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Embedded Random Matrix Ensembles with Lie Symmetries: Results from $U(\Omega)$ Wigner-Racah algebra

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Abstract.

Random matrix ensembles for a system of m number of fermions or bosons in Ω number of single particle levels each r-fold degenerate and interacting with two-body forces are considered. The spectrum generating algebra for these systems is $U(r\Omega)$ and a subalgebra of interest is $U(r\Omega) \supset U(\Omega) \otimes SU(r)$ algebra. Now, for random two-body interactions preserving $SU(r)$ symmetry, one can introduce embedded Gaussian unitary ensemble of random matrices with $U(\Omega)\otimes SU(r)$ embedding and this class of ensembles are denoted by EGUE(2)-$SU(r)$. Ensembles with $r = 1$, 2 and 4 for fermions correspond to spinless fermions, fermions with spin and fermions with Wigner's spin-isospin $SU(4)$ symmetry respectively. Similarly, for bosons $r = 1$, 2 and 3 correspond to spinless bosons, two species boson systems and bosons with spin one respectively. The distinction between fermions and bosons is in the $U(\Omega)$ irreducible representations. General formulation based on Wigner-Racah algebra for lower order moments of the one- and two-point functions in eigenvalues generated by EGUE(2)- $SU(r)$ is briefly reviewed. The final formulas for the moments involve only $SU(\Omega)$ Racah coefficients. For the fourth moment of the one-point function for $r > 1$ and for the higher order (> 4) bivariate moments of the two-point function for $r \geq 1$, formulas are not available for the $SU(\Omega)$ Racah coefficients that are needed. It is necessary to derive analytical formulas for these or develop methods that give asymptotic results (an example for this is given in the paper) or develop methods that allow for their numerical evaluation. This important open problem is discussed in some detail.

1. Introduction

Wigner introduced random matrix theory (RMT) in physics in 1955 and as Wigner stated: *The assumption is that the Hamiltonian which governs the behavior of a complicated system is a random symmetric matrix, with no special properties except for its symmetric nature.* Further, if 'complicated'had been replaced by 'non-integrable', it would have led to the foundations of what is now well known as 'Quantum Chaos'. Depending on the global symmetry properties of the Hamiltonian of a quantum system, namely rotational symmetry and time-reversal symmetry, we have Dyson's tripartite classification of random matrices giving the classical random matrix ensembles, the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles. For example, given the matrix elements $a_{ij} = b_{ij} + ic_{ij}$ of a GUE matrix with $a_{ji} = (a_{ij})^{\dagger}$,


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the b_{ij}, $i \geq j$ and $c_{ij}, i > j$ are independent Gaussian random variables with zero center and variance $v^2(1+\delta_{ij})$ where v is a free parameter. Moreover, $\overline{a_{ij}a_{kl}} = 2v^2(\delta_{il}\delta_{jk})$ where the 'overline'indicates ensemble average. Here after we call this GUE(v^2). In the last three decades, RMT has found applications not only in all branches of quantum physics but also in many other disciplines such as Econophysics, Wireless communication, information theory, multivariate statistics, number theory, neural and biological networks and so on [1, 2, 3, 4].

As stated by H.A. Weidenmüller [5]: *although used with increasing frequency in many branches of Physics, random matrix ensembles sometimes are too unspecific to account for important features of the physical system at hand. One refinement which retains the basic stochastic approach but allows for such features consists in the use of embedded ensembles.* Finite quantum systems such as nuclei, atoms, quantum dots, small metallic grains, interacting spin systems modeling quantum computing core and ultra-cold atoms, share one common property - their constituents (predominantly) interact via two-particle interactions. Therefore, it is more appropriate to represent an isolated finite interacting quantum system by random matrix models generated by random two-body interactions. Then we have random matrix ensembles in m-particle spaces - these ensembles are defined by representing the two-particle Hamiltonian (H) by GOE/GUE/GSE and then the m particle H matrix is generated by the m-particle Hilbert space geometry. The key element here is the recognition that there is a Lie algebra that transports the information in the two-particle spaces to many-particle spaces. As a GOE/GUE/GSE random matrix ensemble in two-particle spaces is embedded in the m-particle H matrix, these ensembles are more generically called embedded ensembles (EE). It is important to note that the classical GOE/GUE/GSE, now almost universally regarded as models for the corresponding chaotic systems [2], are ensemble of multi-body, not two-body interactions. This difference shows up both in one-point (density of states) and two-point (fluctuations and transition strengths) functions as seen first in nuclear shell model examples [6, 7].

Embedded random matrix ensembles ensembles or simply embedded ensembles (EE), though introduced in 1970 and analyzed in detail first in 1975, have received new emphasis beginning from 1996 and since then a wide variety of EGE have been introduced in literature, both for fermion and boson systems [7, 8, 9]. Motivation to study EE in detail has come from: (i) many-body quantum chaos; (ii) statistical spectroscopy (in nuclei and atoms); (iii) generic electron-electron interaction effects on transport properties of mesoscopic systems; (iv) regular structures generated by random interactions in quantum many-body systems; (v) possible applications to quantum gases and in quantum information science (thermalization, entanglement and other aspects). Since 1997, a variety of EE have been introduced in literature and almost all these arise from different Lie algebras that define the embedding. A major hurdle in deriving statistical properties generated by EE is that they are largely intractable analytically. Although numerical experiments have given some interesting and valuable results [7, 8, 9, 10, 11, 12], perturbation theory, trace propagation methods and the so-called binary correlation method are used to derive some analytical results. However, one general formulation for deriving analytical results is to use the Wigner-Racah algebra of the embedding Lie algebra. Though this is indicated in some early papers [13, 14, 15, 16], it is clearly established only in 2005 [17] and applied to a variety of EGUEs thereafter [17, 18, 19, 20]. Our purpose here is to review the progress made in applying Wigner-Racah algebra to a class of EE, the embedded Gaussian unitary ensembles with $U(\Omega) \otimes SU(r)$ embedding and generated by random two-body interactions with $SU(r)$ symmetry. These are called EGUE(2)-$SU(r)$ ensembles. Secondly, the purpose is also to clearly bring out the outstanding unsolved problems. Now we will give a preview.

In Section 2, introduced and defined are EGUE(2)-$SU(r)$ ensembles. Section 3 gives a brief discussion of the Wigner-Racah algebra formulation for the moments of the one- and two-point functions in eigenvalues generated by these ensembles. In Section 4, the formulation for the fourth moment of the density of eigenvalues is presented and discussed in detail the difficulties

associated with the Wigner-Racah formulation. In Section 5, as an example of asymptotic methods, results obtained using an extended binary correlation method for the moments of a bivariate transition strength density are presented. Finally, Section 6 gives future outlook.

