

# A classification of finite quantum kinematics

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**Abstract.** Quantum mechanics in Hilbert spaces of finite dimension  $N$  is reviewed from the number theoretic point of view. For composite numbers  $N$  possible quantum kinematics are classified on the basis of Mackey's Imprimitivity Theorem for finite Abelian groups. This yields also a classification of finite Weyl-Heisenberg groups and the corresponding finite quantum kinematics. Simple number theory gets involved through the fundamental theorem describing all finite discrete Abelian groups of order  $N$  as direct products of cyclic groups, whose orders are powers of not necessarily distinct primes contained in the prime decomposition of  $N$ . The representation theoretic approach is further compared with the algebraic approach, where the basic object is the corresponding operator algebra. The consideration of fine gradings of this associative algebra then brings a fresh look on the relation between the mathematical formalism and physical realizations of finite quantum systems.

## 1. Introduction

Looking back on our papers [1, 2] on symmetries of finite Heisenberg groups, I find it appropriate — at this Symposium — to return to the foundations of our approach to quantum systems with finite-dimensional Hilbert spaces and supplement them with the algebraic treatment.

Non-relativistic quantum mechanics of particle systems can be divided into quantum kinematics and quantum dynamics. First the Hilbert space and the non-commuting operators of complementary observables, positions and momenta, are constructed. This kinematical structure then remains the same for all possible quantum dynamics which are determined by the respective Hamiltonians.

For systems with configuration space  $\mathbb{R}^n$  one can define quantum kinematics according to H. Weyl [3] in terms of the Weyl system — a projective unitary representation of the Abelian group of translations of the corresponding classical phase space  $\mathbb{R}^n \times \mathbb{R}^n$ . Using Weyl's system of unitary operators, J. von Neumann was able to prove the uniqueness theorem stating that there is, up to unitary equivalence, unique irreducible quantum kinematics, commonly taken in the form of the Schrödinger representation. In the most general form the uniqueness theorem was proved



on the basis of G.W. Mackey's Imprimitivity Theorem. In this mathematical generalization the configuration space  $\mathbb{R}^n$  was replaced by an arbitrary locally compact second countable Abelian group  $G$  [4]. The direct product of  $G$  and its Pontryagin dual then plays the role of the phase space.

Quantum mechanics in finite-dimensional Hilbert spaces originally seemed to present only a nice and simple exercise in linear algebra. During the last decades it unexpectedly became the mathematical framework for the development of methods of quantum information processing with numerous applications to quantum cryptography, teleportation and quantum computing. For instance, the mathematical notion of complementary bases lies at the heart of quantum cryptography [5].

Historically, H. Weyl — not successful with the proof of the uniqueness theorem — wanted to present a simple example in  $\mathbb{C}^n$  analogous to one-dimensional quantum particle and at the same time exhibiting the uniqueness feature [3]. This example was further developed by J. Schwinger [6] with the aim to approximate quantum mechanics of particles. He noticed the complementary nature of position and momentum observables which are here built in the finite Weyl-Heisenberg group generated by the generalized Pauli matrices. The elements of this group provide a useful basic set of quantum operators for finite quantum systems.

The special role of the finite Weyl-Heisenberg group has been recognized also in a distant domain of mathematics — the classification of fine gradings of classical Lie algebras [7, 8]. For classical Lie algebras of the type  $\mathfrak{sl}(n, \mathbb{C})$ ,  $n = 2, 3, \dots$ , this classification contains — among others — the Pauli grading based on generalized Pauli matrices [9, 10]. The finite group generated by them has been called the Pauli group — a notion identical with the Weyl-Heisenberg group.

In our paper [11] we succeeded to prove the uniqueness of finite quantum kinematics using a simple version of Mackey's Imprimitivity Theorem. In this way also a geometric interpretation of finite quantum kinematics was obtained: it appears as quantum mechanics where a cyclic group serves as the configuration space. For composite dimensions  $N$  the fundamental theorem on finitely generated Abelian groups immediately leads to the classification of all finite quantum kinematics. This classification was also independently noted in 1995 by V.S. Varadarajan [12].

In this contribution our approach via representation theory is confronted with the algebraic formulation of quantum mechanics [13]. For finite quantum systems, the operator algebras are the associative complex matrix algebras  $M_N(\mathbb{C})$  [15]. Inspired by Lie algebra gradings, we describe all fine gradings of  $M_N(\mathbb{C})$  which are induced by inner automorphisms of  $\mathrm{GL}(N, \mathbb{C})$ .

