

The KP hierarchy with self-consistent sources: construction, Wronskian solutions and bilinear identities

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Abstract. In this paper, we will present some of our results on the soliton hierarchy with self-consistent sources (SHSCSs). The Kadomtsev–Petviashvili (KP) hierarchy will be used as an illustrative example to show the method to construct the SHSCSs. Some properties of the KP hierarchy with self-consistent sources will also be given, such as the dressing approach, the Wronskian solutions (including soliton solutions), its bilinear identities and the tau function.

1. Introduction

The soliton equation with self-consistent sources (SESCSs) was proposed by Mel'nikov [16] and has important applications in hydrodynamics, plasma physics and solid state physics. Later on, some SESCShs were studied by inverse scattering method (*without* explicit Lax pair) [16, 17], Matrix theory [18] and D-bar method [4].

Since the constrained flows of soliton equations can be viewed as the stationary case of the SESCShs, one can derive the auxiliary linear problems for the SESCShs on the basis of the Lax pair for the constrained flows (the latter can be obtained systematically by the adjoint representation) [24]. With the help of the constrained flows, some SESCShs were studied by inverse scattering method (with *explicit* Lax pair), and some soliton solutions were obtained [8, 23]. It was shown that the velocities of the solitons can be changed by the choice of sources [8, 23]. A similar observation can be found in [5, 19].

The KdV equation with self-consistent sources reads [16, 8]

$$u_t = -\frac{1}{4}(6uu_x + u_{xxx}) - \frac{1}{2}\partial_x \sum_{j=1}^N \phi_j^2, \quad (1a)$$

$$\phi_{j,xx} + (\lambda_j + u)\phi_j = 0, \quad j = 1, \dots, N, \quad (1b)$$

where $u = u(x, t)$, $\phi_j = \phi_j(x, t)$, N is a natural number, λ_j 's are parameters.



One can get the soliton solutions for KdV equation with sources (1) by inverse scattering method [8], e.g., the following is its two-soliton solution

$$u(x, t) = \frac{12\{3 + 4\cosh[2x - 2t - 2\int_0^t \beta_2(z) dz] + \cosh[4x - 16t - 2\int_0^t \beta_1(z) dz]\}}{\{\cosh[3x - 9t - \int_0^t (\beta_1(z) + \beta_2(z)) dz] + 3\cosh[x - 7t - \int_0^t (\beta_1(z) - \beta_2(z)) dz]\}^2}, \quad (2a)$$

$$\phi_1(x, t) = \frac{4\sqrt{6\beta_1(t)} \cosh[x - t - \int_0^t \beta_2(z) dz]}{\cosh[3x - 9t - \int_0^t (\beta_1(z) + \beta_2(z)) dz] + 3\cosh[x - 7t - \int_0^t (\beta_1(z) - \beta_2(z)) dz]}, \quad (2b)$$

$$\phi_2(x, t) = \frac{4\sqrt{3\beta_2(t)} \sinh[2x - 8t - \int_0^t \beta_1(z) dz]}{\cosh[3x - 9t - \int_0^t (\beta_1(z) + \beta_2(z)) dz] + 3\cosh[x - 7t - \int_0^t (\beta_1(z) - \beta_2(z)) dz]}, \quad (2c)$$

where $\beta_1(t)$ and $\beta_2(t)$ are arbitrary continuous functions of t . If we choose special $\beta_1(t)$ and $\beta_2(t)$, the soliton with smaller amplitude may propagate faster than that with bigger amplitude (e.g., as plotted in Figure 1 with $\beta_1(t) = 1$ and $\beta_2(t) = 9$). This phenomenon is *completely different* from that of solitons to the original KdV hierarchy (*without sources*). Some other choices of $\beta_i(t)$ can give a great variety of dynamics of soliton solutions. Similar cases were also studied in [2-7,16-19,25].

