An Introduction to Geometric Algebra with some Preliminary Thoughts on the Geometric Meaning of Quantum Mechanics

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Abstract. It is still a great riddle to me why Wolfgang Pauli and P.A.M. Dirac had not fully grasped the meaning of their own mathematical constructions. They invented magnificent, fantastic and very important mathematical features of modern physics, but they only delivered half of the interpretations of their own inventions. Of course, Pauli matrices and Dirac matrices represent operators, which Pauli and Dirac discussed in length. But this is only part of the true meaning behind them, as the non-commutative ideas of Grassmann, Clifford, Hamilton and Cartan allow a second, very far reaching interpretation of Pauli and Dirac matrices.
An introduction to this alternative interpretation will be discussed. Some applications of this view on Pauli and Dirac matrices are given, e.g. a geometric algebra picture of the plane wave solution of the Maxwell equation, a geometric algebra picture of special relativity, a toy model of SU(3) symmetry, and some only very preliminary thoughts about a possible geometric meaning of quantum mechanics.

1. Aim of this paper

"The neglect of exterior algebra is the mathematical tragedy of this century. (…) meanwhile, we have to bear with mathematicians who are exterior algebra-blind," tells us Gian-Carlo Rota [1]. Consequently mathematics teachers and their students at schools and universities are exterior algebra-blind, too. Even today nobody tells them how to deal with exterior algebraic structures. It is the aim of this paper, to change this sad truth a little bit.

The remedy Hestenes and other scientists doing physics and mathematics education research propose against exterior algebra-blindness is called *Geometric Algebra*. Thus this paper gives first and foremost a short introduction to Geometric Algebra. But as the author is spelled by the aesthetical and conceptual strength of Geometric Algebra, some more speculative thoughts about Geometric Algebra will be discussed in the second part of this paper.

2. The space we all live in

Obviously, we live in three-dimensional space: We can go to the left or to the right (x direction), forwards or backwards (y direction), and upwards or downwards (z direction). There is no forth spatial direction we can move into. When moving around or moving our arms we feel and experience just three directions.

But there is a severe problem with the construction of our sensory system. Most of our perceptions we use to characterize and interpret the world around us are visual perceptions. We see with our eyes.

But the light, which carries the information about our three-dimensional surroundings, is absorbed by the retina, which is a two-dimensional, curved surface. Thus the three-dimensional objects outside us are represented as two-dimensional images in our eyes. Neurons then conduct this two-dimensional information into our brain which tries to reconstruct the three-dimensional objects around us. But as only information about a two-dimensional distorted picture reaches our brain, the visual model our brain constructs might be incomplete, unsuited for some situations or even faulty. And if the visual models our brain delivers are not correct, the mathematical models we build on these visual models might be partly wrong too.

There are strong hints that the mathematical models of sighted persons are indeed deficient. As knot theorist Alexei Sossinsky remarks, the most complicated knots were invented by blind mathematicians [2]. And "almost all blind mathematicians are (or were) geometers. (…) A blind person's spatial intuition is primarily the result of tactile and operational experience. It is deeper – in the literal as well as the metaphorical sense" [2].

Children are not able to interpret two-dimensional pictures in a way we do as grown-ups. They can not distinguish properly between illuminated convex or concave objects [3]. They have to learn it. In a similar way we all are not able to interpret rotations correctly. We have not learned it. Usually we all suppose objects to be in the same topological state after a rotation of 2π. But this is wrong. An object attached to its surroundings by some strings exhibits a rotation symmetry of 4π which we do not re-

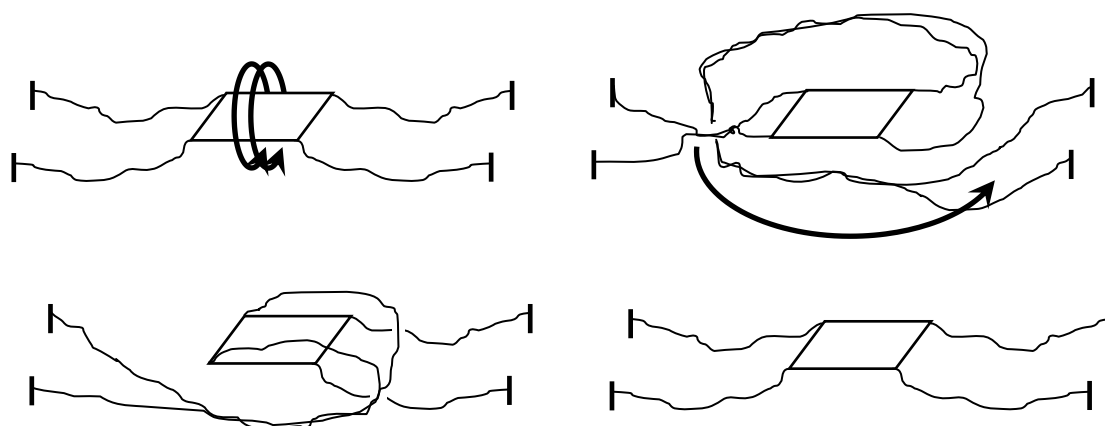**Figure 1.** Visualization of the Dirac belt trick.

cognize due to our visual deficiencies. Thus such an object has a spin of ½. Dirac's belt trick [4], [5], [6], [7], [8] clearly demonstrates this fact (see figure 1). We therefore have to draw two important conclusions:

- 4π symmetries are nothing exclusively quantum mechanical.
- 4π symmetries are an essential part of our everyday world (our space) we live in.
- We need an appropriate mathematical language to describe 4π symmetries.

This language is Clifford Algebra. When applied to didactical situations in physics or other sciences Clifford Algebra $C\ell_{3,0}$ of three-dimensional Euclidean space is called Geometric Algebra [9], and Clifford Algebra $C\ell_{1,3}$ of four-dimensional pseudo-Euclidean spacetime is called Spacetime Algebra [10], [11].