In section 2 we reproduce the group theoretical approach based on representation theory of finite Abelian groups. Sections 3 and 4 are devoted to the classification of finite quantum kinematics and some historical remarks. In section 5 fine gradings of matrix algebras  $M_N(\mathbb{C})$  are obtained and physical interpretation of the Pauli gradings is offered. Section 6 concludes the paper.

## 2. Simple quantum kinematics on cyclic groups

Ordinary quantum mechanics prescribes that the mathematical quantities representing the position and momentum should be self-adjoint operators on the Hilbert space of the system. Their algebraic properties constitute *quantum kinematics*, while *quantum dynamics* of the system

is given by the unitary group generated by the Hamiltonian which is expressed as a function of the position and momentum operators.

According to Mackey, quantum kinematics of a system localized on a homogeneous space  $M = G/H$  is determined by an irreducible transitive system of imprimitivity  $(\mathcal{U}, \mathcal{E})$  for  $G$  based on  $M$  in a Hilbert space  $\mathcal{H}$  [4]. Here  $\mathcal{U} = \{U(g)|g \in G\}$  is a unitary representation of  $G$  in  $\mathcal{H}$  and  $\mathcal{E} = \{E(S)|S \text{ Borel subset of } M\}$  is a projection-valued measure in  $\mathcal{H}$  satisfying the covariance condition

$$U(g)E(S)U(g)^{-1} = E(g^{-1}.S). \quad (1)$$

Given a positive integer  $N \geq 2$ , we assume that the configuration space  $M$  is the finite set

$$M = Z_N = \{\rho|\rho = 0, 1, \dots, N-1\}$$

with additive group law modulo  $N$ . Since there is a natural transitive action of  $Z_N$  on itself, we may consider  $M$  as a homogeneous space of

$$G = Z_N = \{j|j = 0, 1, \dots, N-1\},$$

realized as an additive group modulo  $N$  with the action

$$G \times M \rightarrow M : (j, \rho) \mapsto \rho + j(\text{mod } N)$$

and the isotropy subgroup  $H = \{0\}$ . For the finite set  $M = Z_N$  the covariance condition simplifies to

$$U(j)E(\rho)U(j)^{-1} = E(\rho - j),$$

where  $\rho \in M$ ,  $j \in G$ ,  $E(\rho) = E(\{\rho\})$  and  $E(S) = \sum_{\rho \in S} E(\rho)$ .

Complete classification of transitive systems of imprimitivity up to simultaneous unitary equivalence of both  $\mathcal{U}$  and  $\mathcal{E}$  is obtained from Mackey's Imprimitivity Theorem. Its application to our system yields

**Theorem 2.1.** *If  $(\mathcal{U}, \mathcal{E})$  acts irreducibly in  $\mathcal{H}$ , then there is, up to unitary equivalence, only one system of imprimitivity, where:*

(i)  $\mathcal{H}$  is the Hilbert space  $\mathcal{H}_N = \ell^2(\mathbb{Z}_N)$ , i.e.  $\mathbb{C}^N$  with the inner product

$$(\varphi, \psi) = \sum_{\rho=0}^{N-1} \overline{\varphi_\rho} \psi_\rho,$$

where  $\varphi_\rho, \psi_\rho, \rho = 0, 1, \dots, N-1$ , denote the components of  $\varphi, \psi$  in the standard basis.

(ii)  $\mathcal{U}$  is the induced representation  $\mathcal{U} = \text{Ind}_H^G I$  called the (right) regular representation

$$[U(j)\psi]_\rho = \psi_{\rho+j} \quad (j \in G).$$

Its matrix form in the standard basis is

$$(U(j))_{\rho\sigma} = \delta_{\rho+j,\sigma}.$$

(iii)  $\mathcal{E}$  is given by

$$[E(\rho)\psi]_\sigma = \delta_{\rho\sigma}\psi_\sigma.$$

This unique system of imprimitivity has a simple physical meaning. The localization operators  $E(\rho)$  are projectors on the eigenvectors  $e^{(\rho)} \in \mathcal{H}$  corresponding to the positions  $\rho = 0, 1, \dots, N-1$ . Since the set  $\{e^{(\rho)}\}$  forms the standard basis of  $\mathcal{H}$  in the above matrix realization, this realization may be called position representation. Then in a normalized state  $\psi = (\psi_0, \dots, \psi_{N-1})$  the probability to measure the position  $\rho$  is equal to

