Generalizations of the Ermakov system through the Quantum Arnold Transformation

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Abstract. An Ermakov system consists of a pair of coupled non-linear differential equations which share a joint constant of motion named Ermakov invariant. One of those equations, non-linear, is frequently referred to as the Ermakov-Pinney equation; the other equation may be thought of as describing a dynamical system: a harmonic oscillator with time-dependent frequency. In this paper, we revise the Quantum Arnold Transformation, a unitary operator mapping the solutions of the Schrödinger equation for time-dependent (even damped) harmonic oscillators, described by the Generalized Caldirola-Kanai equation, into solutions for the free particle. With this tool, we elucidate the existence of Ermakov-type invariants in classically linear systems at the classical and quantum levels. We also provide more general Ermakov-type systems and the corresponding invariants, together with a physical interpretation.

1. Introduction

Since 2011, the authors have participated in a series of publications in which the Quantum Arnold Transformation (QAT) was presented [1], developed [2, 3] and applied [4, 5, 6] to several purposes. In particular, in [3], an explicit relation with the Ermakov-Pinney equation and the corresponding Ermakov system was provided. The Ermakov-Pinney equation is related to the Ermakov invariant [7, 8, 9] and appears in many branches of physics, such as Cosmology [10], BEC [11], etc. The Ermakov system appears in BEC [12, 13] and what is known as Kepler-Ermakov systems [14].

The QAT is a unitary map that relates the Hilbert space of solutions of the time-dependent Schrödinger equation for a Generalized Caldirola-Kanai oscillator (a quantum version of a classical system whose equation of motion is a Linear Second-Order Differential Equation, LSODE) into the corresponding Schrödinger equation for the free particle.

The feeling of the authors is that the QAT provides a framework in which many known relations, in particular for the Ermakov System, are better understood and some new ones are found. It is the purpose of this paper to deepen in the relation between the QAT and the Ermakov system, and to present examples that may help the reader to get a better grasp of the approach to the Ermakov system through the QAT. Also, new generalizations of the Ermakov system are possible within the QAT framework in a quite easy way.

The paper is organized as follows: in Section 2 we describe the transformation due to Arnold [15] and set some of the notation. The corresponding quantum version, the QAT, is revised in Section 3. We use the QAT in Section 4 to construct the quantum Arnold-Ermakov-Pinney transformation, which maps solutions of the time-dependent Schrödinger equation for a LSODE-system into solutions of the time-dependent Schrödinger equation for any other LSODE-system; a generalization of the Ermakov-Pinney equation is also provided. Finally, Section 5 is devoted to provide several examples in order to clarify the ideas previously exposed.

2. The Classical Arnold Transformation

The context of the classical Arnold transformation [15] is that of Lie point symmetries of ordinary differential equations. A Lie point symmetry of an ordinary differential equation (ODE) is a coordinate transformation that sends solutions into solutions. The problem of determining the Lie point symmetries of an ODE is rather old, and S. Lie gave the main results at the end of the nineteenth century [16]. One of these results was that a second-order differential equation (SODE) $y'' = F(x, y, y')$ has the maximal number of Lie point symmetries ($sl(3, \mathbb{R})$) if it can be transformed into the free equation by a point transformation:

$$y'' = F(x, y, y') \quad \overset{\substack{\tilde{x} = \tilde{x}(x,y) \\ \tilde{y} = \tilde{y}(x,y)}}{\Longrightarrow} \quad \tilde{y}'' = 0\,. \tag{1}$$

This linearization is possible if the ODE is of the form:

$$y'' = E_3(x, y)(y')^3 + E_2(x, y)(y')^2 + E_1(x, y)y' + E_0(x, y)\,, \tag{2}$$

with $E_i(x, y)$ satisfying some *integrability conditions* (see, for instance, [16, 17, 18]).

There is a nice geometric interpretation of this condition in terms of projective geometry. The non-linear SODE (2) is obtained by projection from the geodesic equations in a two-dimensional Riemannian manifold. The coefficients $E_i(x, y)$ are in one-to-one correspondence with Thomas projective parameters Π, and the integrability conditions that they satisfy are the conditions for the Riemann tensor to be zero (see [17, 18]).