3. Geometric algebra is simple

"Much of Clifford algebra is quite simple" [12] or even "simple-minded" (see page XIII at the introduction of [7]). It is even so simple and easy to understand, that Geometric Algebra "can be explained to the first person you meet in the street" (page 820 of [13]). Its basic features are described in

many introductory books like [14], [15], [16], [17], [18], [19], [20] from a more or less university level perspective. But to my astonishment even university professors who should be competent with modeling geometric situations mathematically confess that they were "finding it so difficult to understand GA" (see page VII at the introduction of [17]). Therefore I decided to present the basic foundations of Geometric Algebra in this paper in a way suitable for school lessons. I myself taught Geometric Algebra and Spacetime Algebra at physics lessons of upper secondary level in school [21], [22]. And the results of the examinations were indeed encouraging [23].

The first step at these lessons has been to introduce oriented line elements (vectors) and oriented area elements (bivectors) with the aim to immediately hit the algebraic heart of Geometric Algebra: anti-commutativity. A simple discussion of the geometrical situation directly reveals the central idea of Geometric Algebra (see figure 2).
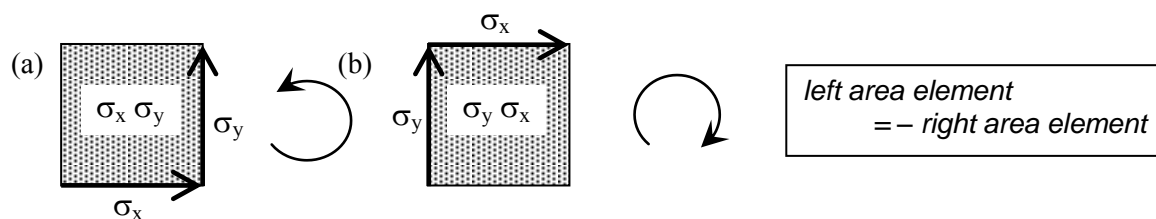

Figure 2. Visualization of the geometric product of two base vectors as base bivector.

If two different vectors are multiplied, we get an oriented area element. This multiplication is called geometric product (or Clifford product if you like) and contains the complete geometric information about the relation of the two vectors [11], [15]. Thus the product of two base vectors results in a base bivector, which represents an oriented unit square. The orientation then of course depends on the order of the multiplication. If we first make a step into the direction of the x-axis and then a step into the direction of the y-axis (figure 2a), we will get a positive or counter-clockwise orientation (which equals the direction of a traffic circle on the continent). If we first make a step into the direction of the y-axis instead and then a step into the direction of the x-axis (figure 2b), we will get a negative or clockwise orientation (which equals the direction of a ghost driver in a traffic circle). This change of orientation is codified in the negative signs of equations (1).

$$\sigma_x \sigma_y = -\sigma_y \sigma_x \qquad \sigma_y \sigma_z = -\sigma_z \sigma_y \qquad \sigma_z \sigma_x = -\sigma_x \sigma_z \qquad (1)$$

Together with the condition, that unit vectors square to one

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \qquad (2)$$

these equations form the algebraic core of Geometric Algebra. This is indeed rather simple, and Parra Serra concludes in his didactical analysis of Geometric Algebra: "The only rules to remember are that different orthogonal generating units (vectors) anticommute and that their square is +1…"[1] [13].

When I read for the first time about these rules in a paper from Cambridge, a bizarre announcement followed: "We have now reached the point which is liable to cause the greatest intellectual shock" (see page 1184 of [24]). And it was indeed a severe shock which influences my scientific work till today: These rules define Pauli algebra. Therefore we are forced to conclude: **Pauli matrices represent base vectors of three-dimensional Euclidean space.**

And with the help of the oriented unit volume element (or base trivector or pseudoscalar) I

$$I = \sigma_x \sigma_y \sigma_z \qquad I^2 = (\sigma_x \sigma_y \sigma_z)^2 = -1 , \qquad (3)$$

which acts as an imaginary unit, the Pauli relations can be rediscovered in standard form:

$$\sigma_x \sigma_y = I \sigma_z \qquad \sigma_y \sigma_z = I \sigma_x \qquad \sigma_z \sigma_x = I \sigma_y . \qquad (4)$$

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[1] …or in spacetime −1 for timelike vectors or 0 in the case of light rays, which will be discussed later.

Every vector r can then be written as linear combination of the Pauli matrices σ_x, σ_y, and σ_z, a fact already noticed by Cartan (see page 43 of [25]):

$$r = x\,\sigma_x + y\,\sigma_y + z\,\sigma_z \;. \tag{5}$$

Following the idea to describe all "geometric objects as real linear combinations of the generating units and their geometric products" (see page 823 of [13]) the most general mathematical object possible in Pauli algebra can be written as multivector

$$M = k + x\,\sigma_x + y\,\sigma_y + z\,\sigma_z + A_{xy}\,\sigma_x\sigma_y + A_{yz}\,\sigma_y\sigma_z + A_{zx}\,\sigma_z\sigma_x + V\,\sigma_x\sigma_y\sigma_z$$

$$M = (k + V\,I) + (x + A_{yz}\,I)\,\sigma_x + (y + A_{zx}\,I)\,\sigma_y + (z + A_{xy}\,I)\,\sigma_z \;. \tag{6}$$

The second version of multivector (6) is called paravector [26], because the real coefficients k, x, y, z, A_{xy}, A_{yz}, A_{zx}, V $\in \mathbb{R}$ are grouped to form a complex-like vectorial structure.

4. The double nature of Pauli matrices

Pauli invented Pauli matrices to model the strange behaviour of electrons in 1927 (page 608 of [27]). He never considered them to represent base vectors. Pauli always interpreted his matrices as operators (or as a substantial part of an operator) and used the sandwich product $\mathbf{P = S\,p\,S^{-1}}$ (page 611 of [27]) in his calculations.

