

The application of path integral for log return probability calculation

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Abstract. Log return probability has been calculated using path integral method. The stock price is assumed obeying the stochastic differential equation of a geometric Brownian motion and the volatility is assumed following Ornstein Uhlenbeck process. The stochastic differential equation of stock price and volatility lead to Fokker-Plank equation. The Fokker-Plank equation is solved using path integral method. Distribution of log return can be used to take the valuation in return stock.

1. Introduction

Study in financial market has been carried out by enormous authors in the framework of physics. One of those author is Scaden [1]. He modeled secondary financial market by making use of the machineries of quantum mechanics in which investors are represented as vectors in a Hilbert space. All possible investors holding securities is taken as the basis of the Hilbert space of market states and the transactions are represented by operators. Bagarello [2] applied the above model introduced by Scaden in the case of finite dimensional investor Hilbert space.

In this work we calculate the log return probability using path integral method, in contrast to the work of Dragulescu and Yakovenko [3] in which they applied the characteristic method to calculate the same probability. The probability of log return plays some important roles in financial market analysis. At the first place the log return probability determines the valuation of the stock proper price. It also leads to the calculation of derivative price such as option and future.

We assumed that the stock price obeys the stochastic equation of a geometric Brownian motion, whereas the volatility is assumed following Ornstein-Uhlenbeck process. The assumption is also implemented in Heston's model [4]. The stochastic differential equation of stock price and volatility lead to a Fokker-Plank equation. The equation can be considered as Schroedinger-like equation and be solved using path integral method.

2. The Stochastic differential of stock price and volatility

Consider a stock whose price follows the stochastic differential equation of geometric Brownian motion as [3,4,5]

$$dS(t) = \phi S(t) dt + \sigma S(t) dW_S, \quad (1)$$



where $S(t)$ is time function of stock price, ϕ drift parameter, and dW_s Wiener process standard. Drift parameter ϕ is assumed to be constant and risk free rate. Some author assumed that ϕ may depend on stock price, volatility as well as on time [6].

The volatility in equation (1) is assumed following Ornstein Uhlenbeck process [7]

$$d\sigma(t) = -\eta\sigma(t)dt + \mu dW_v(t). \quad (2)$$

Using Ito's lemma, variance in equation (2) can be written as [4,6]

$$dv(t) = -\gamma(v - \theta)dt + \kappa\sqrt{v}dW_v, \quad (3)$$

where $v = \sigma^2$ is variance. The factor dW_s and dW_v is Wiener process for stock price and Wiener process for variance respectively. Both of those processes have correlation given by

$$dW_v(t) = \rho dW_s(t) + \sqrt{1 - \rho^2} Z(t)$$

with $\rho \in [-1, 1]$.

There are some variables that can be investigated in financial [8]. One of the interesting variable to be analyzed is log return. In this work we would like to obtain the probability of log return. We use $z = l_r - \phi t$ as log return, with

$$l_r = \ln \frac{S(t)}{S(0)}.$$

Using Ito's calculus, equation (1) can be written as

$$dz = -\frac{v}{2}dt + \sqrt{v}dW_s. \quad (4)$$

When two random variables X_i ($i = 1, 2$) satisfy stochastic differential equations

$$dX_i(t) = g_i(X_i(t))dt + B_{ij}(X_i(t))dW_j(t), \quad (5)$$

then we obtained the following Fokker-Plank equation [9,10]

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x_i} [g_i(x, t) P(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} [D_{ij}(x, t) P(x, t)], \quad (6)$$

where $D_{ij} = B_{ik} B_{jk}$ [9,10]. Equation (3) and (4) then lead to

$$\frac{\partial P}{\partial t} = -\gamma \frac{\partial}{\partial v} [(v - \theta) P] + \frac{1}{2} \frac{\partial}{\partial z} (vP) + \rho \kappa \frac{\partial^2}{\partial z \partial v} (vP) + \frac{1}{2} \frac{\partial^2}{\partial z^2} (vP) + \frac{\kappa}{2} \frac{\partial^2}{\partial z^2} (vP), \quad (7)$$

where $P = P(z, v | v_i)$ is the transition probability from initial state at $t = 0$ with log return $z = 0$ and variance v_i to the final state at time t with variance v and log return z . The initial condition for Fokker-Plank equation which governs time evolution of $P = P(z, v | v_i)$ is given by

$$P(t = 0) = \delta(x) \delta(v - v_i). \quad (8)$$

The probability distribution of variance that satisfy equation (3) or Fokker-Plank equation for variance in equation (3) itself is given by

$$\frac{\partial \wp(t, v)}{\partial t} = \frac{\partial [\gamma(v - \theta) \wp(t, v)]}{\partial v} + \frac{\kappa^2}{2} \frac{\partial^2}{\partial v^2} [v \wp(t, v)]. \quad (9)$$