V.I. Arnold named this process *rectification* or *straightening* of the trajectories, and studied the case of linear SODE (LSODE), giving explicitly the point transformation for this case [15]. Specifically, given a general Linear Second-Order Differential Equation (LSODE):

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda\,, \tag{3}$$

where f, ω and Λ are functions of t, the Classical Arnold Transformation (CAT) is a point transformation that is a local (in time) diffeomorphism:

$$\begin{array}{rccc} A: & \mathbb{R}\times T & \rightarrow & \mathbb{R}\times \mathcal{T} \\ & (x,t) & \mapsto & (\kappa,\tau) \end{array} \;:\; \left\{ \begin{array}{lcl} \tau & = \frac{u_1(t)}{u_2(t)} = \int_{t_0}^{t} \frac{W(t')}{u_2(t')^2}dt' \\ \kappa & = \frac{x - u_p(t)}{u_2(t)} \end{array} \right. \,, \tag{4}$$

where T and $\mathcal{T}$ are, in general, open intervals containing t_0 and 0, respectively, u_1 and u_2 are independent solutions of the homogeneous LSODE satisfying the canonicity conditions:

$$u_1(t_0) = 0 = u_2'(t_0)\,, \qquad u_1'(t_0) = 1 = u_2(t_0)\,, \tag{5}$$

u_p is a particular solution of the inhomogeneous LSODE satisfying $u_p(t_0) = u_p'(t_0) = 0$, and $W(t) = \dot{u}_1 u_2 - u_1 \dot{u}_2 = e^{-f}$ is the Wronskian of the two solutions. Here t_0 is an arbitrary time, conveniently chosen to be $t_0 = 0$ (see [1] for details).

The CAT transforms the original LSODE (3) into that of the free particle, up to a factor:

$$\ddot{x} + \dot{f}\dot{x} + \omega^2 x = \Lambda \quad \overset{A}{\longrightarrow} \quad \frac{W}{u_2^3}\,\ddot{\kappa} = 0\,. \tag{6}$$

The presence of this factor implies that patches of trajectories of (3) are transformed into patches of straight (free) trajectories. In fact, an arbitrary trajectory solution of (3) can be written as $x(t) = Au_1(t) + Bu_2(t) + u_p(t)$, and the CAT sends it to $\kappa(\tau) = A\tau + B$. While t varies in the interval T defined by two consecutive zeros of $u_2(t)$ (containing t_0), τ varies in the range of the map defined by $\frac{u_1(t)}{u_2(t)}$. In the case in which $u_2(t)$ has one zero, T is (left- or right-) unbounded, and, if it has no zeros, T is $\mathbb{R}$.

Even though the CAT is a local (in time) diffeomorphism, it can be defined for an arbitrary time t_0. Thus different CATs can be defined for different times t_0 and cover in this way a complete trajectory of (3). We shall show with the example of the harmonic oscillator how this can be done.

2.1. The example of the harmonic oscillator

For this case, and considering $\Lambda = 0$, the two solutions are:

$$u_1(t) = \frac{1}{\omega}\sin(\omega t)\,, \qquad u_2(t) = \cos(\omega t)\,. \tag{7}$$

The open interval T defined by two consecutive zeros of $u_2(t)$, and containing $t_0 = 0$, is $(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega})$, and the CAT A and its inverse A^{-1} are then written as:

$$A: \quad \kappa \;= \tfrac{x}{u_2(t)} = \tfrac{x}{\cos(\omega t)}\,, \quad \tau \;= \tfrac{u_1(t)}{u_2(t)} = \tfrac{1}{\omega}\tan(\omega t)\,, \tag{8}$$

$$A^{-1}: \quad \begin{aligned} x &= \cos(\arctan(\omega\tau))\kappa\,, \qquad t = \tfrac{1}{\omega}\arctan(\omega\tau)\,. \\ &= \tfrac{\kappa}{\sqrt{1+\omega^2\tau^2}} \end{aligned} \tag{9}$$

In this case $\tau \in \mathbb{R}$. Pictorially, the CAT for the HO can be represented as in Figure 1, where velocities have also been included in the graphic for clarity. Here A maps the solid part of the helix (half a period of a harmonic oscillator trajectory) into the whole line (a free particle trajectory). The horizontal plane represents the space of all possible initial conditions at $t = 0 = \tau$. Note that both trajectories are tangent when projected onto this plane, due to the conditions (5). See [4] for more details in this case. For the CAT to map other patches of the HO trajectories into the free particle trajectories, different branches of the arctan function in the inverse CAT (9) should be used (and a different $t_0 \neq 0$ for the CAT). For each integer k, let us take $T_k = ((k - \frac{1}{2})\frac{\pi}{\omega}, (k + \frac{1}{2})\frac{\pi}{\omega})$ and $t_k = k\frac{\pi}{\omega}$. The solutions verifying conditions (5) at t_k are $u_i^{(k)}(t) = (-1)^k u_i(t) = u_i(t - t_k)\,,\ i = 1, 2$. Define a pair of CAT and inverse CAT from $\mathbb{R} \times T_k$ into $\mathbb{R}^2$ of the form: $A_{(k)}(x, t) = (\frac{x}{u_2^{(k)}(t)}, \frac{u_1^{(k)}(t)}{u_2^{(k)}(t)}) = (\kappa, \tau)$ and $A^{-1}_{(k)}(\kappa, \tau) = (x, t)$, where the k-th branch of the arctan function has been used in $A^{-1}_{(k)}$. An *unfolded* version of the CAT, $\tilde{A}$, can be built by joining all the patches $A_{(k)}$, defining an application that maps a complete trajectory $x(t)$ of the harmonic oscillator into a trajectory $\kappa(\tau)$ of the free particle. $\tilde{A}$ is periodic on t with period $\frac{\pi}{\omega}$, although discontinuous.