Of course this is right, too. Pauli matrices possess a double nature: They can be interpreted as operands (vectors, on which is acted on) as well as operators which act on vectors. As $\sigma_i = \sigma_i^{-1}$ the sandwich product of different Pauli matrices with a vector r results in:

$$r_1 = \sigma_x\,r\,\sigma_x = \quad x\,\sigma_x - y\,\sigma_y - z\,\sigma_z \tag{7}$$

$$r_2 = \sigma_y\,r\,\sigma_y = -\,x\,\sigma_x + y\,\sigma_y - z\,\sigma_z \tag{8}$$

$$r_3 = \sigma_z\,r\,\sigma_z = -\,x\,\sigma_x - y\,\sigma_y + z\,\sigma_z \;. \tag{9}$$

These are reflections: vector r_3 of equation (9) equals the original vector r reflected at the z-axis. If we multiply a vector r from the left and from the right by a base vector, the vector will be reflected at the axis which points in the direction of the base vector. **Pauli matrices represent base reflection in three-dimensional Euclidean space.**

This can be generalized for arbitrary directions. If we multiply a vector r from the left and from the right by a unit vector n, the vector r will be reflected at the axis which points in the direction of the unit vector n:

$$r_{ref} = n\,r\,n \qquad \text{with} \quad n^2 = (n_x\,\sigma_x + n_y\,\sigma_y + n_z\,\sigma_z)^2 = 1 \;. \tag{10}$$

The two interpretations of figure 3 form the geometric core of Geometric Algebra.

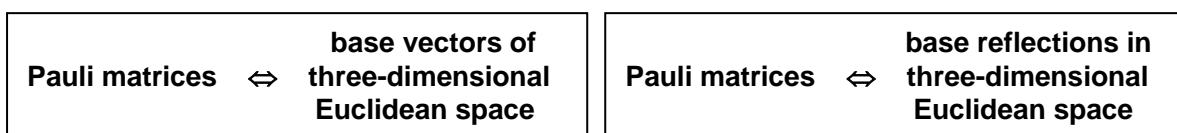| Pauli matrices	⇔	base vectors of three-dimensional Euclidean space		Pauli matrices	⇔	base reflections in three-dimensional Euclidean space

Figure 3. Central ideas of Geometric Algebra.

It is well known that two succeeding reflections always result in a rotation. Or the other way round: every rotation can be split into two (or more even numbered) reflections (see pages 1137/1138 of [5] and pages 45/46 of [25]). Therefore rotations can be modeled by two reflections:

$$r_{rot} = m\,r_{ref}\,m = m\,n\,r\,n\,m \qquad \text{with} \quad m^2 = n^2 = 1 \;. \tag{11}$$

If we multiply a vector r from the left and from the right first by a unit vector n and then multiply the result from the left and from the right by another unit vector m, the vector r will be rotated in the plane spanned by the two unit vectors n and m through twice the angle between the two unit vectors n and m (see page 111 of [9]).

This is extraordinary. Please notice the many exclamation marks Doran and Lasenby use when discussing these points: "This is starting to look extremely simple!" (page 43 of [15]). And: "The rule also works for any grade of multivector!" (page 44 of [15]). Modeling rotations in Geometric Algebra is indeed a complete conceptual highlight and much easier than in other mathematical systems.

5. Historical background: a missed opportunity

Geometric Algebra goes back to the theory of extensions, first formulated by Grassmann in 1844. Unfortunately, Grassmann's theory was written in an obscure mathematical language nearly nobody understood at this time. It seems that only two eminent mathematicians really noticed, what Grassmann had written: Hamilton, the inventor of quaternion theory, and Charles Sanders Peirce.

Dyson describes the reason for this tremendous ignorance of nearly all mathematicians: "Grassmann was an obscure high-school teacher in Stettin, while Hamilton was the world-famous mathematician whose official titles occupy six lines of print after his name (…). So it is regrettable, but not surprising, that quaternions were hailed as a great discovery, while Grassmann had to wait 23 years before his work received any recognition at all from professional mathematicians" (see page 644 of [28]). And Peirce, fully aware of the importance of Grassmann's contribution, translated Grassmann's ideas into his own mathematical language [29], which again nearly nobody understood at this time – except Clifford, who finally succeeded in writing down [30] all these ideas in a language comprehensible by his contemporaries.

In his short, but important paper Peirce clearly states that "in truth, Grassmann has got hold (though he did not say so) of an eightfold algebra" [29], just the algebra of the multivector (6). So it was totally clear in 1877 that the basic entities of Grassmann's algebra are "three rectangular vectors …, three rectangular planes …, one solid …, unity" [29].

In his Josiah Willard Gibbs Lecture [28] Dyson identifies the delay in the reception of Grassmann's ideas as one of several missed opportunities in the history of mathematics. According to him the central question should have been: "How can it happen that the properties of three-dimensional space are represented equally well by two quite different and incompatible algebraic structures?" (page 644 of [28]). Unfortunately the answer Wolfgang Pauli and P.A.M. Dirac gave to this question partly neglected the geometrical meaning. They never fully grasped the meaning of their own mathematical constructions, and they only delivered half of the interpretations of their inventions. In his Oersted Medal Lecture David Hestenes urgently points to this failure: "The most basic of these misconceptions is that the Pauli matrices are intrinsically related to spin. On the contrary, I claim that their physical significance is derived solely from their correspondence with orthogonal directions in space" (p. 115 of [9]).

This is especially sad in the case of the great Dirac, because it seems that he knew about these ideas and renounced them due to aesthetical considerations. He writes: "In this way the physical world is put into correspondence with the scheme of bi-quaternions, instead of with the scheme of quaternions. Now the scheme of bi-quaternions is not of any special interest in mathematical theory" (page 261 of [31]). This lack of special mathematical interest convinced Dirac to follow another route and construct a very interesting, but also very complicated and sort of conformal connection of quaternions with physical reality [31]. And he writes: "Also (bi-quaternions) have eight components, which is rather too many for a simple scheme for describing quantities in space-time" (page 261 of [31]). Here Dirac failed twice: Eight components are not too many but too few for a scheme for describing quantities in four-dimensional spacetime with necessarily $2^4 = 16$ geometric base elements. And eight components are just the right number of components to describe three-dimensional space, as already Peirce has noticed.

Even today the missed opportunity Dyson identified has an effect on the work of scientists. For example field medalist Michael Atiyah writes: "The geometric significance of spinors is still very

mysterious. Unlike differential forms, which are related to areas and volumes, spinors have no such simple explanation. They appear out of some slick algebra, but the geometrical meaning is obscure" (see pages 113/114 of [32]).