The solution of equation (9) in stationary state is

$$\wp_s(v) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} v^\beta \exp(-\alpha v), \quad (10)$$

where $\alpha = \frac{2\gamma}{\kappa^2}$ and $\beta = \alpha\theta - 1$. We can write equation (7) simpler via Fourier transformation

$$P(z, v | v_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_z e^{ip_z z} \tilde{P}(v | v_i) \quad (11)$$

as

$$\frac{\partial \tilde{P}}{\partial t} = \gamma \frac{\partial}{\partial v} \left[(v - \theta) \tilde{P} \right] - \left[\frac{p_z^2 - ip_z}{2} v - i\rho\kappa p_z \frac{\partial}{\partial v} v - \frac{\kappa^2}{2} \frac{\partial^2}{\partial v^2} v \right] \tilde{P}. \quad (12)$$

3. Path integral method for Fokker-Plank solution

Equation (12) can be considered as Schroedinger-like equation [5,11,12]

$$\frac{\partial \tilde{P}(v | v_i)}{\partial t} = -\hat{H} \tilde{P}(v | v_i) \quad (13)$$

which the Hamiltonian is given by

$$\hat{H} = \frac{\kappa^2}{2} \hat{p}_v^2 \hat{v} - i\gamma \hat{p}_v (\hat{v} - \theta) + \frac{p_z^2 - ip_z}{2} \hat{v} + \rho\kappa p_z \hat{p}_v \hat{v},$$

where $\hat{p}_v = -i \frac{\partial}{\partial v}$ is the canonical operator conjugating to \hat{v} which satisfy the commutation relation

$[\hat{v}, \hat{p}_v] = i$. The solution of equation (13) are matrix elements of evolution operator

$$\tilde{P}(v, t | v_i, t_i) = \langle v_f | \exp(-\hat{H}t) | v_i \rangle, \quad (14)$$

if \hat{H} is independent of time. Although the Hamiltonian \hat{H} in equation (14.a) depends on time, this formula can be applied if the interval of time is very small so that \hat{H} is nearly constant. So we can divide the time interval from t_i to t in equation (14.a) with M equal segments given by

$$\varepsilon = \frac{t - t_i}{M}$$

and then calculate $\tilde{P}(v, t | v_i, t_i)$ using path integral method. Now equation (14.a) can be expressed as

$$\tilde{P}(v, t | v_i, t_i) = \langle v_f | \exp(-\hat{H}t) | v_i \rangle = \int \prod_{j=1}^{M-1} dv_j \prod_{j=1}^M \langle v_j | \exp(-\hat{H}(t_j - t_{j-1})) | v_{j-1} \rangle. \quad (15)$$

Consider now the segment $(t_j - t_{j-1})$, inserting the completeness condition of basis $\{|p_{vj}\rangle\}$ into matrix elements $\langle v_j | \exp(-\hat{H}t) | v_{j-1} \rangle$ in this segment yields

$$\langle v_j | \exp(-\hat{H}(t_j - t_{j-1})) | v_{j-1} \rangle = \int dp_{vj} \langle v_j | p_{vj} \rangle \langle p_{vj} | \exp(-\hat{H}(t_j - t_{j-1})) | v_{j-1} \rangle, \quad (16)$$

where

$$\langle v_j | p_{vj} \rangle = \frac{\exp(ip_{vj}v_j)}{\sqrt{2\pi}}$$

and

$$\langle p_{vj} | v_{j-1} \rangle = \frac{\exp(-ip_{vj}v_{j-1})}{\sqrt{2\pi}}.$$