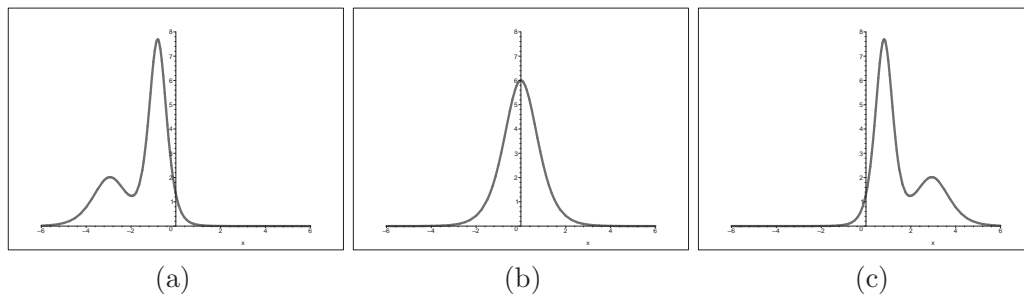


Figure 1. The plot of two-soliton solution $u(x, t)$ in Eq. (2) with $\beta_1(t) = 1$ and $\beta_2(t) = 9$ (for $t = -0.06$ (a), $t = 0$ (b) and $t = 0.06$ (c)).

Later, some explicit solutions (such as solitons, positons, negatons) of some SESCOs were obtained by Darboux transformation (see the references in [9]) and Hirota method [25]. A source generalization method was proposed by Hu and coworkers [6].

In the study on SESCOs, it is found that two types of self-consistent sources can be added to a soliton equation [7, 16]. Let's take the Kadomtsev–Petviashvili (KP) equation as an example. The first type of KP equation with self-consistent sources (KPwS-I) is [7, 16]

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} + 4 \sum_{i=1}^N (q_i r_i)_{xx} = 0, \quad (3a)$$

$$q_{i,y} = q_{i,xx} + 2uq_i, \quad i = 1, \dots, N, \quad (3b)$$

$$r_{i,y} = -r_{i,xx} - 2ur_i, \quad (3c)$$

where N is a natural number. The second type of KP equation with self-consistent sources

(KPwS-II) is [7, 16]

$$4u_t - 12uu_x - u_{xxx} - 3\partial^{-1}u_{yy} = 3 \sum_{i=1}^N \left[q_{i,xx}r_i - q_i r_{i,xx} + (q_i r_i)_y \right], \quad (4a)$$

$$q_{i,t} = q_{i,xxx} + 3uq_{i,x} + \frac{3}{2}q_i\partial^{-1}u_y + \frac{3}{2}q_i \sum_{j=1}^N q_j r_j + \frac{3}{2}u_x q_i, \quad (4b)$$

$$r_{i,t} = r_{i,xxx} + 3ur_{i,x} - \frac{3}{2}r_i\partial^{-1}u_y - \frac{3}{2}r_i \sum_{j=1}^N q_j r_j + \frac{3}{2}u_x r_i, \quad (4c)$$

where ∂^{-1} stands for the inverse of $\partial \equiv \partial_x$.

In the beginning, the two types of SESCSS (3) and (4) were studied individually. In order to find a unified framework to study the two types of SESCSS, a systematical method was proposed by the authors on the basis of Sato's theory [13]. This is a systematic method to generate the SESCSS, and can be used to study the case of BKP, CKP [22], q -deformed KP [10, 11], and some other cases. A generalized dressing method was also derived for these soliton hierarchy with sources, and their Wronskian solutions (including soliton solutions) were obtained [14]. Recently, a bilinear identity for the KP hierarchy with self-consistent sources (KPHwS) and their Hirota's bilinear equations (*in a simpler form*) were obtained [12].

This paper presents some of our results on the SESCSS. In Section 2, the KP hierarchy is used as an illustrative example to show the method to construct a soliton hierarchy with self-consistent sources on the basis of Sato's theory. In Section 3, a dressing approach is given for the KP hierarchy with self-consistent sources (KPHwS) and its Wronskian solutions (including soliton solutions) are shown. In Section 4, the bilinear identities and tau-function for the KPHwS are given, Hirota's bilinear form for the KP equation with sources is obtained, which is in a simpler form compared with the existing result. In Section 5, we will give a conclusion, remarks and some problems for further exploration.