2. EGUE(2)-$SU(r)$ ensembles: Definition

Let us begin with m spinless fermions in Ω number of single particle (sp) states (labeled by $|\nu_i\rangle$, $i = 1, 2, \ldots, \Omega$) and the Hamiltonian for the system is say two-body preserving particle number m. Now, each distribution of the m fermions in the Ω number of sp states defines a basis state. The number of basis states $d_f(\Omega, m)$ is

$$d_f(\Omega, m) = \binom{\Omega}{m} . \tag{1}$$

For example, $d_f(12, 6) = 924$ and $d_f(16, 8) = 12870$. General form of a two-body Hamiltonian is,

$$H(2) = \sum_{i>j,k>\ell} H_{ijk\ell}\, a^\dagger_{\nu_k} a^\dagger_{\nu_\ell} a_{\nu_j} a_{\nu_i} \; ; \quad H_{ijk\ell} = \langle \nu_k \nu_\ell \mid H(2) \mid \nu_i \nu_j \rangle \; . \tag{2}$$

Here, $a^\dagger$ are creation operators for a single fermion and a are annihilation operators. It is easy to write formulas for the matrix elements of H in the m fermion basis states and hence it is easy to construct the m-fermion H matrix. Assuming GUE(1) representation for H in two-particle space will give EGUE(2) in m particle spaces. The embedding algebra here is $U(\Omega)$. Also, this corresponds to $r = 1$ in EGUE(2)-$SU(r)$ with $U(\Omega)$ irreducible representation (irrep) being $\{1^m\}$. See Fig. 1 for the Young tableaux representation of irreps and this is used throughout this paper. Just as above, it is possible to define BEGUE(2) for m bosons in Ω number of sp states. Here, the number of basis sates is given by

$$d_b(\Omega, m) = \binom{\Omega + m - 1}{m} . \tag{3}$$

For example, $d_b(5, 10) = 1001$ and $d_b(8, 20) = 888030$. Similarly, the two-body Hamiltonian is

$$H(2) = \sum_{i\geq j,k\geq \ell} \frac{H_{ijk\ell}}{\sqrt{(1+\delta_{ij})(1+\delta_{k\ell})}}\, b^\dagger_{\nu_k} b^\dagger_{\nu_\ell} b_{\nu_j} b_{\nu_i} \; ; \quad H_{ijk\ell} = \langle \nu_k \nu_\ell \mid H(2) \mid \nu_i \nu_j \rangle \; . \tag{4}$$

Assuming GUE(1) representation for H in two-particle spaces will give BEGUE(2) in m particle spaces. The embedding algebra here also is $U(\Omega)$ and $r = 1$ in EGUE(2)-$SU(r)$ but the $U(\Omega)$ irrep is $\{m\}$. Now we will consider generalization to any $r > 1$.

Consider m number of fermions or bosons in Ω number of sp levels each r-fold degenerate and then the total number of sp states is $N = r\Omega$. Also we assume that the Hamiltonian for these systems is a $SU(r)$ scalar. Then the spectrum generating algebra is $SU(r\Omega) \supset U(\Omega) \otimes SU(r)$. All the m-particle states can be classified according to the irreps of $SU(r\Omega)$, $U(\Omega)$ and $SU(r)$. These in turn can be labeled by the Young tableaux $\{f\}$ with no more than k parts for a given $U(k)$ and $\sum_{i=1}^{k} f_i = m$. Moreover, all m particle states for fermion systems will transform as the $SU(r\Omega)$ irrep $\{1^m\}$ (antisymmetric irrep shown in Fig. 1) and for bosons as the symmetric irrep $\{m\}$. Due to this fact, with respect to $SU(\Omega)$ if a m particle state transforms as $\{f\}$, then with respect to $U(r)$ it will transform as $\{\tilde{f}\}$ irrep for fermions and simply as $\{f\}$ for bosons. Note that we are not making a distinction between $U(k)$ and $SU(k)$ although the $SU(k)$ irreps $\{g\}$ that corresponds to a $U(k)$ irrep $\{f\}$ will be such that $g_i = f_i - f_k$. From now on we will denote m particle $U(\Omega)$ irreps by f_m. Then the corresponding $SU(r)$ irrep for bosons is f_m and for fermions it is $\tilde{f}_m$. Therefore, the $SU(r)$ irrep need not be specified in denoting the symmetry

defined basis states. In addition, we will denote the irrep labels that belong to the subalgebras of $U(\Omega)$ by v_m and those that belong to the subalgebras of $SU(r)$ by β_m. With all these, m particle states with $U(\Omega) \otimes SU(r)$ symmetry can be labeled as $|f_m, v_m, \beta_m\rangle$. Then, a two-body Hamiltonian that is $SU(r)$ scalar can be written as

$$\begin{aligned} H(2) &= \sum_{f_2, v_2^i, v_2^f, \beta_2; f_2=\{2\},\{1^2\}} H_{f_2 v_2^i v_2^f} \; A^\dagger(f_2 v_2^f \beta_2) \, A(f_2 v_2^i \beta_2) \;\; ; \\ H_{f_2 v_2^i v_2^f} &= \left\langle f_2, v_2^f, \beta_2 \mid H(2) \mid f_2, v_2^i, \beta_2 \right\rangle \; . \end{aligned} \tag{5}$$

Note that $H_{f_2 v_2^i v_2^f}(2)$ is independent of β_2 and this ensures that the $H(2)$ operator is $SU(r)$ scalar. Also $A^\dagger$ and A are normalized two-particle creation and annihilation operators. Similarly, the matrix elements of $H(2)$ in m particle space,

$$H_{f_m v_m^i v_m^f} = \left\langle f_m, v_m^f, \beta_m \mid H(2) \mid f_m, v_m^i, \beta_m \right\rangle \tag{6}$$

are also independent of β_m. Now we are in a position to define the EGUE(2)-$SU(r)$ ensemble.

Firstly, the $H(2)$ matrices in two-particle spaces are taken to be independent GUEs, one for each f_2, i.e. real and imaginary parts of the two-particle matrix elements $H_{f_2 v_2^i v_2^f}$ are independent Gaussian variables with zero center and variance given by (with bar representing ensemble average),

$$\overline{H_{f_2 v_2^1 v_2^2} \; H_{f_2' v_2^3 v_2^4}} = \delta_{f_2 f_2'} \delta_{v_2^1 v_2^4} \delta_{v_2^2 v_2^3} \left(\lambda_{f_2}\right)^2 . \tag{7}$$

Using this random $H(2)$ operator, we can construct H matrix in m particle spaces with fixed f_m. Then, we have EGUE(2)-$SU(r)$ for each f_m.