Since ε is very small, we obtain

$$\langle p_{vj} | \exp(-\hat{H}(t_j - t_{j-1})) | v_{j-1} \rangle \approx \exp(-\varepsilon H(p_{vj}, v_{j-1})) \langle p_{vj} | v_{j-1} \rangle \quad (17)$$

and the probability $\tilde{P}(v, t | v_i, t_i)$ in equation (15) can be expressed as

$$\begin{aligned} \tilde{P}(v, t | v_i, t_i) = & \int \prod_{j=1}^{M-1} dv_j \int \prod_{j=1}^M \frac{dp_{vj}}{2\pi} \\ & \times \exp\left(\sum_{j=1}^M (ip_{vj} v_j)\right) \exp\left(\sum_{j=1}^M (-ip_{vj} v_{j-1})\right) \exp\left(-\sum_{j=1}^{M-1} \varepsilon H(p_{vj}, v_{j-1})\right), \end{aligned} \quad (18)$$

$$\text{where } H(p_{vj}, v_{j-1}) = \frac{\kappa^2}{2} p_{vj}^2 v_{j-1} - i\gamma p_{vj} (v_{j-1} - \theta) + \frac{p_z^2 - ip_z}{2} v_{j-1} + \rho\kappa p_z p_{vj} v_{j-1}.$$

The path integral expression for equation (15) is obtained by taking limit $M \rightarrow \infty$ as [5,12]

$$\tilde{P}(v, t | v_i, t_i) = \int Dv Dp_v e^S \quad (19)$$

with the action is given by

$$S = \sum_{j=1}^M \left(ip_{vj} (v_j - v_{j-1}) - \varepsilon \left(\frac{\kappa^2}{2} p_{vj}^2 v_{j-1} - i\gamma p_{vj} (v_{j-1} - \theta) + \frac{p_z^2 - ip_z}{2} v_{j-1} + \rho\kappa p_z p_{vj} v_{j-1} \right) \right). \quad (20)$$

Rearranging S as

$$\begin{aligned} S = & ip_{vM} v_M - ip_{vi} v_i + \sum_{j=1}^{M-1} v_j \left[i(p_{vj-1} - p_{vj}) - \varepsilon \frac{\kappa^2}{2} p_{vj}^2 + i\varepsilon\gamma p_{vj} - \varepsilon \frac{p_z^2 - ip_z}{2} - \varepsilon \rho\kappa p_z p_{vj} \right] \\ & - \sum_{j=1}^M i\varepsilon\gamma p_{vj} \theta \end{aligned} \quad (21)$$

and integrating equation (19) with respect to dv in interval j with S in equation (21) yields

$$\frac{dp_v}{d\tau} = i \frac{\kappa^2}{2} p_v^2 + \Gamma p_v + i \frac{p_z^2 - ip_z}{2} \quad (22)$$

with $\Gamma = \gamma + i\rho\kappa p_z$ and the boundary condition $p_v = \tilde{p}_v$ at $\tau = t$.

The transition probability in equation (14) now remains

$$\tilde{P} = \int \frac{dp_v}{2\pi} \exp[ip_v v - ip_{vi} v_i] \exp\left[-i \int_0^t \gamma p_v \theta d\tau\right]. \quad (23)$$

To obtain the solution of equation (23), we have to solve equation (22) firstly. Equation (22) can be expressed as

$$\frac{dp_v}{d\tau} = ap_v^2 + bp_v + c, \quad (24)$$

with $a = \frac{i\kappa^2}{2}$, $b = \Gamma$, $c = \frac{i}{2}(p_z^2 - ip_z)$ [13]. By substituting $p_v = x + r$ into (24), where r is root of

$ap_v^2 + bp_v + c = 0$, i.e. $r = \frac{-\Gamma \pm \Omega}{i\kappa^2}$, and $\Omega = \sqrt{\Gamma^2 + \kappa^2(p_z^2 - ip_z)}$ we obtain

$$\frac{dx}{d\tau} - (b + 2ar)x = ax^2. \quad (25)$$

Equation (25) is Bernoulli equation

$$\frac{dy}{du} + P(u)y = g(u)y^n,$$

with $P = -(b + 2ar)$, $g(u) = a$, and $n = 2$. There are two kinds of r so we have two solutions of equation (25). Inserting the solutions into $p_v = x + r$ yields

$$p_v = -\frac{i2\Omega}{\kappa^2} \frac{1}{\chi_+ \exp(\Omega(t - \tau)) - 1} + i \frac{\Gamma - \Omega}{\kappa^2} \quad (26.a)$$

and

$$p_v = -\frac{i2\Omega}{\kappa^2} \frac{1}{1 - \chi_- \exp(\Omega(\tau - t))} + i \frac{\Gamma + \Omega}{\kappa^2} \quad (26.b)$$

where χ_+ and χ_- are

$$\chi_+ = \left(1 - \frac{2\Omega i}{((\tilde{p}_v)\kappa^2 - i(\Gamma - \Omega))} \right)$$

and

$$\chi_- = \left(\frac{i2\Omega}{(\kappa^2 \tilde{p} - i(\Gamma + \Omega))} + 1 \right)$$

respectively.