These words are astonishing, and we should compare them with the conclusion David Hestenes published 17 years earlier: "Thus … every spinor uniquely determines a rotation dilatation. Therein lies the operational geometric significance of spinors. The spinors form a subalgebra of Geometric Algebra. The algebra of spinors was discovered independent of the full Geometric Algebra by Hamilton, who gave it the name quaternion algebra. Some readers may want to say, rather, that we have here two isomorphic algebras, but there is no call for any such distinction. A quaternion is a spinor. The identification of quaternions with spinors is fully justified not only because they have equivalent algebraic properties, but more important, because they have the same geometric significance" (see page 1022 of [33]).

An example of the ignorance with respect to such a picture of spinors can be found when solving the vacuum Maxwell equation. Even before the re-formulation of Geometric Algebra by David Hestenes in 1966 [34] it was known (but widely ignored) that the spinorial plane wave solution of the Maxwell equations results in circular polarized electromagnetic waves (see page 14 of [35]). This is shown in the following section.

6. Geometric Algebra calculus & plane electromagnetic waves

In three dimensions geometric calculus builds on the geometric derivative

$$\nabla = \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z} \; . \tag{12}$$

This operator acts on multivectors. In electrodynamics we consider especially the multivector which represents the electromagnetic field strength. As usual the electric field E and the magnetic field B can be combined with the help of an imaginary unit (see page 114 of [9], page 1025 of [33], and page 509 of [36]) into the "complex Faraday vector" F

$$F = E + I\,B = E + \sigma_x \sigma_y \sigma_z\, B \; . \tag{13}$$

But this electromagnetic field strength F should not be looked upon with algebraic eyes only, as the term "complex vector" totally disguises the geometric meaning behind electric and magnetic fields. In Geometric Algebra the electric field strength E is considered as a vector while the magnetic field strength is considered as a bivector I B. Thus the electromagnetic field strength F is always a sum of a vector and an oriented area element.

These two basic ingredients (12) and (13) are all the ingredients needed to dramatically condense the four Maxwell equations into one single Maxwell equation (see again [9], [33], [36])

$$\left(\frac{\partial}{c\,\partial t} + \nabla \right) F = \rho - \frac{1}{c}\,J \qquad \text{or in empty space:} \qquad \left(\frac{\partial}{c\,\partial t} + \nabla \right) F = 0 \; . \tag{14}$$

Instead of four rather unconnected algebraic equations we now possess one equation which expresses all the relevant physical information geometrically as scalar part, vector part, bivector part and trivector part. The great didactical advantage of this geometric interpretation becomes clear when we substitute the purely oscillatory trial solution

$$F = f\,e^{I(\omega t - k z)} \tag{15}$$

for a monochromatic plane wave (with frequency $\omega > 0$ travelling parallel to the z-axis) into the vacuum Maxwell equation (14) and solve it for f. The constant factor f must not be a scalar. According to equation (13) f has again to be the sum of a vector and a bivector, which indicates the electric and the magnetic fields of the origin at time t = 0.

Inserting equation (15) into equation (14) results after some straightforward mathematical steps in the following simple intermediate equation (see page 1026 of [33]):

$$F = \sigma_z \, F \qquad \text{or} \qquad E + \sigma_x \sigma_y \sigma_z \, B = \sigma_z \, E + \sigma_x \sigma_y \, B \,. \qquad (16)$$

The different terms of this equation can now be split into its geometrical constituents with the help of the inner and the outer products. The inner product of two vectors a and b or of a bivector **a** and a vector b is defined as:

$$a \bullet b = \frac{1}{2} (ab + ba) \qquad \mathbf{a} \bullet b = \frac{1}{2} (\mathbf{a}b - b\mathbf{a}) \,. \qquad (17)$$

It reduces the geometrical grades of the constituents: While a and b are vectors, $a \bullet b$ is a scalar (see left equation of (17)). While **a** is a bivector and b is a vector, $\mathbf{a} \bullet b$ is a vector (see right equation of (17)). The outer product of two vectors a and b or of a bivector **a** and a vector b is defined as:

$$a \wedge b = \frac{1}{2} (ab - ba) \qquad \mathbf{a} \wedge b = \frac{1}{2} (\mathbf{a}b + b\mathbf{a}) \,. \qquad (18)$$

It increases the geometrical grades of the constituents: While a and b are vectors, $a \wedge b$ is a bivector (left equation of (18)). While **a** is a bivector and b is a vector, $\mathbf{a} \wedge b$ is a trivector or pseudoscalar (right equation of (18)). And please note the interesting variation of signs in equations (17) and (18). It really depends on the geometrical grades of the factors whether a product is symmetric or antisymmetric!

The sum of inner and outer products now form the geometric product

$$ab = a \bullet b + a \wedge b \qquad \mathbf{a}b = \mathbf{a} \bullet b + \mathbf{a} \wedge b \qquad (19)$$

This is indeed strong stuff, and Sobczyk still today remembers the moment, when he first heard about these products: "I remember my sense of amazement when he (David Hestenes) wrote down the basic identity for the geometric multiplication of vectors … Why hadn't I ever heard of this striking product, and why hadn't I ever heard of a bivector or directed plane segment, since it was the natural generalization of a vector. Twenty-five years later I still find myself asking these same questions" (see page 1291 of [37]).

Now we are able to evaluate equation (16):

$$E + \sigma_x \sigma_y \sigma_z \, B = \sigma_z \bullet E + \sigma_z \wedge E + \sigma_x \sigma_y \bullet B + \sigma_x \sigma_y \wedge B \qquad (20)$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$

vector bivector   scalar bivector   vector   trivector

The scalar part of this equation

$$\sigma_z \bullet E = 0 \qquad (21)$$

tells us that the direction of propagation σ_z and the direction of the electric field vector E are perpendicular. The trivector part of this equation

$$\sigma_x \sigma_y \wedge B = 0 \qquad (22)$$

tells us that the conventional magnetic field vector B is parallel to the xy-plane. Therefore the magnetic bivector I B is perpendicular to this plane. The vector and bivector equations

$$E = \sigma_x \sigma_y \bullet B \qquad \text{and} \qquad \sigma_x \sigma_y \sigma_z \, B = \sigma_z \wedge E \qquad (23)$$