$$(\psi, E(\rho)\psi) = |\psi_\rho|^2.$$

Unitary operators  $U(j)$  act as displacement operators

$$U(j)e^{(\rho)} = e^{(\rho-j)}.$$

In the position representation they are given by unitary matrices equal to the powers  $U(j) = P_N^j = U(1)^j$  of the one-step cyclic permutation matrix

$$P_N \equiv U(1) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

In this way finite-dimensional quantum mechanics can be viewed as quantum mechanics on configuration spaces given by finite sets equipped with the structure of a finite Abelian group [11]. In the above simplest case a single cyclic group  $\mathbb{Z}_N$  was taken as the underlying configuration space. For given  $N \in \mathbb{N}$  we set  $\omega_N := e^{2\pi i/N} \in \mathbb{C}$ . The *generalized Pauli matrices* of order  $N$  are given by

$$Q_N := \text{diag}(1, \omega_N, \omega_N^2, \dots, \omega_N^{N-1}), \quad (P_N)_{\rho, \sigma} = \delta_{\rho+1, \sigma}, \quad \rho, \sigma \in \mathbb{Z}_N.$$

The subgroup of unitary matrices in  $\text{GL}(N, \mathbb{C})$  generated by  $Q_N$  and  $P_N$ ,

$$\Pi_N := \{\omega_N^j Q_N^k P_N^l | j, k, l \in \{0, 1, \dots, N-1\}\}$$

is called the *finite Weyl-Heisenberg group*.

The special role of the generalized Pauli matrices has been confirmed as the cornerstone of finite-dimensional quantum mechanics and of quantum information science. The quantum mechanical operators,  $Q_N$  and  $P_N$  act in the  $N$ -dimensional Hilbert space  $\mathcal{H}_N = \ell^2(\mathbb{Z}_N)$ . Further properties of  $\Pi_N$  are as follows:

- (1) The order of  $\Pi_N$  is  $N^3$ .
- (2) The center of  $\Pi_N$  is  $\{\omega_N^\rho I_N | \rho \in \{0, 1, \dots, N-1\}\}$ , where  $I_N$  is the  $N \times N$  unit matrix.
- (3) The commutation relation  $P_N Q_N = \omega_N Q_N P_N$  is equivalent with (1).

In order to formulate finite quantum kinematics in terms of the equivalent *discrete Weyl system*, we need the dual system of unitary operators  $\mathcal{V} = \{V(\rho)\}$  defined as powers of  $Q_N$ ,

$$V(\rho) = Q_N^\rho.$$

Then the quantum kinematics  $(\mathcal{U}, \mathcal{E})$  can be equivalently replaced by the discrete Weyl system  $(\mathcal{U}, \mathcal{V})$  satisfying

$$U(j)V(\rho) = \omega^{j\rho}V(\rho)U(j).$$

The discrete Weyl displacement operators are defined by unitary operators<sup>1</sup>

$$W(\rho, j) = \omega^{j\rho/2}V(\rho)U(j) = \omega^{-j\rho/2}U(j)V(\rho).$$

They satisfy the composition law for a ray representation of  $\mathbb{Z}_N \times \mathbb{Z}_N$

$$W(\rho, j)W(\rho', j') = \omega^{(\rho'j - \rho j')/2}W(\rho + \rho', j + j').$$

According to Schwinger, the discrete Weyl system  $\mathcal{W}$  consisting of  $N^2$  operators  $W(\rho, j)$  provides an operator basis in the space of all linear operators in  $\mathbb{C}^N$ , i.e. in the full matrix algebra  $M_N(\mathbb{C})$ . The operators  $W(\rho, j)$  are orthogonal with respect to the Hilbert-Schmidt inner product

$$\text{Tr}(W(\rho, j)W(\rho', j')^*) = N\delta_{\rho\rho'}\delta_{jj'}$$

and satisfy the completeness relation

$$\sum_{\rho, j} W(\rho, j)W(\rho', j')^* = N^2 1.$$

This important result can be summarized as follows:

**Theorem 2.2.** *The set of  $N^2$  matrices  $S(\rho, j) = Q_N^\rho P_N^j / \sqrt{N}$ ,  $j, \rho = 0, 1, \dots, N-1$ , constitutes a complete and orthonormal basis for the linear space  $M_N(\mathbb{C})$  of  $N \times N$  complex matrices. Any  $N \times N$  complex matrix can thus be uniquely expanded in this basis. If  $N$  is odd, then the operator basis can be taken in the form  $\omega^{j\rho/2}S(\rho, j) = W(\rho, j)/\sqrt{N}$ .*

### 3. A classification of finite quantum kinematics

The cyclic group  $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$  is a configuration space for  $N$ -dimensional quantum kinematics of a single  $N$ -level system. However, the reasoning on the basis of Mackey's Imprimitivity Theorem allows direct extension to any finite Abelian group as configuration space because of the fundamental theorem describing the structure of finite Abelian groups [14].