2. Construction of KP hierarchy with self-consistent sources

In this section, we will use the KP hierarchy as an example to show how to construct a soliton hierarchy with self-consistent sources [13] on the basis of Sato's theory.

It is known that there is a squared eigenfunction symmetry (or "ghost flow") [1, 20]

$$\partial_z L = \left[\sum_{i=1}^N q_i \partial^{-1} r_i, L \right], \quad (5a)$$

$$\partial_{t_n} q_i = L_+^n(q_i), \quad (5b)$$

$$\partial_{t_n} r_i = -(L_+^n)^*(r_i), \quad i = 1, \dots, N, \quad (5c)$$

to the original KP hierarchy

$$\partial_{t_n} L = [B_n, L], \quad B_n \equiv L_+^n, \quad (5d)$$

where $L = \partial + \sum_{i=1}^{\infty} u_i \partial^{-i}$, and for an pseudo-differential operator $P = \sum_{i=-\infty}^n p_i \partial^i$, we define $P_+ = \sum_{i=0}^n p_i \partial^i$, $P^* = \sum_{i=-\infty}^n (-\partial)^i p_i$. The idea of generating the KP hierarchy with self-consistent

sources (KPHwS) is to modify a specific flow (say t_k -flow) by the squared eigenfunction symmetry to be a new flow (denoted by \bar{t}_k -flow) as

$$\partial_{\bar{t}_k} L = [L_+^k + \sum_{i=1}^N q_i \partial^{-1} r_i, L], \quad (6a)$$

$$\partial_{t_n} L = [L_+^n, L], \quad (n \neq k) \quad (6b)$$

$$\partial_{t_n} q_i = L_+^n(q_i), \quad (6c)$$

$$\partial_{t_n} r_i = -(L_+^n)^*(r_i), \quad i = 1, \dots, N. \quad (6d)$$

It can be shown that the \bar{t}_k -flow commutes with the other t_n -flow ($n \neq k$) which gives the KP hierarchy with self-consistent sources (KPHwS) [13].

Proposition 1. (see [13]) *The community of the \bar{t}_k -flow and the t_n -flow ($n \neq k$) in (6) give rise to the following KP hierarchy with self-consistent sources (KPHwS)*

$$B_{n, \bar{t}_k} - (B_k + \sum_{i=1}^N q_i \partial^{-1} r_i)_{t_n} + [B_n, B_k + \sum_{i=1}^N q_i \partial^{-1} r_i] = 0 \quad (7a)$$

$$q_{i, t_n} = B_n(q_i), \quad (7b)$$

$$r_{i, t_n} = -B_n^*(r_i), \quad i = 1, \dots, N. \quad (7c)$$

For example (see [13]), the system (7) with $n = 2$ and $k = 3$ gives the KPwS-I (3) with $y \equiv t_2$, $t \equiv \bar{t}_k$, $u \equiv u_1$, and the system (7) with $n = 3$ and $k = 2$ gives the KPwS-II (4) with $y \equiv \bar{t}_k$, $t \equiv t_3$, $u \equiv u_1$.

In fact, the two types of reductions on the KPHwS (7) give two types of $(1+1)$ -dimensional soliton hierarchy with self-consistent sources (see [13]), i.e., the k -constrained KP hierarchy and the Gelfand–Dickey hierarchy with sources, which include two types of KdV equations with sources and two types of Boussinesq equations with sources.

3. Dressing approach and Wronskian solutions for the KP hierarchy with sources (7)

Here we denote the Wronskian determinant as

$$\text{Wr}(h_1, \dots, h_N) = \begin{vmatrix} h_1 & h_2 & \cdots & h_N \\ h'_1 & h'_2 & \cdots & h'_N \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(N-1)} & h_2^{(N-1)} & \cdots & h_N^{(N-1)} \end{vmatrix}.$$

Then by modifying the dressing approach for the original KP hierarchy, we can get a dressing approach for the KP hierarchy with self-consistent sources as the following [14]:

Proposition 2. (see [14]) *Let W , q_i and r_i be defined as*

$$W = \frac{\text{Wr}(h_1, \dots, h_N, \partial)}{\text{Wr}(h_1, \dots, h_N)} \equiv \frac{1}{\text{Wr}(h_1, \dots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ h'_1 & h'_2 & \cdots & h'_N & \partial \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_1^{(N)} & h_2^{(N)} & \cdots & h_N^{(N)} & \partial^N \end{vmatrix}, \quad (8a)$$

$$q_i = -\alpha_{i,\bar{t}_k} W(g_i) \quad r_i = (-1)^{N-i} \frac{\text{Wr}(h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_N)}{\text{Wr}(h_1, \dots, h_N)}, i = 1, \dots, N \quad (8b)$$

$$h_i = f_i + \alpha_i(\bar{t}_k)g_i, \quad i = 1, \dots, N, \quad (8c)$$

$$\partial_{t_n} f_i = \partial^n(f_i), \quad \partial_{\bar{t}_k} f_i = \partial^k(f_i), \quad i = 1, \dots, N, \quad (8d)$$

$$\partial_{t_n} g_i = \partial^n(g_i), \quad \partial_{\bar{t}_k} g_i = \partial^k(g_i), \quad (8e)$$

where the α_i 's are arbitrary differentiable function of \bar{t}_k , $\text{Wr}(h_1, \dots, h_N, \partial)$ is understood as an expansion with respect to its last column, in which all sub-determinants are collected on the left of the differential symbols, then $L = W\partial W^{-1}$, q_i and r_i satisfy the KP hierarchy with self-consistent sources (7).

For example, the one-soliton solution for the KP equation with a second type of self-consistent sources (4) with $N = 1$ can be obtained by the dressing approach (Proposition 2)

$$u = \frac{(\lambda_1 - \mu_1)^2}{4} \text{sech}^2(\Omega_1), \quad (9a)$$

$$q_1 = \sqrt{\alpha_{1,y}}(\lambda_1 - \mu_1) e^{\frac{\xi_1 + \eta_1}{2}} \text{sech}(\Omega_1), \quad (9b)$$

$$r_1 = \frac{1}{2\sqrt{\alpha_1}} e^{-\frac{\xi_1 + \eta_1}{2}} \text{sech}(\Omega_1), \quad (9c)$$

where $\Omega_i = \frac{\xi_i - \eta_i}{2} - \frac{1}{2} \ln(\alpha_i)$, $\exp(\lambda_i x + \lambda_i^2 y + \lambda_i^3 t) \equiv e^{\xi_i}$, $\exp(\mu_i x + \mu_i^2 y + \mu_i^3 t) \equiv e^{\eta_i}$, λ_i and μ_i are different parameters (see [14]).

In [14], a gauge transformation between the KP hierarchy with sources and the mKP hierarchy with sources is constructed, and a Wronskian solution (including soliton solutions) for the mKP hierarchy with sources is also obtained with the help of dressing approach (Proposition 2) for the KP hierarchy with sources.

4. The bilinear identities and Hirota's bilinear form for the KP hierarchy with sources (7)

The Sato theory is of fundamental importance in the study of integrable systems (see [3] and references therein). It reveals the infinite dimensional Grassmannian structure of the space of τ -functions, where the τ -functions are solutions of Hirota's bilinear form of the KP hierarchy. The key point to this important discovery is a bilinear residue identity for wave functions called *bilinear identity*. Bilinear identity plays an important role in the proof of existence for τ -functions. It also serves as the generating function of Hirota's bilinear equations for the KP hierarchy [2, 15, 21].

With the help of an auxiliary parameter "z" corresponding to the "ghost flow", we can find the bilinear identities for the KP hierarchy with self-consistent source (7).