Inserting equation (26.a) and (26.b) into (23) then integrating these equations with respect to τ yields

$$\tilde{P}(v | v_i) = \int \frac{dp_v}{2\pi} \exp[ip_v v - ip_{v_i} v_i] \exp \left[-\frac{2\gamma\theta}{\kappa^2} \ln \frac{\chi_+ - e^{-\Omega t}}{\chi_+ - 1} + \frac{\gamma\theta(\Gamma - \Omega)}{\kappa^2} t \right] \quad (27.a)$$

and

$$\tilde{P}(v | v_i) = \int \frac{dp_v}{2\pi} \exp[ip_v v - ip_{v_i} v_i] \exp \left[-\frac{2\gamma\theta}{\kappa^2} \ln \frac{(e^{\Omega t} - \xi_-)}{(1 - \xi_-)} + \frac{\gamma\theta(\Gamma + \Omega)}{\kappa^2} t \right] \quad (27.b)$$

The probability $P(z, v | v_i)$ obtained by inserting equation (27.a) and (27.b) into Fourier transform (11) is given by

$$P(z, v | v_i) = \int \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{dp_v}{2\pi} e^{ip_z z} \exp[ip_v v - ip_{v_i} v_i] \exp \left[-\frac{2\gamma\theta}{\kappa^2} \ln \frac{\chi_+ - e^{-\Omega t}}{\chi_+ - 1} + \frac{\gamma\theta(\Gamma - \Omega)}{\kappa^2} t \right] \quad (28)$$

and

$$P(z, v | v_i) = \int \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{dp_v}{2\pi} e^{ip_z z} \exp[ip_v v - ip_{v_i} v_i] \exp \left[-\frac{2\gamma\theta}{\kappa^2} \ln \frac{(e^{\Omega t} - \chi_-)}{(1 - \chi_-)} + \frac{\gamma\theta(\Gamma + \Omega)}{\kappa^2} t \right] \quad (29)$$

Variance and volatility can not be obtained from financial data directly, and we interest with the log return whatever the volatility, so we concentrate to $P(z)$. Integrating equation (28) and (29) with respect to v and p_v respectively and applying $p_v = 0$ at $\tau = t$ and $p_v(\tau = t) = \tilde{p}_v$, yields the same expression

$$P(z | v_i) = \int_{-\infty}^{\infty} \frac{dp_z}{(2\pi)} \exp \left[\frac{-(p_z^2 - ip_z) v_i}{\Gamma + \Omega \coth\left(\frac{\Omega t}{2}\right)} \right] \exp \left[-\frac{2\gamma\theta}{\kappa^2} \ln \frac{\left(\Gamma \sinh\left(\frac{\Omega t}{2}\right) + \Omega \cosh\left(\frac{\Omega t}{2}\right) \right)}{\Omega} + \frac{\gamma\theta\Gamma t}{\kappa^2} \right] \quad (30)$$

The probability $P(z)$ can not be obtain unless we have the distribution of v_i . When the distribution of v_i at $t = 0$ is stationer we can use equation (10) for v_i as

$$\wp_s(v_i) = \frac{\alpha^{\beta+1}}{\Gamma(\beta+1)} v_i^\beta \exp(-\alpha v_i),$$

and then $P(z)$ will be obtained from

$$P(z) = \int_0^\infty dv_i \wp_s(v_i) P(z | v_i) \quad (31)$$

by integrating with respect to v_i as

$$P(z) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} dp_z e^{ip_z z} \exp \left(\frac{\gamma \theta t}{\kappa^2} - \frac{2\gamma \theta}{\kappa^2} \ln \left(\frac{(\Omega^2 - \Gamma^2) + \Gamma 2\gamma}{\Omega 2\gamma} \sinh \frac{\Omega t}{2} + \cosh \left(\frac{\Omega t}{2} \right) \right) \right). \quad (32)$$

Equation (32) can be integrated easily by numeric.

4. Conclusion.

The probability of log return $P(z)$ can be calculated using path integral in which the initial volatility is set in stationary state. $P(z)$ is obtained from $P(z, v | v_i)$, and $P(z, v | v_i)$ is solution of Fokker Plank two dimensional that is built by stochastic differential equation for log return and volatility. $\tilde{P}(v | v_i)$ that is fourier transformation for $P(z, v | v_i)$ is calculated using path integral.

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