**Theorem 3.1.** *Let  $G$  be a finite Abelian group. Then  $G$  is isomorphic with the direct product  $\mathbb{Z}_{N_1} \times \dots \times \mathbb{Z}_{N_f}$  of a finite number of cyclic groups for integers  $N_1, \dots, N_f$  greater than 1, each of which is a power of a prime, i.e.  $N_k = p_k^{r_k}$ , where the primes  $p_k$  need not be mutually different.*

<sup>1</sup> If  $N$  is odd, the factors  $\omega_N^{j\rho/2}$  are well-defined on  $\mathbb{Z}_N \times \mathbb{Z}_N$ .

The integers  $N_k = p_k^{r_k}$  are called the *elementary divisors* of  $G$ . Two finite Abelian groups are isomorphic if and only if they have the same *elementary divisor decomposition*. In the special case of  $G = \mathbb{Z}_N$  with composite  $N = p_1^{r_1} \dots p_f^{r_f}$  and distinct primes  $p_k > 1$ , the unique decomposition

$$\mathbb{Z}_N \cong \mathbb{Z}_{N_1} \times \dots \times \mathbb{Z}_{N_f}, \quad N_k = p_k^{r_k}$$

is obtained by the Chinese Remainder Theorem.

Now for a general finite Abelian group as configuration space, a transitive system of imprimitivity for  $G = \mathbb{Z}_{N_1} \times \dots \times \mathbb{Z}_{N_f}$  based on  $M = \mathbb{Z}_{N_1} \times \dots \times \mathbb{Z}_{N_f}$  is equivalent to the tensor product

$$\mathcal{U} = \mathcal{U}_1 \otimes \dots \otimes \mathcal{U}_f, \quad \mathcal{E} = \mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_f \quad (2)$$

acting in the Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_f,$$

where the dimensions are  $\dim \mathcal{H}_k = N_k$ ,  $\dim \mathcal{H} = N_1 \dots N_f$ . Each such system of imprimitivity  $(\mathcal{U}, \mathcal{E})$  is irreducible, if and only if each  $(\mathcal{U}_k, \mathcal{E}_k)$  is irreducible, hence if irreducible, it is unique up to unitary equivalence by the Imprimitivity Theorem.

For a given composite dimension  $N$ , all irreducible quantum kinematics in the Hilbert space of dimension  $N$  can be obtained according to Theorem 3.1: just take all inequivalent choices of elementary divisors  $N_k = p_k^{r_k}$  with not necessarily distinct primes  $p_k > 1$ . One can give physical interpretation to these tensor products in the sense that the factors correspond to the *elementary building blocks or constituents* forming a finite quantum system. Our results are summarized in

**Theorem 3.2.** *For a given finite Abelian group  $G$  there is a unique class of unitarily equivalent, irreducible imprimitivity systems  $(\mathcal{U}, \mathcal{E})$  in a finite-dimensional Hilbert space  $\mathcal{H}$ . In the special case  $G = \mathbb{Z}_N$  the irreducible imprimitivity system is unitarily equivalent to the tensor product (2) with  $N_k = p_k^{r_k}$  and distinct primes  $p_1, \dots, p_f > 1$ .*

**Corollary 3.3.** *For a given finite Abelian group  $G$  there is a unique class of unitarily equivalent discrete Weyl systems  $\mathcal{W}$  in a finite-dimensional Hilbert space  $\mathcal{H}$ ,*

$$\mathcal{W} = \mathcal{W}_1 \otimes \dots \otimes \mathcal{W}_f. \quad (3)$$

The above classification conclusively shows that finite quantum kinematics are mathematically composed of elementary constituents which are of standard types associated with finite Abelian groups  $G = \mathbb{Z}_p, \mathbb{Z}_{p^2}, \dots, \mathbb{Z}_{p^r}, \dots$ , where  $p$  runs through the set of prime numbers  $> 1$ .