both allow us to eliminate the magnetic field because they both result in

$$\sigma_z \, E = \sigma_x \sigma_y \sigma_z \, B = I \, B \qquad (24)$$

and therefore

$$F = (1 + \sigma_z) \, E = E \, (1 - \sigma_z) \,. \qquad (25)$$

If the initial condition $E = E_0\,\sigma_x$ holds at $z = 0$ and $t = 0$, then the plane wave solution of the Maxwell equation in empty space is

$$F_{(t,z)} = E_0\,\sigma_x\,(1+\sigma_z)\,e^{I(\omega t - kz)} = E_0\,(\sigma_x - \sigma_z\sigma_x)\,[\cos(\omega t - kz) + I\sin(\omega t - kz)] \qquad (26)$$

which is shown in figure 4 (see also figure 2 at page 510 of [36]).
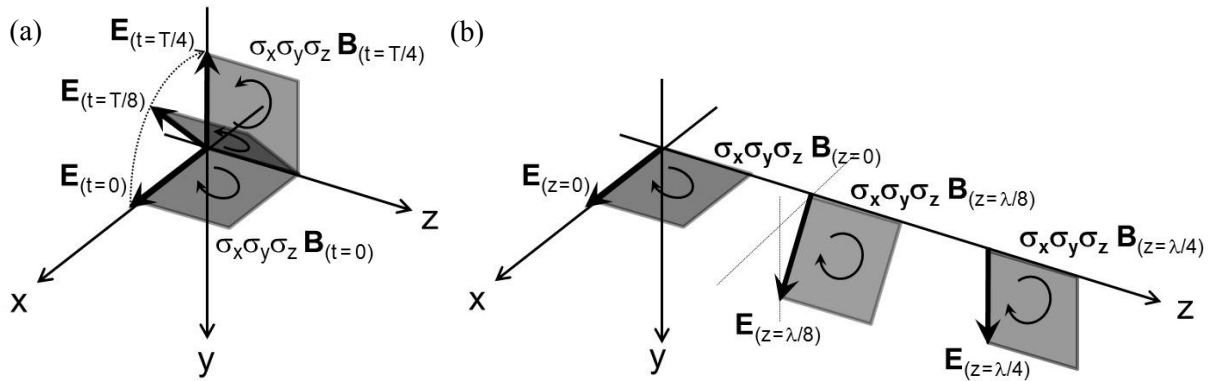

Figure 4. Graphic representations of the plane wave solution of Maxwell's equation in empty space (a) at $z = 0$ and (b) at $t = 0$.

The plane wave solution given in standard textbooks showing linear polarized electromagnetic waves is thus an artificial algebraic solution which neglects the geometric structure of the space we live in. It can only be constructed by a superposition of left and right polarized electromagnetic waves (see also page 2027 of [33]).

7. Four-dimensional spacetimes & special relativity

Obviously, we live in a three-dimensional spatial world. But we only lived there till the invention of special relativity by Einstein in 1905. Since 1905 we live in a four-dimensional spacetime world. This world can be modeled in Geometric Algebra by increasing the number of base vectors – and by changing their quality.

Starting at the origin of a coordinate system light travels the distance x during the time interval $c\,t$ which can be expressed by the spacetime interval

$$+ (c\,t)^2 - x^2 = 0\,. \qquad (27)$$

This is a disturbing equation! The theory of relativity claims that space and time can be treated structurally in the same way which according to Minkowski calls for an "identical treatment of the four coordinates x, y, z, t" (page 83 of [38]). But space and time are not treated identically here. There is a positive sign in front of the time interval square and a negative sign in front of the space interval square in equation (27). Of course we cannot get rid of these different signs. But we can hide them. To do this, we write equation (27) with the use of two base vectors:

$$+ (c\,t\,\gamma_t)^2 + (x\,\gamma_x)^2 = 0\,. \qquad (28)$$

Now we have identical signs in front of space and time interval squares, because we define the time-like base vector γ_t and the three space-like base vectors $\gamma_x, \gamma_y, \gamma_z$ as

$$\gamma_t^2 = 1 \qquad \gamma_x^2 = \gamma_y^2 = \gamma_z^2 = -1\,. \qquad (29)$$

Again different base vectors anti-commute:

$$\gamma_i\,\gamma_k = -\gamma_k\,\gamma_i \qquad \text{with} \quad i \neq k \quad \text{and} \quad i, k \in \{t, x, y, z\}\,. \qquad (30)$$

And we have now again "reached the point which is liable to cause the greatest intellectual shock" (page 1184 of [24]): These rules define Dirac algebra. Therefore we are forced to conclude: **Dirac matrices represent base vectors of four-dimensional spacetime.**

Every spacetime vector r can then be written as linear combination of the Dirac matrices γ_t, γ_x, γ_y, and γ_z, a fact already noticed by Cartan (page 133 of [25]):

$$r = c\,t\,\gamma_t + x\,\gamma_x + y\,\gamma_y + z\,\gamma_z \ . \tag{31}$$

Hestenes does not hide his astonishment about the path the history of mathematics has taken here: "The Dirac matrices are no more and no less than matrix representations of an orthonormal frame of spacetime vectors and thereby they characterize spacetime geometry. But how can this be? Dirac never said any such thing!" (page 694 of [10])

Dirac invented Dirac matrices to model the strange behaviour of relativistic electrons in 1928. He never considered them to represent spacetime base vectors. Dirac and Pauli always interpreted their matrices as operators (or as a substantial part of an operator). This is really surprising: clearly all the mathematics was there, and Pauli and Dirac were masters of it. But instead of following the idea of Cartan to use the spacetime vector (31), Pauli tried to model physics with the complete spacetime multivector (see lemma 5 at page 113 of [39])

$$X = \sum_{A=1}^{16} x_A \gamma_A \qquad \text{(with } \gamma_A \text{ every possible product of several Dirac matrices)} \tag{32}$$

as the most general mathematical object possible in Dirac algebra. And of course again he used sandwich products like $\gamma_\mu' = \mathbf{S}\,\gamma_\mu\,\mathbf{S^{-1}}$ (see equation 1 at page 110 of [39]) or $\gamma_\mu\,X\,\gamma_\mu = \ldots$ (page 114 of [39]) in his calculations. This makes sense, as Dirac matrices possess also a double nature: They can be interpreted as operands as well as operators which act on operands.

