Effective 8-Spinor Model for Leptons

Yunita Umniyati

Department of Mechatronics, Swiss German University, EduTown BSD City Tangerang 15339, Indonesia

E-mail: yunita.umniyati@sgu.ac.id

Abstract. The 8-spinor realization of the chiral Skyrme-Faddeev model to describe leptons is proposed. The lepton charge being identified with the Hopf index. Energy in the model is proved to be estimated from below by the Hopf index to the power of 3/4.

1. Introduction

The Skyrme's idea to describe baryons as topological solitons [1] proved to be fruitful in nuclear physics. In the Skyrme Model the topological charge $Q = \deg(S^3 \to S^3)$ is interpreted as the baryon number B. The similar idea to describe leptons as topological solitons was announced by Faddeev [2], in which the Hopf invariant Q_H is interpreted as the lepton number L. Proposed the unification of Skyrme and Faddeev models through effective 8-spinor field $\mathbf{\Psi}$ as suggested in [3].

If we take into account the existence of the special 8-spinors identity discovered by the Italian geometer Brioschi [4]:

$$2j_\mu j^\mu = s^2 + p^2 + \mathbf{v}^2 + \mathbf{a}^2 + \mathbf{\Delta}^2, \tag{1}$$

where the following quadratic spinor quantities are introduced:

$$s = \bar{\mathbf{\Psi}}\mathbf{\Psi}, \ p = i\bar{\mathbf{\Psi}}\gamma^5\mathbf{\Psi}, \ \mathbf{v} = \bar{\mathbf{\Psi}}\lambda\mathbf{\Psi}, \ \mathbf{a} = i\bar{\mathbf{\Psi}}\lambda\gamma^5\mathbf{\Psi}, \ j^\mu = \bar{\mathbf{\Psi}}\gamma^\mu\mathbf{\Psi}, \ \tilde{j}^\mu = \bar{\mathbf{\Psi}}\gamma^\mu\gamma^5\mathbf{\Psi},$$

$$\mathbf{\Delta}^2 = \left[(\varphi_1^+\varphi_1)(\varphi_2^+\varphi_2) - |\varphi_1^+\varphi_2|^2 + (\chi_1^+\chi_1)(\chi_2^+\chi_2) - |\chi_1^+\chi_2|^2 \right] \geq 0.$$

where $\bar{\mathbf{\Psi}} = \mathbf{\Psi}^+\gamma_0$ and λ standing for Pauli matrices in the flavor (isotopic) space. Here the diagonal (Weyl) representation for $\gamma_5 = \gamma_5^+$ is used and γ_μ, $\mu = 0, 1, 2, 3$, designate the unitary Dirac matrices acting on Minkowsky spinor indices.

8-spinors (as columns) are defined as

$$\mathbf{\Psi} = \mathrm{col}(\psi_1, \psi_2), \ \psi_i = \mathrm{col}(\varphi_i, \chi_i), \ i = 1, \ 2,$$

with φ_i and χ_i being 2-spinors. The structure of the identity (1) leads to the natural conclusion that Higgs potential V in the effective spinor field model can be represented as the function of $j_\mu j^\mu$:

$$V = \frac{\sigma^2}{8}(j_\mu j^\mu - \varkappa_0^2)^2, \tag{2}$$

with σ and $\varkappa_0$ being some constant parameters. If the Hamiltonian of the model is bounded below by some positive function of the topological charge, it is possible to prove the existence of stationary soliton solutions endowed with topological charge and implementing energy minimum.

Note that depending on the choice of the boundary condition at $r \to \infty$ will be obtained different varieties contained in S^8 defined by (1).

2. Spinor realization of chiral models

To ensure the fulfillment of the condition $\bar{\Psi}\lambda_3\Psi_{\text{vac}} = \text{const}$, $Q_H = \pi_3(S^2)$, the easiest way by choosing a nonlinear spinor model with Lagrangian as follows:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3,$$

where:

$$\mathcal{L}_1 = \partial_\mu \bar{\Psi}\gamma^\nu j_\nu \partial^\mu \Psi, \tag{3a}$$

$$\mathcal{L}_2 = g_1[(\bar{\Psi}\partial_\mu\Psi)(\bar{\Psi}\partial_\nu\Psi)^* - (\bar{\Psi}\partial_\nu\Psi)(\bar{\Psi}\partial_\mu\Psi)^*]^2, \tag{3b}$$

$$\mathcal{L}_3 = -g_2[(\bar{\Psi}\lambda_3\psi) - \sigma_0)]^2. \tag{3c}$$

Energy–momentum tensor in this model is defined as:

$$T_\nu^\mu = \Pi_\Psi^\mu \partial_\nu \Psi + \partial_\nu \bar{\Psi}\Pi_{\bar{\Psi}}^\mu - \delta_\nu^\mu \mathcal{L}.$$

The energy density is not difficult to find with the basis of Euler's theorem for homogeneous functions, and energy is obtained by Noether's theorem.

Let the unit vector is determined as $\vec{n} = \frac{\vec{N}}{|\vec{N}|}$, where $\vec{N} = (\bar{\psi}\vec{\lambda}\psi)$ is Goldstone field. It can be shown that energy in this model is bounded below by Hopf index:

$$E > \alpha|Q_H|^{\frac{3}{4}},$$

as in Faddeev model.