As already mentioned, among possible factorizations into elementary constituents special role is played by the factorization associated with the ‘best’ prime decomposition of the dimension  $N = p_1^{r_1} \dots p_f^{r_f}$  with mutually distinct primes  $p_1, \dots, p_f$ . Then the discrete Weyl system of composite dimension  $N$  is equivalent to a composite system where the constituent Weyl subsystems act in the Hilbert spaces of relatively prime dimensions  $p_1^{r_1}, \dots, p_f^{r_f}$ .

#### 4. Quantum degrees of freedom taken seriously

A historical remark is in order: the structure of the finite-dimensional Weyl operators was thoroughly investigated by J. Schwinger. He noted in particular that, if the dimension  $N$  is

a composite number which can be decomposed as a product  $N = N_1 N_2$ , where the positive integers  $N_1, N_2 > 1$  are relatively prime, then all the Weyl operators in dimension  $N$  can be simultaneously factorized in tensor products of the Weyl operators in the dimensions  $N_1$  and  $N_2$ . J. Schwinger then states ([6], pp. 578–579):

*“The continuation of the factorization terminates in*

$$N = \prod_{k=1}^f \nu_k,$$

*where  $f$  is the total number of prime factors in  $N$ , including repetitions. We call this characteristic property of  $N$  the number of degrees of freedom for a system possessing  $N$  states.” And further, “each degree of freedom is classified by the value of the prime integer  $\nu = 2, 3, 5, \dots \infty$ .”*

However, our application of Theorem 3.1 leads to the building blocks  $\mathbb{Z}_{p^r}$  with  $r \geq 1$  as constituent configuration spaces. This fact was independently noted by V.S. Varadarajan in his paper [12] devoted to the memory of J. Schwinger: *“Curiously, Schwinger missed the systems associated to the indecomposable groups  $\mathbb{Z}_N$  where  $N$  is a prime power  $p^r$ ,  $r \geq 2$  being an integer.”* And in another paper: *“In this way he arrived at the principle that the Weyl systems associated to  $\mathbb{Z}_p$  where  $p$  runs over all the primes are the building blocks. Curiously this enumeration is incomplete and one has to include the cases with  $\mathbb{Z}_{p^r}$  where  $p$  is as before a prime but  $r$  is any integer  $\geq 1$ .”*

Thus the elementary building blocks of finite quantum kinematics are given by the elementary divisor decomposition. In mathematics also exists equivalent *invariant factor decomposition* [14]. In that approach a finite Abelian group  $G$  is uniquely determined by an ordered finite list of integers  $n_1 \geq n_2 \geq \dots \geq n_s$  greater than 1 determining the invariant factors  $\mathbb{Z}_{n_i}$  of  $G$  such that  $n_{i+1}$  divides  $n_i$  and  $N = n_1 n_2 \dots n_s$ . For instance, if  $N = 180$ , the full list of non-isomorphic Abelian groups of order 180 consists of  $\mathbb{Z}_{180}$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_{90}$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_{60}$ ,  $\mathbb{Z}_6 \times \mathbb{Z}_{30}$ . However, to exhibit the elementary building blocks of the corresponding composite system, the elementary divisor decomposition is indispensable. In the above example all inequivalent choices of elementary divisors  $N_k = p_k^{r_k}$  with not necessarily distinct primes  $p_k > 1$  are  $2^2.3^2.5$ ,  $2.2.3^2.5$ ,  $2^2.3.3.5$ ,  $2.2.3.3.5$ .

The task of enumerating all finite quantum kinematics in dimension  $N$  then amounts to the determination of all finite Abelian groups of order  $N$ . It starts with the factorization of  $N = p_1^{r_1} \dots p_f^{r_f}$  with mutually distinct primes  $p_1, \dots, p_f$ . First one finds all permissible lists for groups of orders  $p_i^{r_i}$  for each  $i$ . For a prime power  $p_i^{r_i}$  the problem of determining all permissible lists is equivalent to finding all *partitions of the exponent  $r_i$* , and does not depend on  $p_i$ . Recall that the number of partitions of a natural number  $r$  is called *Bell’s number  $B(r)$* . Then the total number of groups of order  $N$  is equal to the product of Bell’s numbers  $B(r_1)B(r_2) \dots B(r_f)$ .