Other simple examples where this construction can be done are the damped particle and the damped harmonic oscillator, see [2] for details.

It should be stressed that the CAT can be used to find invariant expressions for the harmonic oscillator performing the transformation A on invariant expressions for the free particle (the reversed procedure can be realized using A^{-1}). We call this simple strategy "*importing*" conserved quantities from the free particle to the harmonic oscillator, a procedure that can be employed for different pairs of systems.

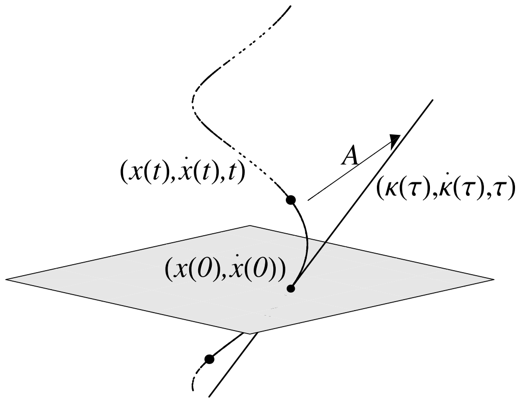

Figure 1. Depiction of the CAT for the harmonic oscillator (adapted from [4]).

3. The Quantum Arnold Transformation

An arbitrary LSODE system (3) can be derived from the Lagrangian (we take $\Lambda = 0$ for simplicity but the whole formalism can be developed with $\Lambda \neq 0$, see [1]):

$$L = \frac{1}{2} m e^f \left(\dot{x}^2 - \omega^2 x^2\right) , \tag{10}$$

and from this the Hamiltonian

$$H = \frac{p^2}{2m} e^{-f} + \frac{1}{2} m\omega^2 x^2 e^f \tag{11}$$

is derived, which is known as the Generalized Caldirola-Kanai (GCK) Hamiltonian for a damped oscillator (see [1] and references therein). The case in which $\dot{f} = \gamma$ and ω are constants corresponds to the original Caldirola-Kanai Hamiltonian for a damped harmonic oscillator [19, 20], and whose corresponding Lagrangian was given for the first time by Bateman [21]. Canonical quantization of the GCK Hamiltonian leads to the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \phi}{\partial t} = \hat{H}\phi = -\frac{\hbar^2}{2m} e^{-f} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 e^f \phi \,. \tag{12}$$

The CAT A is a local (in time) diffeomorphism between the space of solutions of the LSODE system (3) and the space of solutions of the free particle. It is possible to extend it to a unitary transformation $\hat{A}$, the Quantum Arnold Transformation (QAT), between the Hilbert space of solutions $\phi(x,t)$ of the time-dependent Schrödinger equation for the GCK oscillator (12) at time t, $\mathcal{H}_t$, into the Hilbert space of solutions $\varphi(\kappa,\tau)$ of the time-dependent Schrödinger equation for the Galilean free particle

$$i\hbar \frac{\partial \varphi}{\partial \tau} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial \kappa^2} \,, \tag{13}$$

at time τ, $\mathcal{H}^G_\tau$. The desired extension is given by:

$$\begin{aligned} \hat{A}: \quad \mathcal{H}_t \quad &\longrightarrow \quad \mathcal{H}^G_\tau \\ \phi(x,t) &\longmapsto \varphi(\kappa,\tau) = \hat{A}\left(\phi(x,t)\right) \\ &\qquad = A^*\left(\sqrt{u_2(t)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t)}\frac{\dot{u}_2(t)}{u_2(t)}x^2} \phi(x,t)\right) . \end{aligned} \tag{14}$$

Here A^* is the pullback of the CAT A, acting on functions (i.e. $A^*(f(x,t)) = f(A^{-1}(\kappa,\tau))$). The QAT can be diagrammatically represented as:

$$\begin{tikzcd} \mathcal{H}_\tau^G & \mathcal{H}_t \arrow[l, "\hat{A}"'] \\ \mathcal{H}_0^G \equiv \mathcal{H} \arrow[u, "\hat{U}_G(\tau)"] \arrow[r, "\hat{1}"'] & \mathcal{H} \equiv \mathcal{H}_0 \arrow[u, "\hat{U}(t)"'] \end{tikzcd} \tag{15}$$

where $\mathcal{H}_0 \equiv \mathcal{H}_0^G \equiv \mathcal{H}$ is the common Hilbert space of solutions of the Schrödinger equation for both systems at $t = \tau = 0$ (we take, for simplicity, $t_0 = 0$, as before), $U(t)$ is the unitary time-evolution operator for the GCK oscillator and $\hat{U}_G(\tau)$ is the corresponding one for the Galilean free particle. The map at the bottom of the diagram is the identity due to conditions (5); otherwise a non-trivial unitary transformation appears (see [1]).