In RMT, one is interested in deriving the form for the ensemble averaged eigenvalue density or the one-point function,

$$\overline{\rho^{m, f_m}(E)} = \overline{\langle \delta(H(2) - E) \rangle^{m, f_m}} \; , \tag{8}$$

and it is defined by its moments M_p,

$$M_p = \overline{\langle [H(2)]^p \rangle^{m, f_m}} \; . \tag{9}$$

Similarly, the two-point function that gives information about fluctuations is

$$S^{m f_m : m' f_{m'}}(E, E') = \overline{\rho^{m, f_m}(E) \rho^{m', f_{m'}}(E')} - \overline{\rho^{m, f_m}(E)} \;\; \overline{\rho^{m', f_{m'}}(E')} \tag{10}$$

and it is defined by the bivariate moments $\Sigma_{pq}\left(m, f_m : m', f_{m'}\right)$,

$$\Sigma_{pq}\left(m, f_m : m', f_{m'}\right) = \overline{\langle [H(2)]^p \rangle^{m, f_m} \;\; \langle [H(2)]^q \rangle^{m', f_{m'}}} - \overline{\langle [H(2)]^p \rangle^{m, f_m}} \;\; \overline{\langle [H(2)]^q \rangle^{m', f_{m'}}} \; . \tag{11}$$

Then, the scaled or reduced bivariate moments are,

$$\hat{\Sigma}_{pq}\left(m, f_m : m', f_{m'}\right) = \frac{\Sigma_{pq}\left(m, f_m : m', f_{m'}\right)}{\left[\overline{\langle [H(2)]^2 \rangle^{m, f_m}}\right]^{p/2} \;\; \left[\overline{\langle [H(2)]^2 \rangle^{m', f_{m'}}}\right]^{q/2}} \; . \tag{12}$$

Thus, in general we need the following ensemble averaged bivariate moments

$$M_{pq}\left(m, f_m : m', f_{m'}\right) = \overline{\langle [H(2)]^p \rangle^{m, f_m} \;\; \langle [H(2)]^q \rangle^{m', f_{m'}}} \; . \tag{13}$$

It is useful to note that $M_p = M_{p0}$ or M_{0p}.

In addition to the correlation functions in eigenvalues, one is also interested in transition strength densities and fluctuations in transition strengths generated by the action of a transition operator on eigenstates. We will discuss these with an example in Section 5.

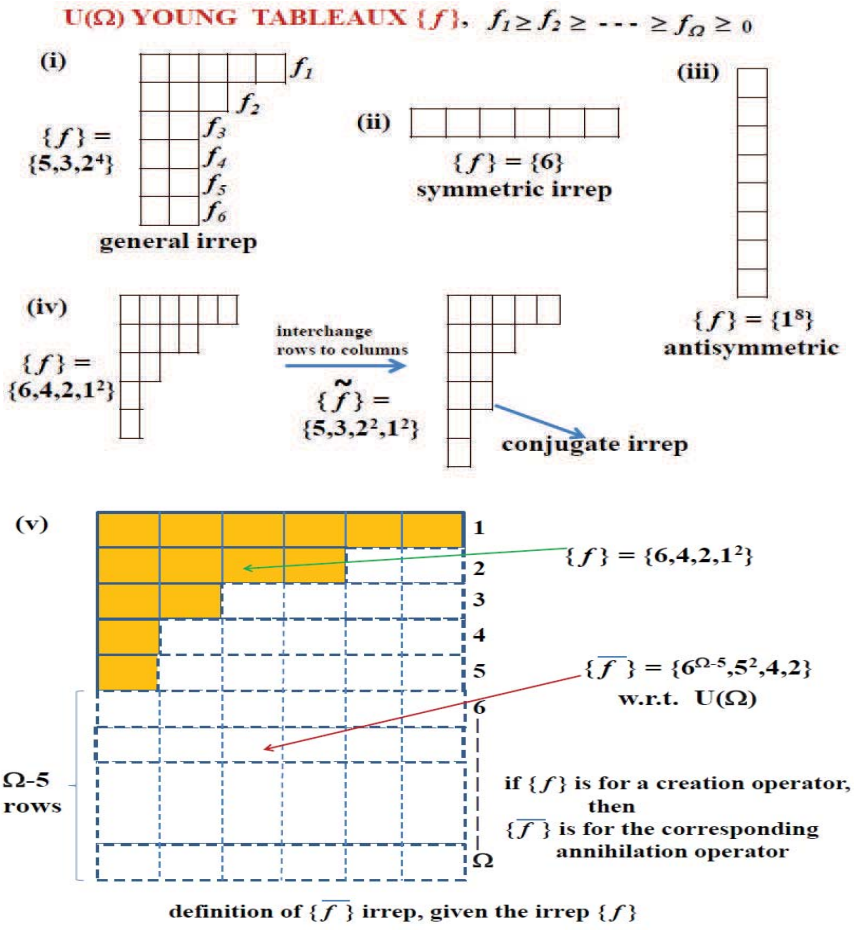


Figure 1. Young tableaux representation of the irreps of $U(\Omega)$. Shown are: (i) a general irrep $\{f\}$; (ii) symmetric irrep $\{m\}$; (iii) antisymmetric irrep $\{1^m\}$; (iv) conjugate irrep $\{\tilde{f}\}$ that corresponds to a given $\{f\}$; (v) irrep $\overline{\{f\}}$ that corresponds to a given $\{f\}$. Note the importance of Ω in defining $\overline{\{f\}}$.

3. Wigner-Racah algebra for EGUE(2) with SU(r) symmetry

Wigner-Racah algebra of the $U(\Omega) \otimes SU(r)$ algebra can be used to derive formulas for the moments M_p and Σ_{pq}. As we shall see ahead, they will lead to tractable formulas involving only $SU(\Omega)$ Racah coefficients for the lower order moments. However, for moments of higher order, evaluation of the needed Racah coefficients will be challenging. We will discuss in this Section, the formulation for lower order moments. First key point in deriving analytical results is tensorial decomposition of $H(2)$ with respect to (w.r.t.) $U(\Omega) \otimes SU(r)$. As $H(2)$ is a $SU(r)$ scalar, it will transform as the irrep $\{0\}$ w.r.t. $SU(r)$. As a result, w.r.t. $SU(\Omega)$ the operator $H(2)$ transforms as the irrep F_ν where $f_2 \times \overline{f_2} \rightarrow F_\nu$; the multiplication here is the Kronecker multiplication. Note that $f_2 = \{2\}$ or $\{1^2\}$ and see Fig. 1 for the definition of $\overline{f_2}$ irrep that corresponds to a given f_2 irrep. For $f_2 = \{2\}$, we have $F_\nu = \{0\}$, $\{21^{\Omega-2}\}$ and $\{42^{\Omega-2}\}$. Similarly, for $f_2 = \{1^2\}$, we have $F_\nu = \{0\}$, $\{21^{\Omega-2}\}$ and $\{2^21^{\Omega-4}\}$. Unit tensors B's can be

defined as,

$$B(f_2 F_\nu \omega_\nu) = \sqrt{\frac{1}{d_r(f_2^{(r)})}} \sum_{v_2^i, v_2^f, \beta_2} A^\dagger(f_2 v_2^f \beta_2)\, A(f_2 v_2^i \beta_2)\, \left\langle f_2 v_2^f\ \overline{f_2}\, \overline{v_2^i} \mid F_\nu \omega_\nu \right\rangle . \tag{14}$$