Proposition 3. (see [12]) *The bilinear identity for the KP hierarchy with self-consistent sources (7) with $N = 1$ is given by the following sets of residue identities with auxiliary variable "z":*

$$\text{res}_\lambda w(z - \bar{t}_k, \mathbf{t}, \lambda) \cdot w^*(z - \bar{t}'_k, \mathbf{t}', \lambda) = 0, \quad (10a)$$

$$\text{res}_\lambda w_z(z - \bar{t}_k, \mathbf{t}, \lambda) \cdot w^*(z - \bar{t}'_k, \mathbf{t}', \lambda) = q(z - \bar{t}_k, \mathbf{t})r(z - \bar{t}'_k, \mathbf{t}'), \quad (10b)$$

$$\text{res}_\lambda w(z - \bar{t}_k, \mathbf{t}, \lambda) \cdot \partial^{-1} (q(z - \bar{t}'_k, \mathbf{t}')w^*(z - \bar{t}'_k, \mathbf{t}', \lambda)) = -q(z - \bar{t}_k, \mathbf{t}), \quad (10c)$$

$$\text{res}_\lambda \partial^{-1} (r(z - \bar{t}_k, \mathbf{t})w(z - \bar{t}_k, \mathbf{t}, \lambda)) \cdot w^*(z - \bar{t}'_k, \mathbf{t}', \lambda) = r(z - \bar{t}'_k, \mathbf{t}'), \quad (10d)$$

where the residue with respect to λ can be simply considered as the coefficient of λ^{-1} in the Laurent expansion, the inverse of ∂ is understood as pseudo-differential operator acting on an exponential function, e.g., $\partial^{-1}(rw) = (\partial^{-1}rW)(e^\xi)$, $W = \sum_{i \geq 0} w_i \partial^{-i}$ ($w_0 = 1$), $\xi(\mathbf{t}, \lambda) = \bar{t}_k \lambda^k + \sum_{i \neq k} t_i \lambda^i$, $\mathbf{t} = (t_1, t_2, \dots, t_{k-1}, \bar{t}_k, t_{k+1}, \dots)$, $\mathbf{t}' = (t'_1, t'_2, \dots, t'_{k-1}, \bar{t}'_k, t'_{k+1}, \dots)$, $w(z, \mathbf{t}, \lambda) = W e^{\xi(\mathbf{t}, \lambda)}$, $w^*(z, \mathbf{t}, \lambda) = (W^*)^{-1} e^{-\xi(\mathbf{t}, \lambda)}$.

Remark 4.1. In this section, we will only show the formulae on the KP hierarchy with self-consistent sources (7) with $N = 1$, $q \equiv q_1$ and $r \equiv r_1$. The same idea can be used to study the case of $N > 1$.

The existence of the τ -function for the original bilinear identity of the KP hierarchy is proved in [3]. In our case, the wave functions $w(z - \bar{t}_k, \mathbf{t}, \lambda)$ and $w^*(z - \bar{t}_k, \mathbf{t}, \lambda)$ satisfy exactly the same bilinear identity (10a) as the original KP case if one considers z as an additional parameter. So it is reasonable to assume the existence of a τ -function and make the following ansatz:

$$w(z - \bar{t}_k, \mathbf{t}, \lambda) = \frac{\tau(z - \bar{t}_k + \frac{1}{k\lambda^k}, \mathbf{t} - [\lambda])}{\tau(z - \bar{t}_k, \mathbf{t})} \cdot \exp \xi(\mathbf{t}, \lambda), \quad (11a)$$

$$w^*(z - \bar{t}_k, \mathbf{t}, \lambda) = \frac{\tau(z - \bar{t}_k - \frac{1}{k\lambda^k}, \mathbf{t} + [\lambda])}{\tau(z - \bar{t}_k, \mathbf{t})} \cdot \exp(-\xi(\mathbf{t}, \lambda)), \quad (11b)$$

where $[\lambda] = (\frac{1}{\lambda}, \frac{1}{2\lambda^2}, \frac{1}{3\lambda^3}, \dots)$. According to [2], we should make further assumptions:

$$q(z, \mathbf{t}) = \frac{\sigma(z, \mathbf{t})}{\tau(z, \mathbf{t})}, \quad r(z, \mathbf{t}) = \frac{\rho(z, \mathbf{t})}{\tau(z, \mathbf{t})}. \quad (11c)$$