The sandwich products of the Dirac matrices with a spacetime vector r are:

$$r_0 = \quad \gamma_t\,r\,\gamma_t = \quad c\,t\,\gamma_t - x\,\gamma_x - y\,\gamma_y - z\,\gamma_z \tag{33}$$

$$r_1 = -\,\gamma_x\,r\,\gamma_x = -\,c\,t\,\gamma_t + x\,\gamma_x - y\,\gamma_y - z\,\gamma_z \tag{34}$$

$$r_2 = -\,\gamma_y\,r\,\gamma_y = -\,c\,t\,\gamma_t - x\,\gamma_x + y\,\gamma_y - z\,\gamma_z \tag{35}$$

$$r_3 = -\,\gamma_z\,r\,\gamma_z = -\,c\,t\,\gamma_t - x\,\gamma_x - y\,\gamma_y + z\,\gamma_z \tag{36}$$

These are spacetime reflections. In equation (33) the vector r is reflected at the time axis, and in the other equations (34) to (36) the vector r is reflected at the spatial axis. **Dirac matrices represent base reflections in four-dimensional spacetime.**

This can be generalized for arbitrary directions. If we multiply a vector r from the left and from the right by a unit vector n, the vector r will be reflected at the axis which points in the direction of the unit vector n:

$$r_{ref} = n\,r\,n^{-1} = \pm\,n\,r\,n \ . \tag{37}$$

with $n^2 = 1$ if n is a time-like unit vector ($n^{-1} = +\,n$) and $n^2 = -1$ if n is a space-like unit vector ($n^{-1} = -\,n$). The two interpretations of figure 5 form the geometric core of Spacetime Algebra.
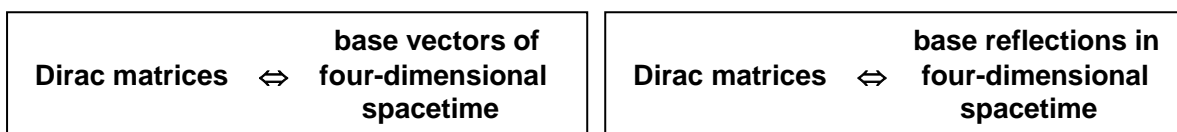

| Dirac matrices | ⇔ | base vectors of four-dimensional spacetime | Dirac matrices | ⇔ | base reflections in four-dimensional spacetime |

Figure 5. Central ideas of Spacetime Algebra.

With all that in mind, Lorentz transformations are "starting to look extremely simple" (page 43 of [15]), because they are nothing else than spacetime rotations or two successive spacetime reflections. Therefore they can be modeled with the following very easy to apply formula:

$$r_{rot} = m \, r_{ref} \, m^{-1} = \pm \, m \, n \, r \, n \, m \qquad \text{with} \quad m, n \text{ spacetime unit vectors.} \qquad (38)$$

Some Geometric Algebra examples for Lorentz transformations, special relativistic length contractions or time dilations, and other special relativistic effects which can be used for secondary school or introductory university courses can be found in [19], [21], [22], [23], [40].

8. Algebraic and geometric relations between Pauli and Dirac matrices

Usually the three Pauli matrices are considered as (2 x 2) matrices (page 44 of [25]):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (39)$$

In contrast to that Dirac matrices are considered as (4 x 4) matrices (page 278 of [15]):

$$\gamma_t = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \qquad \gamma_x = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} \qquad \gamma_y = \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix} \qquad \gamma_z = \begin{pmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{pmatrix} \qquad (40)$$

Applying the direct product of Zehfuss and Kronecker [41], [42], [43], Dirac matrices can be composed of Pauli matrices. They then have the signature $(+, -, -, -)^2$:

$$\gamma_t = \sigma_z \otimes \mathbf{1} \qquad \gamma_x = -(\sigma_z \sigma_x) \otimes \sigma_x \qquad \gamma_y = -(\sigma_z \sigma_x) \otimes \sigma_y \qquad \gamma_z = -(\sigma_z \sigma_x) \otimes \sigma_z \qquad (41)$$

The main algebraic message is: Two (2 x 2) matrices directly multiplied result in one (4 x 4) matrix. A Dirac matrix is algebraically composed of two Pauli matrices (or Pauli matrix products). Or colloquially: There are two Pauli matrices inside one Dirac matrix.

But there is a deep geometric relationship between Pauli algebra and Dirac algebra which contrasts this sort of superficial algebraic picture: The spatial base vectors of Pauli algebra have to be constructed not only by the spatial base vectors of Dirac algebra, but also by its time-like base vector, as the base vectors of an observer in three-dimensional space depend on the time direction of this observer in four-dimensional spacetime. This base-vector of time is relevant for the three base vectors of space we construct.

This can be seen, if we compare the geometric derivative of three-dimensional space (12) with the geometric derivative of four-dimensional spacetime

$$\Box = \gamma_t \, \frac{\partial}{c \, \partial t} + \gamma_x \, \frac{\partial}{\partial x} + \gamma_y \, \frac{\partial}{\partial y} + \gamma_z \, \frac{\partial}{\partial z} \; . \qquad (42)$$

Obviously the two geometric derivatives of a multivector are not equal, if we add the time direction according to the successful procedure of equation (14):

$$\left(\frac{\partial}{c \, \partial t} + \nabla \right) M \neq \Box \, M \qquad \text{but:} \quad \left(\frac{\partial}{c \, \partial t} + \nabla \right) M = \Box \, \gamma_t \, M \; . \qquad (43)$$

At the left equation of (43) the time derivatives have different grades. To adjust this, the geometric derivative (also called Dirac operator) has to be post-multiplied by the time-like base vector γ_t. This results in the balanced equation (44).