From energy density, we find $[\partial_i|\vec{N}|^2\vec{n}^2 + |\vec{N}|^2(\partial_i\vec{n})^2] > |\vec{N}|^2(\partial_i\vec{n})^2$ or $\xi^\dagger\gamma^0 A\xi + \eta^\dagger\gamma^0 A\eta \geq \lambda_{\min}|\xi^\dagger\xi + \eta^\dagger\eta|$. Next, we consider the reference system in which $\Psi_1 \neq 0$; $\Psi_2 = 0$. With some algebras we find that:

$$(\partial_i N^a \partial_k N^b N^c \varepsilon^{abc})^2 = \{2(\bar{\Psi}_1\Psi_1)\partial_i(\bar{\Psi}_1\Psi_1)\partial_k(\bar{\Psi}_1\Psi_1)\}^2 \leq |\partial_i\Psi|^2|\Psi|^2|\partial_k\Psi|^2|\Psi|^2.$$

As the final result, it is shown that Hamiltonian in this model is bounded below by Hamiltonian Faddeev

$$E(\Psi, \partial_\mu\Psi) > \alpha(\nabla n^a)^2 + \beta f_{ik}^2 > \text{const}|Q_H|^{\frac{3}{4}}.$$

3. The effective nonlinear model of 8-spinor field

In this paper we construct a Lagrangian density of our model, that is analog with Skyrme and Faddeev models:

$$\mathcal{L} = \frac{1}{2\lambda^2}\overline{D_\mu\Psi}\gamma^\nu j_\mu D^\mu\Psi + \frac{\varepsilon^2}{4}f_{\mu\nu}f^{\mu\nu} - V, \tag{4}$$

where λ and ε being constant parameters of the model. In (4) the extended covariant derivative is written as:

$$D_\mu = \partial_\mu - ie_0 A_\mu \Gamma_e. \tag{5}$$

Γ_e in (5) stands for the electric charge operator, with e_0 being constant interaction with the electromagnetic field. The standard form of Γ_e reads as follow:

$$\Gamma_e = \frac{1}{2}(\lambda_3 - I). \tag{6}$$

In second term of (4) $f_{\mu\nu}$ stands for the antisymmetric tensor of Faddeev-Skyrme type:

$$f_{\mu\nu} = \left(\bar{\psi}\gamma^{\alpha}D_{[\mu}\psi\right)\left(\overline{D_{\nu]}\psi}\gamma_{\alpha}\psi\right). \qquad (7)$$

It should be stressed that the first term in (4) generalizes the $\sigma-$model term in Skyrme Model, and the second term gives the generalization of Skyrme (or Faddeev) term.

By taking $\varphi_i \to \chi_i$ and using approximate hydrogen substitution for separate angles, as well as an approximation of $|\Psi|^2 = 4[A\tanh(\alpha x)+B]$ in isotropic spherical coordinates $x = \log(r/r_0)$, we obtain functionals of energy $\mathcal{E}$ and topological charge Q_H in the implementation of the chiral spinor Skyrme- Faddeev model:

$$\mathcal{E} = \int_{-\infty}^{+\infty} dx \left\{ \frac{e^x}{\lambda^2}r_0 \left[\frac{1}{4}R'^2 + R^2(\Theta'^2 + \xi'^2 + \eta'^2 + 2\xi'\eta'\cos 2\Theta + 2\sin^2\Theta) \right] + \right.$$

$$+ \quad 4\frac{\epsilon^2}{r_0}e^{-x}R'^2 R^2 \sin^2\Theta(\cos^2\Theta\cos^2 2\eta + \sin^2\Theta\cos^2 2(\xi - \eta)) +$$

$$+ \quad \left. r_0^3\sigma^2 e^{3x}\left(R^2 - \frac{\varkappa_0^2}{4} \right)^2 \right\}. \qquad (8)$$

To estimate the energy $\mathcal{E}$ we take into account $Q_H = 1$ and use trial function:

$$R = A\tanh(\alpha x) + B, \quad \Theta = 2\arctan(e^{-\alpha x}), \; \alpha \approx \sqrt{2}, \; \eta = 0, \qquad (9)$$

where A and B are defined as:

$$A = \frac{1}{2}\left(\frac{\varkappa_0}{2} - R_0 \right), \quad B = \frac{1}{2}\left(\frac{\varkappa_0}{2} + R_0 \right),$$

After some algebras we obtain energy $\mathcal{E}$:

$$\mathcal{E} = \frac{r_0\alpha}{3\lambda^2}(5A^2 + 4B^2) + \frac{64A^2\alpha\epsilon^2}{15r_0}\left(\frac{A^2}{7} + B^2 \right) + \frac{4r_0^3\sigma^2 A^2}{3\alpha}\left[\frac{A^2}{5} + (A + 2B)^2 \right],$$

with α, r_0, and R_0 are parameters of minimization. By varying the energy functional on A provided $A + B = \varkappa_0/2$ and assuming that $\varkappa_0 \ll 1$ and $\sigma\lambda^2 \ll 1$, we find:

$$A \approx \frac{2}{9}\varkappa_0, \quad B \approx \frac{5}{18}\varkappa_0.$$

Further, by varying ε on r_0, we obtain estimation of the form:

$$r_0 \approx 0,3\varkappa_0\lambda\epsilon, \quad \mathcal{E} \approx \frac{\varkappa_0^3\epsilon}{6\lambda}.$$

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References
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