Then, similar to [2], we have the following results:

$$\partial^{-1}(r(z - \bar{t}_k, \mathbf{t})w(z - \bar{t}_k, \mathbf{t}, \lambda)) = \frac{\rho(z - \bar{t}_k + \frac{1}{k\lambda^k}, \mathbf{t} - [\lambda])}{\lambda \tau(z - \bar{t}_k, \mathbf{t})} e^{\xi(\mathbf{t}, \lambda)}, \quad (12a)$$

$$\partial^{-1}(q(z - \bar{t}_k, \mathbf{t})w^*(z - \bar{t}_k, \mathbf{t}, \lambda)) = \frac{-\sigma(z - \bar{t}_k - \frac{1}{k\lambda^k}, \mathbf{t} + [\lambda])}{\lambda \tau(z - \bar{t}_k, \mathbf{t})} e^{-\xi(\mathbf{t}, \lambda)}. \quad (12b)$$

After substituting (11) into (10) and some calculation, we get the following systems with Hirota bilinear derivatives \tilde{D} and D_i 's:

$$\sum_{i \geq 0} p_i(2\mathbf{y}) p_{i+1}(-\tilde{D}) \exp\left(\sum_{j \geq 1} y_j D_j\right) \bar{\tau}(z, \mathbf{t}) \cdot \bar{\tau}(z, \mathbf{t}) = 0, \quad (13a)$$

$$\begin{aligned} & \sum_{i \geq 0} p_i(2\mathbf{y}) p_{i+1}(-\tilde{D}) \exp\left(\sum_{j \geq 1} y_j D_j\right) \bar{\tau}_z(z, \mathbf{t}) \cdot \bar{\tau}(z, \mathbf{t}) \\ & - \sum_{i \geq 0} p_i(2\mathbf{y}) \left(\partial_z \log \bar{\tau}(z, \mathbf{t} + \mathbf{y})\right) p_{i+1}(-\tilde{D}) \bar{\tau}(z, \mathbf{t} + \mathbf{y}) \cdot \bar{\tau}(z, \mathbf{t} - \mathbf{y}) \\ & = \exp\left(\sum_{j \geq 1} y_j D_j\right) \bar{\sigma}(z, \mathbf{t}) \cdot \bar{\rho}(z, \mathbf{t}) \end{aligned} \quad (13b)$$

$$\sum_{i \geq 0} p_i(2\mathbf{y}) p_i(-\tilde{D}) \exp\left(\sum_{j \geq 1} y_j D_j\right) \bar{\tau}(z, \mathbf{t}) \cdot \bar{\sigma}(z, \mathbf{t}) = \exp\left(\sum_{j \geq 1} y_j D_j\right) \bar{\sigma}(z, \mathbf{t}) \cdot \bar{\tau}(z, \mathbf{t}), \quad (13c)$$

$$\sum_{i \geq 0} p_i(2\mathbf{y}) p_i(-\tilde{D}) \exp\left(\sum_{j \geq 1} y_j D_j\right) \bar{\rho}(z, \mathbf{t}) \cdot \bar{\tau}(z, \mathbf{t}) = \exp\left(\sum_{j \geq 1} y_j D_j\right) \bar{\tau}(z, \mathbf{t}) \cdot \bar{\rho}(z, \mathbf{t}), \quad (13d)$$

where the bar $\bar{\cdot}$ over a function $f(z, \mathbf{t})$ is defined as $\bar{f}(z, \mathbf{t}) \equiv f(z - \bar{t}_k, \mathbf{t})$, e.g., $\bar{\tau}(z, \mathbf{t} - [\lambda]) \equiv \tau(z - (\bar{t}_k - \frac{1}{k\lambda^k}), \mathbf{t} - [\lambda])$, $\bar{\tau}(z, \mathbf{t}' + [\lambda]) \equiv \tau(z - (\bar{t}'_k + \frac{1}{k\lambda^k}), \mathbf{t}' + [\lambda])$, $\mathbf{y} = (y_1, y_2, \dots)$, $\tilde{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \dots)$, D_i is the well-known Hirota bilinear derivative $D_i f \cdot g = f_{t_i} g - f g_{t_i}$, and $p_i(\mathbf{t})$ is the i -th Schur polynomial, whose generating function is given by

$$\exp \sum_{i=1}^{\infty} y_i \lambda^i = \sum_{i=0}^{\infty} p_i(\mathbf{y}) \lambda^i.$$