From the commutative diagram, it is clear that $\hat{A}$ is unitary given the unitarity of the evolution operators. However, it can also be checked explicitly that the scalar product of two states in $\mathcal{H}_\tau^G$ at a given time τ is the same than that of the transformed states by $\hat{A}$ in $\mathcal{H}_t$ at the corresponding time t:

$$\begin{aligned} \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_\tau^G} &= \int_{-\infty}^{+\infty} \mathrm{d}\kappa\, \varphi_1(\kappa,\tau)^* \varphi_2(\kappa,\tau) \\ &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{u_2(t)} \left(\sqrt{u_2(t)}\, e^{\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t)}\frac{\dot{u}_2(t)}{u_2(t)}x^2} \phi_1(x,t)^* \right) \\ &\qquad\qquad \times \left(\sqrt{u_2(t)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t)}\frac{\dot{u}_2(t)}{u_2(t)}x^2} \phi_2(x,t) \right) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}x\, \phi_1(x,t)^* \phi_2(x,t) = \langle \phi_1, \phi_2 \rangle_{\mathcal{H}_t}\,, \end{aligned} \tag{16}$$

where τ, κ, $\mathrm{d}\kappa$ and the integration limits have been transformed according to the CAT.

The QAT inherits from the CAT the local character in time, in the sense that it is valid only for $t \in T$ and $\tau \in \mathcal{T}$, although it can be defined for an arbitrary initial time t_0. To extend the QAT beyond T, we can proceed as in the classical case for the harmonic oscillator, considering the different branches of the inverse function of $\tau(t)$, defining an *unfolded* QAT, $\hat{\hat{A}}$.

It should be stressed that, if in the different branches of the *unfolded* CAT proper solutions verifying (5) are not used, changes in signs of the solutions can appear, which result in changes in phases in the different branches of the *unfolded* QAT. This phenomenon is related to the Maslov correction (see for instance [22]). In fact, it can be checked that, for the case of the harmonic oscillator previously considered, $\hat{A}_{(k)}(\phi(x,t)) = e^{ik\frac{\pi}{2}}\hat{A}(\phi((-1)^k x, t))$.

From the commutative diagram (15) and from (16) it is clear that the QAT is a unitary operator, and this has interesting and far-reaching consequences. Mimicking the process of "*importing*" that we mentioned for the CAT between the free particle and the harmonic oscillator at the end of Subsection 2.1, the QAT can be used to find invariant expressions for operators as well as symmetry generators from one system to the other, importing wave functions, scalar product, computing the time evolution operator, etc. One might say that the quantum free particle is somehow "linked" to any LSODE-type quantum system through the QAT. This is just a consequence of the fact that all those classically linear systems share a common symmetry group at the quantum level: the so called Schr̆odinger group [23].

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4. The Arnold-Ermakov-Pinney transformation

Having in mind the diagram (15), one might wonder what happens when two different LSODE-systems are related by QATs with the free-particle system as a middleman, that is, when a QAT and an inverse QAT are composed, as follows:

$$
\begin{array}{ccccc}
\mathcal{H}_{t_1}^{(1)} & \xrightarrow{\hat{A}_1} & \mathcal{H}_{\tau}^{G} & \xleftarrow{\hat{A}_2} & \mathcal{H}_{t_2}^{(2)} \\
\hat{U}^{(1)}(t_1)\Big\uparrow & & \hat{U}_G(\tau)\Big\uparrow & & \Big\uparrow\hat{U}^{(2)}(t_2) \\
\mathcal{H}\equiv\mathcal{H}_0^{(1)} & \xrightarrow[\hat{1}]{} & \mathcal{H}_0^{G}\equiv\mathcal{H} & \xrightarrow[\hat{1}]{} & \mathcal{H}\equiv\mathcal{H}_0^{(2)}
\end{array}
\tag{17}
$$

That was shown in [3]: for the underlying classical construction, let A_1 and A_2 denote the CATs relating the LSODE-system 1 and LSODE-system 2 to the free particle, respectively, then $E = A_1^{-1}A_2$ relates LSODE-system 2 to LSODE-system 1. E can be written as:

$$
\begin{aligned}
E : &\mathbb{R}\times T_2 \to \mathbb{R}\times T_1 \\
&(x_2,t_2)\mapsto (x_1,t_1) = E(x_2,t_2)\,.
\end{aligned}
\tag{18}
$$