In physics, the dimensions of constituent Hilbert spaces are primarily fixed by the numbers of levels of physical subsystems. It follows that  $G$  is isomorphic to a direct product of cyclic groups of the respective orders. Whenever this is the case, in order to be able to work with the elementary building blocks of the corresponding quantum kinematics, we should find the

elementary divisors. For example, if  $G = \mathbb{Z}_6 \times \mathbb{Z}_{15}$ , we factor  $6 = 2 \cdot 3$  and  $15 = 3 \cdot 5$ . Then the elementary divisor decomposition of  $G$  is  $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ .

## 5. Algebraic approach

In quantum information theory as in algebraic quantum theory, the operators of a finite quantum system belong to the full matrix algebra  $M_N(\mathbb{C})$  of  $N \times N$  complex matrices. In this contribution we started with finite Abelian groups and, via representation theory expressed by Mackey's systems of imprimitivity, arrived at the system of  $N^2$  unitary operators forming a complete orthonormal basis of the linear space  $M_N(\mathbb{C})$  with the Hilbert-Schmidt inner product.

Now we shall pass the opposite journey starting from the operator algebra  $M_N(\mathbb{C})$  and equip it as associative algebra with additional structure of a fine grading induced by commuting inner automorphisms of  $\text{GL}(N, \mathbb{C})$ .

A *grading* of an associative algebra  $\mathcal{A}$  is defined as a direct sum decomposition of  $\mathcal{A}$  as a vector space

$$\Gamma : \quad \mathcal{A} = \bigoplus_{\alpha} \mathcal{A}_{\alpha} \quad (4)$$

satisfying the property

$$x \in \mathcal{A}_{\alpha}, \quad y \in \mathcal{A}_{\beta} \quad \Rightarrow \quad xy \in \mathcal{A}_{\gamma}.$$

Similarly as for Lie algebras [7], the linear subspaces  $\mathcal{A}_{\alpha}$  can be determined as eigenspaces of automorphisms of the associative algebra  $\mathcal{A}$ . For an automorphism  $g$  of  $\mathcal{A}$ ,  $g(xy) = g(x)g(y)$  holds for all  $x, y \in \mathcal{A}$ . Now if  $g(x) = \lambda_{\alpha}x$  defines the subspace  $\mathcal{A}_{\alpha}$  and  $g(y) = \lambda_{\beta}y$  defines the subspace  $\mathcal{A}_{\beta}$ , then

$$g(xy) = g(x)g(y) = \lambda_{\alpha}\lambda_{\beta}xy$$

defines the subspace  $\mathcal{A}_{\gamma}$  with  $\lambda_{\gamma} = \lambda_{\alpha}\lambda_{\beta}$ . In this way  $g$  induces the grading decomposition  $\mathcal{A} = \bigoplus_{\alpha} \text{Ker}(g - \lambda_{\alpha})$ . Further commuting automorphisms may refine the grading. If  $gh = hg$ , we have

$$g(h(x)) = h(g(x)) = h(\lambda_{\alpha}x) = \lambda_{\alpha}h(x)$$

implying that the grading decomposition using both  $g$  and  $h$  may lead to a refinement. By extending the set of commuting automorphisms one arrives at so called *fine gradings* where the grading subspaces have the lowest possible dimension.

For the associative algebra  $M_N(\mathbb{C})$  we shall look for fine gradings induced by the inner automorphisms. For  $M \in \text{GL}(N, \mathbb{C})$  we denote  $\text{Ad}_M \in \text{Int}(M_N(\mathbb{C}))$  be the *inner automorphism* of  $M_N(\mathbb{C})$  induced by operator  $M \in \text{GL}(N, \mathbb{C})$ , i.e.

$$\text{Ad}_M(X) = MXM^{-1} \quad \text{for} \quad X \in M_N(\mathbb{C}).$$

The relevant properties of  $\text{Ad}_M$  are: for  $M, N \in \text{GL}(N, \mathbb{C})$

- (i)  $\text{Ad}_M \text{Ad}_N = \text{Ad}_{MN}$ .
- (ii)  $(\text{Ad}_M)^{-1} = \text{Ad}_{M^{-1}}$ .
- (iii)  $\text{Ad}_M = \text{Ad}_N$  if and only if there is a constant  $0 \neq \alpha \in \mathbb{C}$  such that  $M = \alpha N$ .