In Eq. (14), $\langle - - - | \rangle$ is a $U(\Omega)$ Wigner coefficient and $d_r(f_2^{(r)})$ is dimension of the irrep $f_2^{(r)}$ w.r.t. $SU(r)$. Note that for boson systems $f_2^{(r)} = f_2$ and for fermions $f_2^{(r)} = \tilde{f}_2$. It is possible to expand $H(2)$ defined by Eq. (5) in terms of the B's. This gives,

$$H(2) = \sum_{f_2, F_\nu,\, \omega_\nu} W(f_2 F_\nu \omega_\nu)\, B(f_2 F_\nu \omega_\nu) \tag{15}$$

and the expansion coefficients W's will be zero centered independent Gaussian variables with

$$\overline{W(f_2 F_\nu \omega_\nu) W(f_2' F_\nu' \omega_\nu')} = \delta_{f_2 f_2'} \delta_{F_\nu F_\nu'} \delta_{\omega_\nu \omega_\nu'}\, (\lambda_{f_2})^2 d_r(f_2^{(r)}) . \tag{16}$$

Eqs. (15) and (16) are used to derive formulas for M_{pq} and thereby M_p and $\hat{\Sigma}_{pq}$.

Firstly, general form of $M_{pq}(m_1, f_{m_1} : m_2, f_{m_2})$ is [with avg$\{- - -\}$ indicating ensemble average of $\{- - -\}$ and using H for $H(2)$],

$$\begin{aligned}
M_{pq}(m_1, f_{m_1} : m_2, f_{m_2}) &= \text{avg}\left\{ \langle H^p \rangle^{m_1, f_{m_1}}\, \langle H^q \rangle^{m_2, f_{m_2}} \right\} \\
&= \sum_{\alpha_1, \alpha_2, \ldots, \alpha_p : \gamma_1, \gamma_2, \ldots, \gamma_q} \text{avg}\left\{ \left\langle f_{m_1} v_{m_1}^{\alpha_1} \beta_{m_1}^{\alpha_1} \mid H \mid f_{m_1} v_{m_1}^{\alpha_2} \beta_{m_1}^{\alpha_2} \right\rangle \right. \\
&\times \left\langle f_{m_1} v_{m_1}^{\alpha_2} \beta_{m_1}^{\alpha_2} \mid H \mid f_{m_1} v_{m_1}^{\alpha_3} \beta_{m_1}^{\alpha_3} \right\rangle \ldots \left\langle f_{m_1} v_{m_1}^{\alpha_p} \beta_{m_1}^{\alpha_p} \mid H \mid f_{m_1} v_{m_1}^{\alpha_1} \beta_{m_1}^{\alpha_1} \right\rangle \\
&\times \left\langle f_{m_2} v_{m_2}^{\gamma_1} \beta_{m_2}^{\gamma_1} \mid H \mid f_{m_2} v_{m_2}^{\gamma_2} \beta_{m_2}^{\gamma_2} \right\rangle \left\langle f_{m_2} v_{m_2}^{\gamma_2} \beta_{m_2}^{\gamma_2} \mid H \mid f_{m_2} v_{m_2}^{\gamma_3} \beta_{m_2}^{\gamma_3} \right\rangle \\
&\times \left. \ldots \left\langle f_{m_2} v_{m_2}^{\gamma_q} \beta_{m_2}^{\gamma_q} \mid H \mid f_{m_2} v_{m_2}^{\gamma_1} \beta_{m_2}^{\gamma_1} \right\rangle \right\} .
\end{aligned} \tag{17}$$

Now, substituting for H the expansion in terms of the unit tensor operators B, the M_{pq} will reduce into an ensemble average involving only the W coefficients and matrix elements of B in (m_1, m_2) spaces. The ensemble average can be carried out using Eq. (16). Similarly, each matrix elements of B can be written, using Wigner-Eckart theorem, as a product of a $SU(\Omega)$ Wigner coefficient involving the labels of the subalgebras of $SU(\Omega)$ and a reduced matrix element that depends only on $SU(\Omega)$ irreps and the f_2 and F_ν of B. The reduced matrix elements (denoted by triple barred matrix elements) in turn can be written in terms of $SU(\Omega)$ Racah (or U-) coefficients. This result follows by combining the form of the operator B given by Eq. (14), Wigner-Eckart theorem and sum rules involving Wigner coefficients and U-coefficients; see for example [21]. The final formula is,

$$\langle f_m \,|||\, B(f_2 F_\nu) \,|||\, f_m \rangle_\rho = \sum_{f_{m-2}} \frac{-m(m-1)}{2} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{U(f_m \overline{f_2} f_m f_2; f_{m-2} F_\nu)_\rho}{U(f_m \overline{f_2} f_m f_2; f_{m-2}\{0\})} . \tag{18}$$

Note that, $\mathcal{N}_{f_k}$ is the dimension of the irrep f_k of k particles with respect to the symmetric group S_k. Similarly, ρ is the multiplicity label for the occurrence of f_m more than once in the Kronecker product $f_m \times F_\nu \to f_m$. With all these, finally M_{pq} will involve a sum of terms with each term containing a product of Wigner and Racah coefficients of $SU(\Omega)$. As the ensemble

average over W's will give several delta-functions [see Eq. (16)], formulas for M_{pq} will simplify into sums involving terms with products of U-coefficients.

