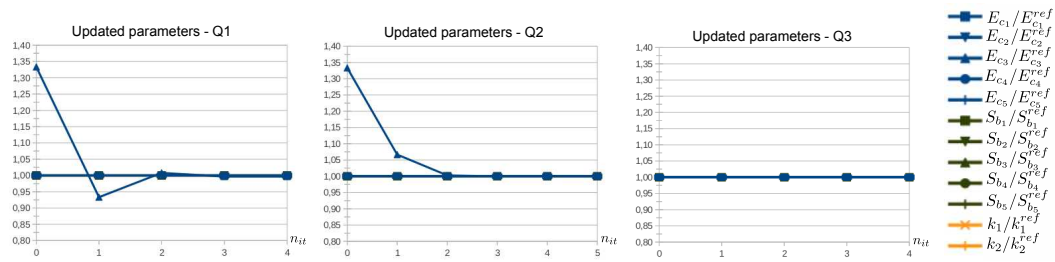


FIGURE 2. Updated beam parameters at each iteration of the updating goal-oriented approach for quantities of interest Q_1 , Q_2 , and Q_3 .

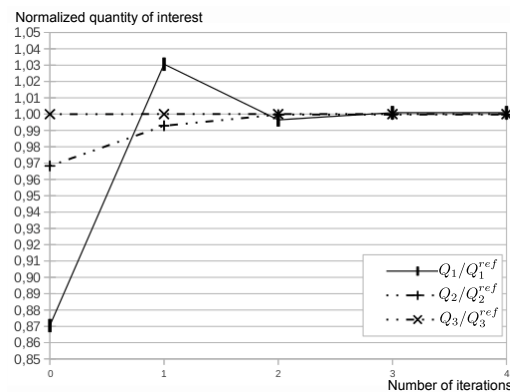


FIGURE 3. Ratio between the updated and the reference values of the quantity of interest at each iteration of the updating goal-oriented approach for Q_1 , Q_2 and Q_3 .

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