The explicit form of the transformation can be easily computed by composing the two CATs, resulting in:

$$
x_1 = \frac{x_2}{b(t_2)} \qquad W_1(t_1)dt_1 = \frac{W_2(t_2)}{b(t_2)^2}dt_2\,,
\tag{19}
$$

where $b(t_2) = \frac{u_2^{(2)}(t_2)}{u_2^{(1)}(t_1)}$ satisfies the non-linear SODE:

$$
\ddot{b} + \dot{f}_2\dot{b} + \omega_2^2 b = \frac{W_2^2}{W_1^2}\frac{1}{b^3}\left[\omega_1^2 + \dot{f}_1\frac{\dot{u}_2^{(1)}}{u_2^{(1)}}(1-b^2\frac{W_1}{W_2})\right]\,,
\tag{20}
$$

and where $u_i^{(j)}$ refers to the i-th particular solution for system j; W_j, $\dot{f}_j$ and ω_j stand for the Wronskian and the LSODE coefficients for system j; and the dot means derivation with respect to the corresponding time variable.

Equation (20) constitutes a *generalization* of the well-known *Ermakov-Pinney equation*. That equation, together with the LSODE of system 2, is a *generalized Ermakov pair*. Also, any (quadratic) conserved quantity corresponding to the Schrödinger group of symmetries, which is shared by the two LSODE-systems, constitutes a *generalized Ermakov invariant*. Below, we consider specific examples in which the usual Ermakov invariant is constructed.

An important point needs to be made: equation (20) actually *defines* a generalized Arnold transformation, to be named (classical) *Arnold-Ermakov-Pinney transformation*, which transforms solutions of the LSODE 1 into solutions of the LSODE 2. However, it *does not describe* any of the two LSODE-systems themselves. Therefore, the Lie point symmetries of (20) need not to be those of the LSODEs in the general case.

The quantum version of the Arnold-Ermakov-Pinney transformation, $\hat{E}$, can be obtained computing the composition of a QAT and an inverse QAT with the diagram (17) in mind, to give:

$$
\begin{aligned}
\hat{E}:\ \ \mathcal{H}_{t_2}^{(2)} &\longrightarrow \ \ \mathcal{H}_{t_1}^{(1)} \\
\phi(x_2,t_2) &\longmapsto \ \varphi(x_1,t_1) = \hat{E}\left(\phi(x_2,t_2)\right) \\
&\qquad = E^*\left(\sqrt{b(t_2)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W_2(t_2)}\frac{\dot{b}(t_2)}{b(t_2)}x_2^2}\phi(x_2,t_2)\right)\,.
\end{aligned}
\tag{21}
$$

The Quantum Arnold-Ermakov-Pinney transformation (QAEPT) is a unitary map importing solutions of a GCK Schrödinger equation from solutions of a different, auxiliary GCK Schrödinger equation which, in particular, might be the one corresponding to a harmonic oscillator, as in the next Section.

5. Examples

5.1. Ermakov System and interpretation of the Ermakov Invariant

Consider the particular case where LSODE-system 1 is a harmonic oscillator ($\omega_1(t_1) \equiv \omega_0$ and $\dot{f}_1 = 0$), which can be described by the Hamiltonian

$$H_{HO} = \frac{p_1^2}{2m} + \frac{1}{2} m\omega_0^2 x_1^2 \,, \tag{22}$$

and LSODE-system 2 is a time-dependent harmonic oscillator with frequency $\omega_2(t_2) \equiv \omega(t)$ and $\dot{f}_2 = 0$, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega(t)^2 x^2 \,. \tag{23}$$

Then, expression (20) simplifies to:

$$\ddot{b} + \omega(t)^2 b = \frac{\omega_0^2}{b^3} \,. \tag{24}$$

That is just the Ermakov-Pinney (also known as Milne-Pinney) equation [7, 24, 25]. Obviously, for $\omega_0 = 0$ the Arnold-Ermakov-Pinney transformation reduces to the ordinary CAT, i.e. $E = A$.

Now, note that LSODE 1 Hamiltonian, H_{HO}, is conserved, and that it is so on both sides of the transformation E, given by (see (19)):

$$x_1 = \frac{x}{b} \,, \quad dt_1 = \frac{1}{b^2} dt \,. \tag{25}$$

Computing the momentum $p_1 = m\dot{x}_1 = m\frac{dx_1}{dt_1} = m\frac{dt}{dt_1}\frac{d}{dt}(\frac{x}{b}) = m(\dot{x}b - \dot{b}x)$, we can write H_{HO} in variables corresponding to system 2:

$$H_{HO} = \frac{1}{2m}(pb - m\dot{b}x)^2 + \frac{1}{2} m\omega_0^2 (\frac{x}{b})^2 \equiv I \,. \tag{26}$$

That is easily recognized as the usual Ermakov invariant I. Thus, we have found a way to characterize it through the CAT (or, more precisely, the CAEPT): I corresponds to the conserved quantity H_{HO} imported from the simple harmonic oscillator, which is used as an auxiliary system. Because the auxiliary system is arbitrary, I is conserved for any ω_0, provided (24) is satisfied.