The procedure described above has been carried out for M_2, M_{11}, M_{22}, M_{13} and M_{31} for any r; see [20] and references therein for details. Here, we will discuss briefly some of these results. It should be noted that all M_{pq} with $p+q$ odd are zero by the definition of EGUE(2)-$SU(r)$ ensemble. Thus, the first non-trivial moments are of second order ($p+q=2$). With $p=q=1$ we have M_{11} and the formula for this involves only the dimensions $\mathcal{N}_{f_k}$. Going further, with $p=2, q=0$ or $p=0, q=2$ we have M_2 and formula for this is,

$$\begin{aligned} M_2(m,f_m) &= \left\langle H^2 \right\rangle^{m,f_m} = \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{\nu=0,1,2} \mathcal{Q}^\nu(f_2:m,f_m)\ ; \\ \mathcal{Q}^\nu(f_2:m,f_m) &= [m(m-1)/2]^2 \sum_{f_{m-2},f'_{m-2}} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{\mathcal{N}_{f'_{m-2}}}{\mathcal{N}_{f_m}} X_{UU}(f_2;f_{m-2},f'_{m-2};F_\nu)\ . \end{aligned} \tag{19}$$

The X_{UU} function involves $SU(\Omega)$ Racah coefficients,

$$X_{UU}(f_2;f_{m-2},f'_{m-2};F_\nu) = \sum_\rho \frac{U(f_m,\overline{f_2},f_m,f_2;f_{m-2},F_\nu)_\rho U(f_m,\overline{f_2},f_m,f_2;f'_{m-2},F_\nu)_\rho}{U(f_m,\overline{f_2},f_m,f_2;f_{m-2},\{0\}) U(f_m,\overline{f_2},f_m,f_2;f'_{m-2},\{0\})}\ . \tag{20}$$

Proceeding to $p+q=4$, we have the moments M_{22}, M_{31}, M_{13}, M_{40} and M_{04}. Firstly, it is possible to write a formula for M_{22} and this involves X_{UU} given above and also a Y_{UU} function,

$$Y_{UU}(f_{m-2},f'_{m-2};F_\nu) = \sum_\rho \frac{U(f_m,\{1^{\Omega-2}\},f_m,\{1^2\};f_{m-2},F_\nu)_\rho\, U(f_m,\{2^{\Omega-1}\},f_m,\{2\};f'_{m-2},F_\nu)_\rho}{U(f_m,\{1^{\Omega-2}\},f_m,\{1^2\};f_{m-2},\{0\})\, U(f_m,\{2^{\Omega-1}\},f_m,\{2\};f'_{m-2},\{0\})}\ . \tag{21}$$

In addition, it is easy to prove that

$$\begin{aligned} M_{31}(m,f_m:m',f_{m'}) &= 3M_{11}(m,f_m:m',f_{m'})M_2(m,f_m)\ , \\ M_{13}(m,f_m:m',f_{m'}) &= 3M_{11}(m,f_m:m',f_{m'})M_2(m,f_{m'})\ . \end{aligned} \tag{22}$$

Using the above results and the tables for products of U-coefficients given in [19, 20, 22], M_2 (giving spectral variances), $\hat{\Sigma}_{11}$ (giving variance of the fluctuations in eigenvalue centroids for $m=m'$ and cross correlations for $m\neq m'$ and/or $f_m \neq f_{m'}$) and $\hat{\Sigma}_{22}$ (giving variance of the fluctuations in spectral variances for $m=m'$ and cross correlations for $m\neq m'$ and/or $f_m \neq f_{m'}$) are calculated in many examples. Using the exact formulas, obtained are also results in the asymptotic limit (note that $\hat{\Sigma}_{13}=\hat{\Sigma}_{31}=3\hat{\Sigma}_{11}$). Interesting asymptotic limit for fermions is the dilute limit defined by $m\to\infty$, $N\to\infty$ and $m/N\to 0$ and for bosons it is the dense limit defined by $m\to\infty$, $N\to\infty$ and $m/N\to\infty$. The exact and asymptotic formulas have given many new insights into the statistical properties of interacting fermion and boson systems [17, 18, 19, 20].

With $p=4, q=0$ or $p=0, q=4$, we have $M_4(f_m)$. Except for $r=1$ as given in detail in [14, 15, 16, 17], analytical formulas for M_4 are not available for $r>1$. Thus, with $p+q=4$, we are left with the unsolved problem of M_4 and we will now discuss what is unknown here.

4. Fourth moment in terms of Racah coefficients: Open questions

Hamiltonian operator H for EGUE(2)-$SU(r)$ is a sum of Hamiltonian operators in $f_2=\{2\}$ and $\{1^2\}$ spaces. Let us denote these two parts as H_{f_2}. Then,

$$M_4(m,f_m) = \overline{\left\langle H^4 \right\rangle^{m,f_m}} = \overline{\left\langle (H_{\{2\}}+H_{\{1^2\}})^4 \right\rangle^{m,f_m}}\ . \tag{23}$$

Using the cyclic invariance of the averages and applying the property that terms with odd powers of H_{f_2} will vanish will give,

$$\begin{aligned}\overline{\langle H^4\rangle^{m,f_m}} &= \overline{\left\langle (H_{\{2\}})^4\right\rangle^{m,f_m}} + \overline{\left\langle (H_{\{1^2\}})^4\right\rangle^{m,f_m}} + 4\overline{\left\langle (H_{\{2\}})^2 (H_{\{1^2\}})^2\right\rangle^{m,f_m}} \\ &+2\overline{\left\langle H_{\{2\}}H_{\{1^2\}}H_{\{2\}}H_{\{1^2\}}\right\rangle^{m,f_m}} \ .\end{aligned} \tag{24}$$

Writing H in terms of the unit tensors B's will give,

$$\begin{aligned}\overline{\left\langle (H_{f_2})^4\right\rangle^{m,f_m}} &= [d_\Omega(f_m)]^{-1} \sum_{v_i,F_{\nu_i},\omega_{\nu_i}\,:\,i=1}^{4} \langle f_m v_1 \mid B(f_2 F_{\nu_1}\omega_{\nu_1}) \mid f_m v_2\rangle \\ &\times \langle f_m v_2 \mid B(f_2 F_{\nu_2}\omega_{\nu_2}) \mid f_m v_3\rangle \langle f_m v_3 \mid B(f_2 F_{\nu_3}\omega_{\nu_3}) \mid f_m v_4\rangle \langle f_m v_4 \mid B(f_2 F_{\nu_4}\omega_{\nu_4}) \mid f_m v_1\rangle \\ &\times \overline{W(f_2 F_{\nu_1}\omega_{\nu_1}) W(f_2 F_{\nu_2}\omega_{\nu_2}) W(f_2 F_{\nu_3}\omega_{\nu_3}) W(f_2 F_{\nu_4}\omega_{\nu_4})} \ .\end{aligned} \tag{25}$$