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^{2} Of course other systems of (4 x 4) matrices comply with Dirac algebra, too. For instance, Cartan proposed the following matrices as spacetime base vectors (see page 133 in connection with page 82 of [25]) with signature $(+, +, +, -)$: $A_1 = \sigma_z \otimes \sigma_x$, $A_2 = -\sigma_z \otimes \sigma_y$, $A_3 = \sigma_x \otimes \mathbf{1}$, and $A_4 = c (\sigma_z \sigma_x) \otimes \mathbf{1}$

$$\left(\frac{\partial}{c\,\partial t} + \sigma_x \frac{\partial}{\partial x} + \sigma_y \frac{\partial}{\partial y} + \sigma_z \frac{\partial}{\partial z} \right) M = \left(\frac{\partial}{c\,\partial t} + \gamma_x \gamma_t \frac{\partial}{\partial x} + \gamma_y \gamma_t \frac{\partial}{\partial y} + \gamma_z \gamma_t \frac{\partial}{\partial z} \right) M \quad (44)$$

The relation between Pauli and Dirac matrices can be found now by comparing the base vectors in front of the spatial derivatives:

$$\sigma_x = \gamma_x \gamma_t \qquad \sigma_y = \gamma_y \gamma_t \qquad \sigma_z = \gamma_z \gamma_t \qquad (45)$$

Geometry tells us, that there are not (colloquially spoken) two Pauli matrices inside one Dirac matrix. There are two Dirac matrices inside every Pauli matrix instead! Of course, the Pauli matrices of equation (45) are written as (4 x 4) matrices. Nevertheless, they totally follow the rules of Pauli algebra[3].

"Consequently, a vector in the Pauli algebra of space is a bivector in the Dirac algebra of space-time" (page 1292 of [37]). Whenever we move along a spatial line, we actually move along a spacetime area element. And this fact is so important, that Sobczyk included it as special highlight in his birthday address for David Hestenes [37].

9. A toy model of SU(3) symmetry

In a sandwich-like product of Geometric Algebra more or less the same factors have to be pre- and post-multiplied. In this section products will be discussed which have different pre- and post-multi-plication factors. As an example we try to translate the eight Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (46)$$

of SU(3) symmetry into a toy model of Geometric Algebra. To mimic the multiplication of the first Gell-Mann matrix λ_1 by a complex column vector

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x + i\,A_{yz} \\ y + i\,A_{zx} \\ z + i\,A_{xy} \end{pmatrix} = \begin{pmatrix} y + i\,A_{zx} \\ x + i\,A_{yz} \\ 0 \end{pmatrix} \quad (47)$$

we have to find a Geometric Algebra transformation of a multivector M which gets rid of the third paravector component $z + A_{xy} I$. And of course the scalar part should disappear too, because it is not part of the resulting column vector of equation (47). This can be achieved by adding the following equations:

$$\sigma_x M \sigma_y = \sigma_x \left((k + V I) + (x + A_{yz} I) \sigma_x + (y + A_{zx} I) \sigma_y + (z + A_{xy} I) \sigma_z \right) \sigma_y$$
$$= (k + V I) \sigma_x \sigma_y + (x + A_{yz} I) \sigma_y + (y + A_{zx} I) \sigma_x - (z + A_{xy} I) \sigma_x \sigma_y \sigma_z \quad (48)$$

$$\sigma_y M \sigma_x = - (k + V I) \sigma_x \sigma_y + (x + A_{yz} I) \sigma_y + (y + A_{zx} I) \sigma_x + (z + A_{xy} I) \sigma_x \sigma_y \sigma_z \quad (49)$$

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[3] In five-dimensional spacetimes or spacetimevelocities this works in a similar way. Then Pauli matrices (base vectors of three-dimensional space) can be identified as products of three Dirac matrices (base vectors of five-dimensional spacetimes or spacetimevelocities, see equations 24-26 of [44]) as $\sigma_x' = \gamma_x \gamma_t \gamma_v$, $\sigma_y' = \gamma_y \gamma_t \gamma_v$, $\sigma_z' = \gamma_z \gamma_t \gamma_v$, if together with the Dirac matrices (41) the time-like Dirac matrix $\gamma_v = \sigma_x \otimes \mathbf{1}$ (see equation 45 at page 497 of [45]) is used as additional base vector for the direction of the fifth coordinate.

This can be done for the other Gell-Mann matrices in a similar way. The results are in three-dimensional space (see following equations on the left side) and according to transformation (45) in four-dimensional spacetime (equations on the right side):

$$\lambda_1(M) = \frac{1}{2}\,(\sigma_x\, M\, \sigma_y + \sigma_y\, M\, \sigma_x) \qquad \lambda_1(M) = -\frac{1}{2}\,\gamma_t\,(\gamma_x\, M\, \gamma_y + \gamma_y\, M\, \gamma_x)\,\gamma_t \qquad (50)$$

$$\lambda_2(M) = \frac{1}{2}\,(\sigma_z\, M\, \mathbf{1} - \mathbf{1}\, M\, \sigma_z) \qquad \lambda_2(M) = -\frac{1}{2}\,\gamma_t\,(\gamma_z\, M\, \gamma_t - \gamma_t\, M\, \gamma_z)\,\gamma_t \qquad (51)$$

$$\lambda_3(M) = \frac{1}{2}\,(\sigma_x\, M\, \sigma_x - \sigma_y\, M\, \sigma_y) \qquad \lambda_3(M) = -\frac{1}{2}\,\gamma_t\,(\gamma_x\, M\, \gamma_x - \gamma_y\, M\, \gamma_y)\,\gamma_t \qquad (52)$$

$$\lambda_4(M) = \frac{1}{2}\,(\sigma_z\, M\, \sigma_x + \sigma_x\, M\, \sigma_z) \qquad \lambda_4(M) = -\frac{1}{2}\,\gamma_t\,(\gamma_z\, M\, \gamma_x + \gamma_x\, M\, \gamma_z)\,\gamma_t \qquad (53)$$

$$\lambda_5(M) = \frac{1}{2}\,(\mathbf{1}\, M\, \sigma_y - \sigma_y\, M\, \mathbf{1}) \qquad \lambda_5(M) = -\frac{1}{2}\,\gamma_t\,(\gamma_t\, M\, \gamma_y - \gamma_y\, M\, \gamma_t)\,\gamma_t \qquad (54)$$

$$\lambda_6(M) = \frac{1}{2}\,(\sigma_y\, M\, \sigma_z + \sigma_z\, M\, \sigma_y) \qquad \lambda_6(M) = -\frac{1}{2}\,\gamma_t\,(\gamma_y\, M\, \gamma_z + \gamma_z\, M\, \gamma_y)\,\gamma_t \qquad (55)$$

$$\lambda_7(M) = \frac{1}{2}\,(\sigma_x\, M\, \mathbf{1} - \mathbf{1}\, M\, \sigma_x) \qquad \lambda_7(M) = -\frac{1}{2}\,\gamma_t\,(\gamma_x\, M\, \gamma_t - \gamma_t\, M\, \gamma_x)\,\gamma_t \qquad (56)$$

$$\lambda_8(M) = \frac{1}{2\sqrt{3}}\,(\sigma_x M \sigma_x + \sigma_y M \sigma_y - 2\sigma_z M \sigma_z) \quad \lambda_8(M) = -\frac{1}{2\sqrt{3}}\,\gamma_t(\gamma_x M \gamma_x + \gamma_y M \gamma_y - 2\gamma_z M \gamma_z)\gamma_t \quad (57)$$