The reader might wonder about the comparison of b (adimensional), satisfying (24) and providing the invariant (26), and the function α satisfying the usual Ermakov-Pinney equation $\ddot{\alpha} + \omega(t)^2\alpha = \frac{1}{\alpha^3}$, which has the dimensions of the square root of time (see e.g. [26]). The relation of b and α is simply $b = \sqrt{\omega_0}\alpha$. In Quantum Mechanics, α is sometimes interpreted physically as the width of a wave packet, up to a factor $\sqrt{2m/\hbar}$ [26]. b may also be interpreted as the width of a wave packet for the quantum time-dependent harmonic oscillator over the natural length $\sqrt{2m\omega_0/\hbar}$ of the harmonic oscillator chosen as LSODE-system 1.

Using the explicit form of the inverse of (21) $\hat{E}^{-1}$ in this case, it is straightforward to arrive at solutions $\phi(x,t)$ of the Schrödinger equation of the time-dependent harmonic oscillator in terms of solutions of the Schrödinger equation for the simple harmonic oscillator $\varphi(x_1, t_1)$:

$$\phi(x,t) = \frac{1}{\sqrt{b}} e^{\frac{i}{2}\frac{m}{\hbar}\frac{\dot{b}}{b}x^2} \varphi\Big(\frac{x}{b}, \int \frac{1}{b^2} dt\Big) \,, \tag{27}$$

where b is any solution of (24). Although the explicit computations are left to the reader, note that, if $\varphi(x_1, t_1)$ is chosen to be, for instance, an eigenfunction of the quantum operator corresponding to (22), $\hat{H}_{HO}$, then the transformed wavefunction $\phi(x,t)$ is an eigenfunction of the quantum operator $\hat{I}$ corresponding to the invariant (26) (the explicit form of such operators is easily obtained from their classical counterpart by the canonical quantization prescription). That shows that $\hat{I}$ has discrete spectrum.

Moreover, $\hat{I}$ belongs to the $sl(2,\mathbb{R})$ subalgebra of the Schrödinger algebra of point variational symmetries of the system described by (23) (specifically, the one generating the compact subgroup). Even beyond that, the process can be repeated for any other operator representing an invariant in the simple harmonic oscillator (LSODE 1), showing the usefulness of the QAEPT to perform quick computations (see also Subsec. 5.2 for the case of creation and annihilation operators).

In the case here considered, the transformation $\hat{E}^{-1}$ together with (25) turns out to be very similar to the one used in BEC, known as scaling transformation, to transform the time-dependent potential (oscillator traps with time-dependent frequencies) into a time-independent harmonic oscillator potential [12, 13]. Also, in that case (i.e. for $\dot{f}_2 = 0$, $W_2 = 1$, $\dot{f}_1 = 0$, $W_1 = 1$) equation (21) reduces to the transformation given by Hartley and Ray [27] (this was already given by Lewis and Riesenfeld in [9]). However, the Quantum Arnold-Ermakov-Pinney transformation allows to choose in a suitable way the auxiliary system from which the solutions may be imported.

The Ermakov-Pinney equation entails a kind of nonlinear superposition principle, in the sense that its solutions can be written in terms of the solutions $y_1(t), y_2(t)$ of the corresponding linear equation (with $\omega_0 = 0$):

$$b(t)^2 = c_1 y_1(t)^2 + c_2 y_2(t)^2 + 2c_3 y_1(t) y_2(t)\,, \quad c_1 c_2 - c_3^2 = \omega_0^2\,. \tag{28}$$

The other way round, the general solution $y(t)$ of the linear equation can be written in terms of a particular solution $\rho(t)$ of the Ermakov-Pinney equation (24) as:

$$y(t) = c_1 \rho(t) \cos(\omega_0\, \theta(t) + c_2)\,, \tag{29}$$

where c_1, c_2 are arbitrary constants and $\theta(t) = \int^t \rho^{-2} dt'$. Note that this equation is just (25) for $t_1 = \theta(t)$, $\rho = b$, $x = y(t)$ and $x_1 = y(t)/b(t) = c_1 \cos(\omega_0 t_1 + c_2)$. As a result, the general solution of (24) can be determined from a particular solution $\rho(t)$ using (29) and (28). A similar situation holds for more general versions of the Ermakov-Pinney equation.