Now, using the property that W's are independent Gaussian variables will lead to

$$\begin{aligned}\overline{\left\langle (H_{f_2})^4\right\rangle^{m,f_m}} &= 2\left[\overline{\left\langle (H_{f_2})^2\right\rangle^{m,f_m}}\right]^2 + \lambda_{f_2}^4 \left[d_r(f_2^{(r)})\right]^2 d_\Omega(f_m) \\ &\times \sum_{F_{\nu_1},F_{\nu_2},\rho_1,\rho_2,\rho_3,\rho_4} \frac{1}{\sqrt{d_\Omega(F_{\nu_1}) d_\Omega(F_{\nu_2})}} U(f_m \overline{f_m} f_m f_m; (F_{\nu_1})_{\rho_1\rho_3} (F_{\nu_2})_{\rho_2\rho_4}) \\ &\times \langle f_m \mid\mid B(f_2 F_{\nu_1}) \mid\mid f_m\rangle_{\rho_1} \langle f_m \mid\mid B(f_2 F_{\nu_2}) \mid\mid f_m\rangle_{\rho_2} \\ &\times \langle f_m \mid\mid B(f_2 F_{\nu_1}) \mid\mid f_m\rangle_{\rho_3} \langle f_m \mid\mid B(f_2 F_{\nu_2}) \mid\mid f_m\rangle_{\rho_4} \ .\end{aligned} \tag{26}$$

Similarly, we have

$$\overline{\left\langle H_{\{2\}}^2 H_{\{1^2\}}^2\right\rangle^{m,f_m}} = \left\{\overline{\left\langle H_{\{2\}}^2\right\rangle^{m,f_m}}\right\} \ \left\{\overline{\left\langle H_{\{1^2\}}^2\right\rangle^{m,f_m}}\right\} \ , \tag{27}$$

and

$$\begin{aligned}\overline{\left\langle H_{\{2\}}H_{\{1^2\}}H_{\{2\}}H_{\{1^2\}}\right\rangle^{m,f_m}} &= \lambda_{\{2\}}^2 \lambda_{\{1^2\}}^2 d_r(\{2\}) d_r(\{1^2\}) d_\Omega(f_m) \\ &\times \sum_{F_{\nu_1},F_{\nu_2},\rho_1,\rho_2,\rho_3,\rho_4} \frac{1}{\sqrt{d_\Omega(F_{\nu_1}) d_\Omega(F_{\nu_2})}} U(f_m \overline{f_m} f_m f_m; (F_{\nu_1})_{\rho_1\rho_3} (F_{\nu_2})_{\rho_2\rho_4}) \\ &\times \langle f_m \mid\mid B(\{2\}F_{\nu_1}) \mid\mid f_m\rangle_{\rho_1} \left\langle f_m \mid\mid B(\{1^2\}F_{\nu_2}) \mid\mid f_m\right\rangle_{\rho_2} \\ &\times \langle f_m \mid\mid B(\{2\}F_{\nu_1}) \mid\mid f_m\rangle_{\rho_3} \left\langle f_m \mid\mid B(\{1^2\}F_{\nu_2}) \mid\mid f_m\right\rangle_{\rho_4} \ .\end{aligned} \tag{28}$$

Substituting the results in Eqs. (25)-(28) in Eqs. (24) will give $\overline{\langle H^4\rangle^{m,f_m}}$. This involves $SU(\Omega)$ Racah coefficients with multiplicity labels and evaluation of these is in general complicated. Similarly, evaluation of the reduced matrix elements is also complicated. It is important to note that

$$\gamma_2(m,f_m) = \frac{M_4(m,f_m)}{[M_2(m,f_m)]^2} - 3 \tag{29}$$

gives the excess parameter and if it is close to zero (asymptotically or otherwise), then we have Gaussian form for the eigenvalue density. This result was established only for spinless EGUE(2) but not yet for EGUE(2)-$SU(r)$ with $r > 1$ as the formulas for the reduced matrix elements and the U-coefficients appearing in Eqs. (26) and (28) are not available for $r > 1$.

One simple situation where there is some hope of deriving formulas for M_4 is for the f_m irreps that give multiplicity labels all unity. We denote the $U(\Omega)$ irreps that satisfy this as $f_m^{(g)}$ and we have verified that one of these irreps is $\{r^k\}$ for fermions and $\{k^r\}$ for bosons where $m = kr$ and others are $\{m\}$ and $\{1^m\}$. For these irreps, the expression for γ_2 is

$$\left[\gamma_2(m, f_m^{(g)}) + 1\right] = \left[\overline{\langle H^2\rangle^{m,f_m^{(g)}}}\right]^{-2}$$
$$\times \left\{ \sum_{f_a, f_b = \{2\},\{1^2\}} \frac{\lambda_{f_a}^2 \lambda_{f_b}^2}{d_\Omega(f_a) d_\Omega(f_b)} \sum_{F_{\nu_1}, F_{\nu_2}} \frac{d_\Omega(f_m^{(g)})}{\sqrt{d_\Omega(F_{\nu_1}) d_\Omega(F_{\nu_2})}} \right. \tag{30}$$
$$\left. \times\, U(f_m^{(g)} \overline{f_m^{(g)}} f_m^{(g)} f_m^{(g)}; F_{\nu_1} F_{\nu_2})\; \mathcal{Q}^{\nu_1}(f_a : m, f_m^{(g)})\; \mathcal{Q}^{\nu_2}(f_b : m, f_m^{(g)}) \right\} .$$

The $\mathcal{Q}^\nu(f_2 : m, f_m)$ in Eq. (30) are defined by Eq. (19) and they can be calculated using X_{UU} discussed before. Therefore, the only unknown in Eq. (30) is the $SU(\Omega)$ Racah coefficient $U(f_m^{(g)} \overline{f_m^{(g)}} f_m^{(g)} f_m^{(g)}; F_{\nu_1} F_{\nu_2})$. To the best of our knowledge, formulas or a viable numerical procedure for evaluating this type of Racah coefficients is not available in literature. As understanding the shape of the eigenvalue density for EGUE(2)-$SU(r)$ for any $r > 1$ rests on these Racah coefficients, certainly efforts should be made to derive formulas (exact or asymptotic or purely a numerical procedure) for these. This is one of the most important open problem in the subject of EE.