These results are nearly identical to results which can be found in the literature. Only the transformations for the diagonal Gell-Mann matrices $\lambda_5(M)$ and $\lambda_8(M)$ slightly differ from these results, see table 1 at page 168 of [46] for the equations given above on the left side and equation 26 a-h at page 7 of [47] for the equations on the right side.

10. Preliminary Thoughts on the Geometric Meaning of Quantum Mechanics

The introduction to Geometric Algebra given in the preceding sections follows ideas published by many other authors. All these things described in these sections are not new, but already known since decades – or even longer.

The main focus of my paper might be a little bit different compared to standard presentations, because for didactical reasons I didn't emphasize coordinate-free methods in the way other authors would expect [48]. But the mathematical and physical interpretations are surely identical.

This changes when we reach quantum mechanics. Very probably no human being till now has actually understood quantum mechanics in a way we understand physical laws from other domains. "I don't understand it. Nobody does," Feynman tells about quantum mechanics (page 9 of [49]).

Therefore the interpretation of equations describing quantum mechanical situations is unclear and controversial. But these equations obviously describe experiments to a very high accuracy. The algebra of quantum mechanics is understood. It's the geometry of quantum mechanics which accounts for nearly all open questions. For this reason a mathematical language which uniquely combines algebra and geometry might be suited to shed more light on the meaning of quantum mechanics.

Then the basic question is: What is the geometry of the world we live in? We can describe this geometry in the non-relativistic case by the 16 basic elements of Pauli algebra: One base scalar which equals the (2 x 2) unit matrix, three base vectors σ_x, σ_y, σ_z, given in equation (39), three base bivectors or oriented unit area elements $\sigma_x\sigma_y$, $\sigma_y\sigma_z$, $\sigma_z\sigma_x$ and one base trivector or oriented unit volume element

$\sigma_x \sigma_y \sigma_z$. But then we are stuck: There are two possibilities for an oriented volume element in nature. In the preceding sections we always decided to take $I = \sigma_x \sigma_y \sigma_z$ as oriented volume element. But does nature really decides for a positive sign if all base vectors are multiplied? This positive sign is nothing else than the consequence of an arbitrary decision we as human beings make. But there is another possibility: It might as well be that nature decides for a negatively oriented unit volume element $-I$.

Or in other words: Do we live in a left-handed or a right-handed coordinate system designed by nature? Obviously we do not know. And the more interesting question is: What will happen, if even nature does not know?

A possible answer to this question could be, that at every measurement there will be a probability $\rho(1)$, that the sign is positive. And there will be a probability $\rho(-1) = 1 - \rho(1)$, that the sign is negative. The consequences are clear: Whenever we measure a physical entity U which should distinguish between the different possible signs of the oriented volume element (for example spin, magnetic moment) we will get a statistical result which represents the mean value of U:

$$\langle U \rangle = (-1)\,\rho(-1) + (1)\,\rho(1) \tag{58}$$

Therefore the hypothesis is: **It is not the spin of the electron, which is uncertain in quantum mechanics, but the orientation or handedness of the space we live in.**

As said, the algebra of quantum mechanics is understood and is completely available for all of us. For instance you can find the algebra of equation (58) in Jordan's book (page 73 of [50]), where this equation is the starting point for his discussion of quantum mechanics. Jordan develops an interpretation of quantum mechanics, in which "all quantities (are) made from spin" (chapter 10 of [58]). We should try to follow his mathematical route, which really offers an "easy way to look rather deeply into quantum mechanics" (pager VIII of [58]). But at the same time we should also try to change the interpretative frame of Jordan's presentation. Quantum mechanical quantities are made from geometry and might be a consequence of uncertainty in the handedness of the space[4].

11. Final remarks: we should teach Geometric Algebra
The following two points, David Hestenes again and again emphasized (see pages 120/121 of [9]):

- Geometric Algebra provides a unified language for the whole of physics that is conceptually and computationally superior to alternative mathematical systems in every application domain,
- Reforming the mathematical language of physics is the single most essential step toward simplifying physics education at all levels from high school to graduate school,

should have become clear by this paper. We therefore should not hesitate to promote and enlarge the knowledge about Geometric Algebra. This surely is no complicated task, because Seneca is still right today: "Non quia difficilia sunt, non audemus, sed quia non audemus, difficilia sunt" [51] – It is not because Geometric Algebra is difficult that we do not dare, but because we do not dare, Geometric algebra is difficult.

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[4] Perhaps this is not the right place to admit that I have a severe problem with general relativity: It delivers far too much solutions. In our eyes today, most solutions of Einstein's field equation seem to be irrelevant to physics, and especially solutions with closed time-like curves are considered as suspicious. Therefore one of three alternatives should be correct: 1. Einstein's field equation is wrong. The correct field equations should only give us solutions which describe situations of physical reality. 2. Einstein's field equation is incomplete. There should be an additional equation which tells us whether a solution of Einstein's field equation describes a situation possible in physical reality or not. 3. Einstein's field equation is correct and all solutions of the field equation describe situations possible in physics. Then solutions with closed time-like curves are just solutions which describe quantum mechanical phenomena. An elementary particle following a time-like curve has a probability $\rho(1)$, that its coordinate system is left-handed. And there will be a probability $\rho(-1) = 1 - \rho(1)$, that its coordinate system is right-handed, if the timelike curve changes the orientation of space within the elementary particles time travel like the orientation of an ant moving along a Moebius strip. This would explain quantum mechanics as a general relativistic effect according to equation (58).

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