Example 1 (First type of KP equation with a source (KPwS-I) [7, 13, 16, 18], i.e., the KPHwS (7) for $n = 2$ and $k = 3$). The Hirota equations for the KPwS-I (3) can be obtained as

$$D_x \tau_z \cdot \tau + \sigma \rho = 0, \quad \text{by (13b) with } y_j = 0, \quad (14a)$$

$$(D_x^4 + 3D_{t_2}^2 - 4D_x(D_{\bar{t}_3} - D_z))\tau \cdot \tau = 0, \quad \text{by (13a) in } y_3, \quad (14b)$$

$$(D_{t_2} + D_x^2)\tau \cdot \sigma = 0, \quad \text{by (13c) in } y_2, \quad (14c)$$

$$(D_{t_2} + D_x^2)\rho \cdot \tau = 0, \quad \text{by (13d) in } y_2, \quad (14d)$$

where D_z is Hirota's derivative, i.e., $D_z f(z) \cdot g(z) = f_z g - f g_z$. Note that from the definition of $\bar{\tau}$, we know that $D_{\bar{t}_3} \bar{\tau} \cdot \bar{\tau} = (D_{\bar{t}_3} - D_z)\tau \cdot \tau$, which interprets the appearance of this term in the second equation.

Example 2 (Second type of KP with a source (KPwS-II) [7, 13, 16, 18], i.e., the KPHwS (7) for $n = 3$ and $k = 2$). The Hirota equations for the KPwS-II (4) can be obtained as

$$D_x \tau_z \cdot \tau + \sigma \rho = 0, \quad \text{by (13b) with } y_j = 0, \quad (15a)$$

$$(D_x^4 + 3(D_{\bar{t}_2} - D_z)^2 - 4D_x D_{t_3})\tau \cdot \tau = 0, \quad \text{by (13a) in } y_3, \quad (15b)$$

$$((D_{\bar{t}_2} - D_z) + D_x^2)\tau \cdot \sigma = 0, \quad \text{by (13c) in } y_2, \quad (15c)$$

$$((D_{\bar{t}_2} - D_z) + D_x^2)\rho \cdot \tau = 0, \quad \text{by (13d) in } y_2, \quad (15d)$$

$$(4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z))\tau \cdot \sigma = 0, \quad \text{by (13c) in } y_3, \quad (15e)$$

$$(4D_{t_3} - D_x^3 + 3D_x(D_{\bar{t}_2} - D_z))\rho \cdot \tau = 0, \quad \text{by (13d) in } y_3. \quad (15f)$$

It seems that the Hirota bilinear equations (15) obtained here for KPwS-II are simpler than the results by Hu and Wang [7].

5. Conclusion and discussions

In this paper, we presented some results on the soliton hierarchy with self-consistent sources, especially on the method based on the symmetry of the soliton hierarchy. The Kadomtsev–Petviashvili (KP) hierarchy is used as an illustrative example to show the method to construct the soliton hierarchy with self-consistent sources (SHSCSs) and some properties of the SHSCSs, such as dressing approach, Wronskian solutions (including soliton solutions), bilinear identities and tau function.

There are some important applications of the bilinear identities for the KP hierarchy with self-consistent sources (KPHwS). As we know, the quasi-periodic solutions for the KP hierarchy can be constructed by using a method in algebraic geometry, where the construction of wave functions (or Baker–Akhiezer functions as in quasi-periodic cases) are intimately related to the bilinear identities, Riemann surfaces and divisors on it. It is very interesting to consider the quasi-periodic solutions for the KPHwS (7) when bilinear identities have been obtained in this paper. Another interesting problem is to consider the bilinear identities for other soliton hierarchies with sources, such as BKP, 2D Toda and discrete KP, etc. We will investigate these problems in the future.

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