5. Binary Correlation Approximation

Bivariate moments Σ_{pq} for $p + q$ up to a large value are needed to derive the form of the two-point function for eigenvalues and thereby determine properties of level fluctuations. Formulas for these higher order moments are not available for any r. It is unlikely that exact formulas for U-coefficients will help in solving this problem and it appears that asymptotic methods (methods giving directly results valid in the asymptotic limit) need to be developed for further analysis of EGUE(2)-$SU(r)$. One asymptotic method originally used by Wigner for GOE and found to be useful [13] for spinless EGOE(2) [and also for EGOE(k), $k \geq 2$] is the so called binary correlation approximation (BCA). As an example for asymptotic results, we will briefly discuss here some results for the bivariate moments of the transition strength density $I_{biv:O} = I(E_f)|\langle E_f \mid O \mid E_i\rangle|^2 I(E_i)$, generated by a transition operator O, for a system of spinless fermions in two different orbits with the Hamiltonian preserving the number of fermions in each orbit [with a suitable interpretation, this has applications to beta decay and neutrinoless double-beta decay nuclear transition matrix elements (NTME)]. Note that $I(E)$ are eigenvalue densities. We are hopeful that the results presented in this Section will prompt development of asymptotic methods for EGUE and EGOE with Lie symmetries.

Let us consider $N_1 + N_2$ number of single particle (sp) states forming two orbits with #1 orbit having N_1 sp states and #2 orbit has N_2 sp states. Say there are m_1 number of spinless fermions in the N_1 sp states and m_2 number of spinless fermions in the N_2 sp states. Also, say the Hamiltonian $H(k_H)$ is a k_H-body operator and the transition operator $O(k_O)$ changes k_O

number of fermions from space #2 to fermions in space #1. Then,

$$H(k_H) = \sum_{i+j=k_H;\alpha,\beta,\gamma,\delta} \left[v_H^{\alpha\beta\gamma\delta}(i,j)\right] \alpha_1^\dagger(i)\beta_1(i)\gamma_2^\dagger(j)\delta_2(j)\,, \;\; O(k_O) = \sum_{\gamma,\delta} v_O^{\gamma\delta}(k_O)\gamma_1^\dagger(k_O)\delta_2(k_O)\,. \tag{31}$$

In general $\alpha_k^\dagger(l)$ is normalized l-particle creation operator in space $\#k$ and $\alpha_k(l)$ is the corresponding annihilation operator. It is important to note that $H(k_H)$ preserves (m_1, m_2) but $O(k_O)$ does not; $O(k_O)\,|m_1, m_2\rangle = |m_1 + k_O, m_2 - k_O\rangle$. The bivariate moments of $I_{biv:O}$ are defined by

$$\widetilde{M}_{PQ}(m_1, m_2) = \left\langle O^\dagger(k_O)[H(k_H)]^Q O(k_O)[H(k_H)]^P \right\rangle^{m_1,m_2} \,. \tag{32}$$

To proceed further, $H(k_H)$ and $O(k_O)$ in (m_1, m_2) spaces are represented by independent EGOEs. Then, the v_H and the v_O matrix elements in Eq. (31) are independent zero centered Gaussian variables with

$$\overline{\left[v_H^{\alpha\beta\gamma\delta}(i,j)\right]^2} = v_H^2(i,j)\,, \quad \overline{\left[v_O^{\gamma\delta}\right]^2} = v_O^2\,. \tag{33}$$

It is useful to note that the EGOE ensemble for H carries $U(N_1 + N_2) \supset U(N_1) \oplus U(N_2)$ direct sum subalgebra symmetry ($\oplus$ denotes direct sum) with $U(N_1)$ and $U(N_2)$ generating m_1 and m_2 respectively. Also, the $H(k_H)$ matrix in a given (m_1, m_2) space is a matrix with dimension $\binom{N_1}{m_1}\binom{N_2}{m_2}$. However, the O matrix will be a rectangular matrix with $\langle m_1 + k_O, m_2 - k_O, \beta \mid O \mid m_1, m_2, \alpha\rangle$ matrix elements (α and β are additional labels needed for specifying the states completely). Asymptotic formulas for the ensemble average (average over both H and O ensembles) of $\widetilde{M}_{PQ}(m_1, m_2)$ are derived using BCA (assumed is $k_H, k_O << m_1$ and $k_H, k_O << m_2$). Substituting in Eq. (32), the forms for H and O from Eq. (31), we will have terms containing ensemble averages of products of v_H^{----} and V_O^{--} and (m_1, m_2) space matrix elements of products of creation and annihilation operators. In BCA, one retains terms that involve only squares of the defining space matrix elements v_H^{----} and V_O^{--}. Simplifying these terms using the independent Gaussian property of these variables and Eq. (33), formulas for $\widetilde{M}_{PQ}(m_1, m_2)$ with $P + Q \leq 4$ are derived. Final results are [23, 24],

$$\begin{aligned}
\widetilde{M}_{00}(m_1, m_2) &= \overline{\langle O^\dagger(k_O)O(k_O)\rangle^{m_1,m_2}} = v_O^2 \tbinom{\widetilde{m}_1}{k_O} \tbinom{m_2}{k_O}\,, \\
\widetilde{M}_{11}(m_1, m_2) &= v_O^2 \sum_{i+j=k_H} v_H^2(i,j) \binom{\widetilde{m}_1 - i}{k_O} \binom{m_2 - j}{k_O} T(m_1, N_1, i) T(m_2, N_2, j)\,, \\
\widetilde{M}_{20}(m_1, m_2) &= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^2(k_H)\rangle^{m_1,m_2}}\,, \\
\widetilde{M}_{02}(m_1, m_2) &= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^2(k_H)\rangle^{m_1+k_O,m_2-k_O}}\,, \\
\widetilde{M}_{40}(m_1, m_2) &= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^4(k_H)\rangle^{m_1,m_2}}\,, \\
\widetilde{M}_{04}(m_1, m_2) &= \widetilde{M}_{00}(m_1, m_2) \overline{\langle H^4(k_H)\rangle^{m_1+k_O,m_2-k_O}}\,, \\
\widetilde{M}_{31}(m_1, m_2) &= 2\overline{\langle H^2(k_H)\rangle^{m_1,m_2}} \widetilde{M}_{11}(m_1, m_2) + v_O^2 \sum_{i+j=k_H, t+u=k_H} v_H^2(i,j) v_H^2(t,u) \\
&\times \tbinom{m_2 - j}{k_O} \tbinom{\widetilde{m}_1 - i}{k_O} F(m_1, N_1, i, t) F(m_2, N_2, j, u)\,,
\end{aligned}$$