The process followed to find the Ermakov system is fairly easy to reproduce for more involved systems playing the role of LSODE-system 2, and the interpretation is exactly the same. For instance, consider a damped harmonic oscillator with time-varying frequency and damping (now we omit the time dependence) characterized by a GCK Hamiltonian:

$$H = \frac{p^2}{2m} e^{-f} + \frac{1}{2} m \omega^2 x^2 e^f\,. \tag{30}$$

As before, the auxiliary system is a simple harmonic oscillator. We arrive at a generalization of the Ermakov-Pinney equation:

$$\ddot{b} + \dot{f}\dot{b} + \omega^2 b = \frac{e^{-2f}}{b^3} \omega_0^2\,. \tag{31}$$

The AEP transformation is given by: $x_1 = \frac{x}{b}$, $dt_1 = \frac{e^{-f}}{b^2} dt$, so that we can compute $p_1 = m\dot{x}_1 = m\frac{dx_1}{dt_1} = m\frac{dt}{dt_1}\frac{d}{dt}(\frac{x}{b}) = me^f(\dot{x}b - \dot{b}x) = pb - m\dot{b}xe^f$ (in the last step we have taken into account that the canonical momentum for the GCK oscillator is $p = m\dot{x}e^{f}$). We then write H_{HO} in GCK variables:

$$H_{HO} = \frac{1}{2m}(pb - m\dot{b}xe^{f})^2 + \frac{1}{2}m\omega_0^2(\frac{x}{b})^2 \equiv I\,, \tag{32}$$

which is an invariant for the GCK oscillator. This way, we have "imported" an invariant from an auxiliary system through the Arnold-Ermakov-Pinney transformation. Note that in [28] the authors also arrived at the result $I = H_{HO}$ using canonical transformations to link both systems (the reader may also note the resemblance of the canonical transformation there employed and the Arnold-Ermakov-Pinney transformation). The explicit construction of solutions of the Schrödinger equation proceeds as before and similar observations can be made.

5.2. Creation-Annihilation operators

The "importing" process discussed before can be repeated for any conserved quantity of the harmonic oscillator when it is used as an auxiliary system. Such an strategy may be quite relevant when going to the quantum theory. For instance, the classical version of the (conserved) annihilation (or annihilation) operators of the simple harmonic oscillator:

$$a = \left(\sqrt{\frac{m\omega_0}{2\hbar}}x_1 + \frac{i}{\sqrt{2\hbar m\omega_0}}p_1\right)e^{i\omega_0 t_1}\,, \tag{33}$$

can be used to find the corresponding conserved function for the GKC oscillator seen in the previous example. Performing the same transformation, we easily arrive at:

$$a = \left(\sqrt{\frac{m\omega_0}{2\hbar}}\frac{x}{b} + \frac{i}{\sqrt{2\hbar m\omega_0}}(pb - m\dot{b}xe^{f})\right)e^{i\omega_0\int_0^t \frac{e^{-f}}{b^2}dt}\,. \tag{34}$$

We have used that $p = m\dot{x}e^{f}$ once more. A similar expression for a has been obtained recently in [26].

The QAEPT works in a similar way and the quantum version of the creation and annihilation operators are found straightforwardly:

$$\begin{aligned}
\hat{a}\phi(x,t) &= \hat{E}^{-1}\hat{a}_1\hat{E}\phi(x,t) = \hat{E}^{-1}\hat{a}_1 E^*\left(\sqrt{b}\,e^{-\frac{i}{2}\frac{m}{\hbar}e^{f}\frac{\dot{b}}{b}x^2}\phi(x,t)\right) = \\
&= \frac{1}{\sqrt{b}}e^{+\frac{i}{2}\frac{m}{\hbar}e^{f}\frac{\dot{b}}{b}x^2}E^{*-1}\left(\left(\sqrt{\frac{m\omega_0}{2\hbar}}x_1 + \sqrt{\frac{\hbar}{2m\omega_0}}\frac{\partial}{\partial x_1}\right)e^{i\omega_0 t_1}E^*\left(\sqrt{b}\,e^{-\frac{i}{2}\frac{m}{\hbar}e^{f}\frac{\dot{b}}{b}x^2}\phi(x,t)\right)\right) = \\
&= \frac{1}{\sqrt{b}}e^{+\frac{i}{2}\frac{m}{\hbar}e^{f}\frac{\dot{b}}{b}x^2}\left(\left(\sqrt{\frac{m\omega_0}{2\hbar}}\frac{x}{b} + \sqrt{\frac{\hbar}{2m\omega_0}}b\frac{\partial}{\partial x}\right)e^{i\omega_0\int_0^t\frac{e^{-f}}{b^2}dt}\left(\sqrt{b}\,e^{-\frac{i}{2}\frac{m}{\hbar}e^{f}\frac{\dot{b}}{b}x^2}\phi(x,t)\right)\right) = \\
&= \left(\sqrt{\frac{m\omega_0}{2\hbar}}\frac{x}{b} + \frac{i}{\sqrt{2\hbar m\omega_0}}(-i\hbar b\frac{\partial}{\partial x} - m\dot{b}xe^{f})\right)e^{i\omega_0\int_0^t\frac{e^{-f}}{b^2}dt}\phi(x,t)\,.
\end{aligned} \tag{35}$$