Since the commuting inner automorphisms form an Abelian subgroup of  $\text{Int}(M_N(\mathbb{C}))$ , *fine gradings* of  $M_N(\mathbb{C})$  can be obtained using the *maximal Abelian groups of diagonalizable automorphisms* – as subgroups of  $\text{Int}(M_N(\mathbb{C}))$  – which have been called the *MAD-groups* [8]. Thus we are looking for fine gradings which are induced via diagonalizable elements of maximal Abelian subgroups of  $\text{Int}(M_N(\mathbb{C}))$ .

The MAD-groups of inner automorphisms of  $M_N(\mathbb{C})$  can be derived by following [8]. The result is straightforward and is summarized in the following theorem:

**Theorem 5.1.** *Any MAD-group contained in  $\text{Int}(M_N(\mathbb{C}))$  is conjugated to one and only one of the groups of the form*

$$\mathcal{P}_{N_1} \otimes \mathcal{P}_{N_2} \otimes \dots \otimes \mathcal{P}_{N_f} \otimes D(m), \quad (5)$$

where  $N_i = p_i^{r_i}$  are powers of primes,  $N = N_1 N_2 \dots N_f$ ,  $m$  and  $D(m)$  is the image in  $\text{Int}(M_N(\mathbb{C}))$  of the group of  $m \times m$  complex diagonal matrices under the adjoint action.

Here  $\mathcal{P}_N$  is defined as the group

$$\mathcal{P}_N = \{\text{Ad}_{Q_N^i P_N^j} | (i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N\}.$$

It is an Abelian subgroup of  $\text{Int}(M_N(\mathbb{C}))$  and is generated by two commuting automorphisms  $\text{Ad}_{Q_N}$ ,  $\text{Ad}_{P_N}$ , each of order  $N$ . A geometric view is sometimes useful that  $\mathcal{P}_N$  is isomorphic to the *quantum phase space* identified with the Abelian group  $\mathbb{Z}_N \times \mathbb{Z}_N$ .

For illustration, we give a list of MAD-groups in low dimensions:

- $n = 2$ :  $\mathcal{P}_2 \otimes D(1)$ ,  $D(2)$
- $n = 3$ :  $\mathcal{P}_3 \otimes D(1)$ ,  $D(3)$
- $n = 4$ :  $\mathcal{P}_4 \otimes D(1)$ ,  $\mathcal{P}_2 \otimes \mathcal{P}_2 \otimes D(1)$ ,  $\mathcal{P}_2 \otimes D(2)$ ,  $D(4)$
- $n = 5$ :  $\mathcal{P}_5 \otimes D(1)$ ,  $D(5)$
- $n = 6$ :  $\mathcal{P}_3 \otimes \mathcal{P}_2 \otimes D(1)$ ,  $\mathcal{P}_3 \otimes D(2)$ ,  $\mathcal{P}_2 \otimes D(3)$ ,  $D(6)$
- $n = 7$ :  $\mathcal{P}_7 \otimes D(1)$ ,  $D(7)$
- $n = 8$ :  $\mathcal{P}_8 \otimes D(1)$ ,  $\mathcal{P}_4 \otimes \mathcal{P}_2 \otimes D(1)$ ,  $\mathcal{P}_2 \otimes \mathcal{P}_2 \otimes \mathcal{P}_2 \otimes D(1)$ ,  $\mathcal{P}_4 \otimes D(2)$ ,  $\mathcal{P}_2 \otimes \mathcal{P}_2 \otimes D(2)$ ,  $\mathcal{P}_2 \otimes D(4)$ ,  $D(4)$

A part of the obtained MAD-groups containing the trivial diagonal subgroup  $D(1)$  induce exactly all our Pauli decompositions (2), (3). However, there are still fine gradings induced by  $D(m)$ ,  $m = 2, 3, \dots$ . They include partial or complete decompositions which have the form of the Cartan root decompositions of Lie algebras  $\mathfrak{sl}(m, \mathbb{C})$  extended by the unit matrix. They contain the Abelian Cartan subalgebra of dimension  $m - 1$ , the unit matrix and one-dimensional root subspaces spanned by nilpotent matrices. Concerning physical interpretation of these Cartan parts of the decompositions, one can speculate that they may reflect extra degrees of freedom corresponding to some internal symmetries [16]. Leaving these decompositions aside, we are left with the *Pauli gradings* which decompose  $M_N(\mathbb{C})$  in direct sums of  $N^2$  one-dimensional subspaces. For these Pauli decompositions with given  $N$  one should realize that  $M_N(\mathbb{C})$  encompasses all operators of any quantum system with  $N$ -dimensional Hilbert space.