$$
\begin{aligned}
\widetilde{M}_{13}(m_1,m_2) &= 2\overline{\langle H^2(k_H)\rangle^{m_1+k_O,m_2-k_O}}\widetilde{M}_{11}(m_1,m_2) + v_O^2 \sum_{i+j=k_H,t+u=k_H} v_H^2(i,j)v_H^2(t,u) \\
&\times\ G(t,u)\tbinom{\widetilde{m}_1-k_O-t+i}{i}\tbinom{m_1+k_O-t}{i}\tbinom{\widetilde{m}_2-u+k_O+j}{j}\tbinom{m_2-k_O-u}{j}\ , \\
\widetilde{M}_{22}(m_1,m_2) &= \widetilde{M}_{00}(m_1,m_2)\overline{\langle H^2(k_H)\rangle^{m_1+k_O,m_2-k_O}}\ \overline{\langle H^2(k_H)\rangle^{m_1,m_2}} \\
&+\ v_O^2 \sum_{i+j=k_H,t+u=k_H} v_H^2(i,j)v_H^2(t,u)\binom{\widetilde{m}_1-i-t}{k_O}\binom{m_2-u-j}{k_O} \\
&\times\ [F(m_1,N_1,i,t)F(m_2,N_2,j,u) \\
&+\ T(m_1,N_1,i)T(m_2,N_2,j)T(m_1,N_1,t)T(m_2,N_2,u)]\ ,
\end{aligned}
\tag{34}
$$

with,

$$
\begin{aligned}
T(m,N,k_H) &= \tbinom{m}{k_H}\left[\tbinom{\widetilde{m}+k_H}{k_H}+1\right]\ , \\
F(m,N,k_H,k_G) &= \tbinom{m-k_H}{k_G}\tbinom{m}{k_H}\tbinom{N}{k_H}\tbinom{N}{k_G}\ , \\
G(t,u) &= \tbinom{\widetilde{m}_1-t}{k_O}\tbinom{m_2-u}{k_O}T(m_1,N_1,t)T(m_2,N_2,u)\ , \\
\overline{\langle H^2(k_H)\rangle^{m_1,m_2}} &= \sum_{i+j=k_H} v_H^2(i,j)T(m_1,N_1,i)T(m_2,N_2,j)\ , \\
\overline{\langle H^4(k_H)\rangle^{m_1,m_2}} &= 2\left[\sum_{i+j=k_H} v_H^2(i,j)T(m_1,N_1,i)T(m_2,N_2,j)\right]^2 \\
&+ \sum_{i+j=k_H,t+u=k_H} v_H^2(i,j)v_H^2(t,u)F(m_1,N_1,i,t)F(m_2,N_2,j,u)\ , \\
\widetilde{m} &= N-m,\quad \widetilde{m}_1 = N_1-m_1,\quad \widetilde{m}_2 = N_2-m_2\ .
\end{aligned}
\tag{35}
$$

These formulas are applied to some nuclei and it is seen that the bivariate transition strength density generated by a two-body Hamiltonian ($k_H = 2$) approaches bivariate Gaussian form for the transition operators appropriate for beta decay ($k_O = 1$) and neutrinoless double-beta decay ($k_O = 2$) [23, 24]. Finally, Eqs. (34) and (35) also show that the one-point function $\rho^{m_1,m_2}(E) = \langle\delta(H-E)\rangle^{m_1,m_2}$ will be asymptotically a Gaussian for $H = H(2)$ and the corresponding EE here is EGOE(2)-$[U(N_1)\oplus U(N_2)]$.

6. Future outlook: Further open questions

Although some advances are made in deriving analytical results for certain quantities using group theory and BCA, mathematical tractability of embedded ensembles remains a challenge. We are hopeful that the present paper will prompt new efforts in developing further Wigner-Racah algebra for general $U(\Omega)$ algebras so that one may obtain a better understanding of one- and two-point functions for EE with Lie symmetries. Besides this, another important future direction is to develop further BCA or develop some other appropriate asymptotic method for

EGEs with symmetries such as $U(\Omega) \otimes SU(r)$. In this quest, the so called p-expansion method sketched by French [25] may need re-examination and further exploration. Certainly, asymptotic results for fermions with spin, for spinless boson systems and for boson systems with spin (single boson spin $\mathbf{s} = \frac{1}{2}$ and 1) will be interesting and may prove to be useful in mesoscopic physics and in the studies of ultra-cold atoms.

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References

[1] Mehta M L 2004 *Random Matrices* (Netherlands: Elsevier B.V.)
[2] Haake F 2010 *Quantum signatures of Chaos* (Heidelberg: Springer-Verlag)
[3] Akemann G, Baik J and Di Francesco P 2011 *The Oxford Handbook of Random Matrix Theory* (Oxford: Oxford University Press)
[4] Couillet R and Debbah M 2012 *Random Matrix Methods for Wireless Communications* (New York: Cambridge University Press)
[5] Weidenmüller H A, private communication to V.K.B. Kota
[6] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
[7] Kota V K B 2001 *Phys. Rep.* **347** 223
[8] Papenbrock T and Weidenmüller H A 2007 *Rev. Mod. Phys.* **79** 997
[9] Gómez J M G, Kar K, Kota V K B, Molina R A, Relaño A and Retamosa J 2011 *Phys. Rep.* **499** 103
[10] Vyas Manan, Kota V K B and Chavda N D 2010 Phys. Rev. E **81** 036212
[11] Vyas Manan, Chavda N D, Kota V K B and Potbhare V 2012 *J. Phys. A: Math. Theor.* **45** 265203
[12] Deota H, Chavda N D, Kota V K B, Potbhare V and Vyas Manan 2013 *Phys. Rev. E* **88** 022130
[13] Mon K K and French J B 1975 *Ann. Phys. (N.Y.)* **95** 90
[14] Benet L, Rupp T and Weidenmüller H A 2001 *Ann. Phys. (N.Y.)* **292** 67
[15] Agasa T, Benet L, Rupp T and Weidenmüller H A 2002 *Ann. Phys. (N.Y.)* **298** 229
[16] Pluhař Z and Weidenmüller H A 2002 *Ann. Phys. (N.Y)* **297** 344
[17] Kota V K B 2005 *J. Math. Phys.* **46** 033514
[18] Kota V K B 2007 *J. Math. Phys.* **48** 053304
[19] Vyas Manan and Kota V K B 2010 *Ann. Phys. (N.Y.)* **325** 2451
[20] Vyas Manan and Kota V K B 2012 *J. Math. Phys.* **53** 123303
[21] Hecht K T and Draayer J P 1974 *Nucl. Phys.* A **223** 285
[22] Hecht K T 1974 *J. Math. Phys.* **15** 2148
[23] Vyas Manan and Kota V K B (2011) *Preprint* nucl-th/1106.0395
[24] Vyas Manan 2011 *Some studies on two-body random matrix ensembles* (Ph. D. thesis, M. S. University of Baroda, Vadodara, India)
[25] French J B 1980 *Theory and Applications of Moment Methods in Many Fermion Systems*, eds B J Dalton, S M Grimes, J P Vary and S A Williams (New York: Plenum) pp 91