We would like to remark that there are many possible "importable" constructions for which analytical expressions can be found, such as the displacement operator $\hat{D}(a) = e^{a\hat{a}^\dagger - a^*\hat{a}}$, the radial squeezing operator $\hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi(\hat{a}^\dagger)^2)}$, etc. The fundamental reason for the mere existence of these constructions is again the fact that classically linear systems share the same set of symmetries as the harmonic oscillator (and the free particle), which can be shown by the QAT. Some of these constructions can still be possible in classically non-linear systems, such as those presenting a potential of the form $\frac{1}{x^2}$ [28] (in our setup, that would amount to replace

LSODE 2 by the non-linear equation with a $\frac{1}{x^3}$ term). However, some of the symmetries may be lost in such systems with respect to the linear ones, so that linear $\hat{a}$ and $\hat{a}^\dagger$ may not exist as first-order operators, for instance, although they are still present as pseudo-differential operators [29].

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References

[1] Aldaya V, Cossío F, Guerrero J and López-Ruiz F F 2011 *J. Phys. A* **44** 065302
[2] Guerrero J, Aldaya V, López-Ruiz F F and Cossío F 2012 *Int. J. Geom. Meth. Mod. Phys.* **9** 1260011
[3] Guerrero J and López-Ruiz F F 2013, *Phys. Scr.* **87**, 038105
[4] Guerrero J, López-Ruiz F F, Aldaya V and Cossío F 2011 *J. Phys. A* **44** 445307
[5] López-Ruiz F F, Guerrero J and Aldaya V 2011 *J. Russ. Laser Res.* **32** 372
[6] Guerrero J, López-Ruiz F F, Aldaya V and Cossío F 2012, *J. Phys. A* **45**, 475303
[7] Ermakov V P 1880 *Univ. Izv. Kiev* **20** 1
[8] Ermakov V P 2008 *Appl. Anal. Discrete Math.* **2**, 123
[9] Lewis H R Jr. and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[10] Hawkins R M and Lidsey J E 2002 *Phys. Rev. D* **66** 023523
[11] Schuch D and Kaushal R S 2011 *J. Phys.: Conf. Series* **306** 012032
[12] Kagan Y, Surkov E L and Shlyapnikov G V 1996 *Phys. Rev. A* **54** R1753
[13] Castin Y and Dum R 1996 *Phys. Rev. Lett.* **77** 5315
[14] Leach P G L and Karasu A (Kalkanli) 2004 *J. Nonl. Math. Phys.* **11** 269
[15] Arnold V I, *Supplementary Chapters to the Theory of Ordinary Differential Equations* (Nauka, Moscow, 1978); English transl., *Geometrical Methods in the Theory of Ordinary Differential Equations*, (Springer-Verlag, New York-Berlin, 1983)
[16] Lie S 1883 *Arch. Math.* **VIII,IX** 187
[17] Mahomed F M, Qadir A 2009 *J. Nonlinear Math. Phys.* **16** 283
[18] Aminova A V and Aminov N A M 2010 *Sb. Math.* **201** 631
[19] Caldirola P 1941 *Nuovo Cimento* **18** 393
[20] Kanai E 1948 *Prog. Theor. Phys.* **3** 440
[21] Bateman H 1931 *Phys. Rev.* **38** 815
[22] Horvathy P A 1979 *Int. J. Theor. Phys* **18** 245
[23] Niederer U 1972 Helv. Phys. Acta **45** 802; Niederer U 1973 Helv. Phys. Acta **46** 191; Niederer U 1974 Helv. Phys. Acta **47** 167
[24] Milne W E 1930 *Phys. Rev.* **35** 86367
[25] Pinney E 1950 *Proc. A.M.S.* **1** 681
[26] Castaños O, Schuch D and Rosas-Ortiz O 2013, J. Phys. A **46**, 075304
[27] Hartley J G and Ray J R 1982, Phys. Rev. D **25**, 382
[28] Leach P G L and Andriopoulos K 2008, Phys. Scr. **77**, 015002
[29] Dodonov V V, Malkin I A and Man'ko V I 1974, Physica **74**, 597