In this way we have got a new view on the relation between the general mathematical formalism and physical realizations of finite quantum systems. The apparent contradiction that  $M_N(\mathbb{C})$  represents any  $N$ -dimensional quantum system and at the same time there is a multitude of inequivalent quantum kinematics for given  $N$ , is simply resolved: from the physical point of view the same algebra  $M_N(\mathbb{C})$  is the operator algebra not only for a single  $N$ -level system, but also for all other members of the set of inequivalent quantum kinematics for this  $N$ .<sup>2</sup> They just correspond to different physical realizations of composite quantum systems. Of course, each such system has its preferred quantum operators (3).

## 6. Conclusions

In our study unexpectedly rich structures were obtained from number-theoretic properties connected with prime decompositions of numbers  $N$ . Our studies may also shed light on a long-standing unsolved problem related to complementary observables in finite-dimensional quantum mechanics. There the notion of complementarity of observables  $A, B$  with non-degenerate eigenvalues is equivalently reformulated in terms of their eigenvectors forming mutually unbiased bases: if the system is prepared in any eigenstate of  $A$ , then the transition probabilities to all eigenstates of the complementary observable  $B$  are the same (equal to  $1/N$ ). It is known that the maximal set of mutually unbiased bases contains at most  $N + 1$  bases and that this maximal number is attained for  $N$  prime or a power of a prime. For composite numbers  $N$  the maximal number of mutually unbiased bases is unknown. The needed bases can be constructed as common eigenvectors of suitable subsets formed by commuting Pauli operators. In some cases the decomposition of the set of all Pauli operators into subsets of commuting operators can be reflected in the finite geometry [17]. The study of mutually unbiased bases may have implications for quantum information and communication science, since mutually unbiased bases are indispensable ingredients of quantum key distribution protocols [5].

There exist numerous studies of various aspects of the finite Weyl-Heisenberg group over finite fields for Hilbert spaces of prime or prime power dimensions, e.g. [18, 19, 20, 21, 22]. However, our main motivation to study finite quantum kinematics not in prime or prime power dimensions but for arbitrary dimensions stems from our previous research where we obtained results valid for arbitrary dimensions [10, 11, 23]. There are also other papers supporting our motivation [24, 25, 26].

Finite quantum systems are basic constituents of quantum information processing. Except single 2- and  $d$ -level quantum systems — qubits and qudits — many authors pay their attention to finitely composed systems, where the basic operators are formed by tensor products. Multiple qubits with Hilbert spaces  $\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$  are routinely employed in quantum algorithms, while multiple qudits with Hilbert spaces  $\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d$  may be interesting for quantum error-correction codes and for multipartite communication. What about general systems? They are sometimes called ‘mixtures of multiple qudits’. We have shown the general classification of finite quantum kinematics and their physical interpretation: for given dimension there exists a broad variety of finitely composed distinct quantum kinematics involving inequivalent tensorial factorizations into elementary constituents. It is remarkable that in spite of the boom of quantum information

<sup>2</sup> Note that from the mathematical point of view  $M_{N_1}(\mathbb{C}) \otimes \dots \otimes M_{N_f}(\mathbb{C})$  is isomorphic with  $M_N(\mathbb{C})$ .

science the community of quantum information and communication has not noticed this general classification up to now. Only the visible aspect of the finite-dimensional operator formalism is perceived in terms of examples constructed from generalized Pauli matrices in an *ad hoc* manner. In this way the underlying general structure remains hidden. In our approach we explicitly exhibit the elementary quantum degrees of freedom, since their relevance has not been emphasized in most of the literature on finite quantum systems.

It is clear that automorphisms or symmetries of the finite Weyl-Heisenberg group play very important role in the investigation of Lie algebras on the one hand [10, 27, 28] and of quantum mechanics in finite dimensions on the other [29, 21]. These symmetries find proper expression in the notion of the quotient group of a certain normalizer [7]. The groups of symmetries given by inner automorphisms were described in [10] as isomorphic to  $SL(2, \mathbb{Z}_N)$  for arbitrary  $N \in \mathbb{N}$  and as  $Sp(4, \mathbb{Z}_p)$  for  $n = p^2$ ,  $p$  prime in [28] (see also [30]). For the complete description of these symmetries — in quantum information conventionally called Clifford groups [31] — we refer to our papers [1, 2].

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