Symmetries in Science XVI IOP Publishing
Journal of Physics: Conference Series **538** (2014) 012009 doi:10.1088/1742-6596/538/1/012009

Primary classification of symmetries from the solution manifold in Classical Mechanics

J Guerrero[a,b], **V Aldaya**[b], **F Cossío**[b], **and F F López-Ruiz**[c]

[a] Departamento de Matemática Aplicada, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain

[b] Instituto de Astrofísica de Andalucía (IAA - CSIC), Apartado Postal 3004, 18080 Granada, Spain

[c] Departamento de Física Aplicada, Universidad de Cádiz, Campus de Puerto Real, 11510, Puerto Real, Cádiz, Spain

E-mail: `juguerre@um.es`,`valdaya@iaa.es`,`fcossiop@gmail.com`, `paco.lopezruiz@uca.es`

Abstract. The symmetries of the equations of motion of a classical system are characterized in terms of vector field subalgebras of the whole diffeomorphism algebra of the solution manifold (the space of initial constants endowed with a symplectic structure). Among them, naturally arises the subalgebra of Hamiltonian (contact) vector fields corresponding to (jet-prolongued) point symmetries, those not corresponding to point symmetries and the remaining symmetries being associated with non-Hamiltonian (hence non-symplectic) non-strict contact symmetries.

1. Introduction

In this work we aim at finding symmetries of a general (non-linear) system from the diffeomorphisms of its solution manifold, the manifold of the constants of motion, endowed with a symplectic structure. The interest in finding (basic) symmetries of a system is twofold: classically, the symmetries allow for the reduction of order of the equations of motion (EoM), and even for its complete solution when enough symmetries are known, and at the quantum level symmetries allow for the possibility of applying a non-canonical, group-theoretical quantization method like that of [1].

In 1978 V.I. Arnold [2] introduced a transformation that maps an arbitrary Linear Second Order Ordinary Differential Equation (LSODE) into that of the one-dimensional free particle. This was an explicit realization of a particular case of the transformations introduced by S. Lie, who proved that certain systems given by a non-linear SODE whose EoM are up to cubic in the derivatives can be mapped to the free particle [3].

One of the most important properties of the Arnold transformation (also shared by the more general Lie transformations) is that, being a local diffeomorphism, it maps point symmetries of the EoM (or Lagrangian) of the one-dimensional free particle into point symmetries of the EoM (or Lagrangian) of the LSODE system, resulting in the surprising fact that they have the same point symmetries.

This construction is not easily generalizable to more complex systems not even to higher dimensions (where the Arnold transformation does not exist even in the case of LSODE systems unless the system is isotropic). However, inspired by the Arnold transformation, we shall try to find symmetries of general non-linear systems by importing them from those of a simpler system like the free particle. The rationale under the symmetry *tour* of the Arnold transformation is that the LSODE system has the same solution manifold as the free particle. Thus, we can skip the free-particle step, and import directly the symmetries from the solution manifold. These symmetries will be a suitable subset of the diffeomorphisms of the solution manifold (depending on the structure that we wish to preserve).

The passage to the solution manifold is accomplished by a transformation where the new coordinates and momenta are constants of motion. We shall denote a transformation of this kind a Hamilton-Jacobi (HJ) transformation, since, when the new coordinates and momenta are chosen to be canonical, it is generated by Hamilton's principal function S satisfying the Hamilton-Jacobi equation (see below).

It should be stressed that a HJ transformation provides the general solution to the equations of motion, since it expresses the coordinates and momenta in terms of some (integration) constants and time.

To map the symmetries from the solution manifold to the original system, the Jacobian of this transformation is required. This is the main obstacle to accomplish our task, since for an arbitrary non-linear system the HJ transformation is not available in closed form. However, for the basic symmetries, a power series expansion in terms of the adjoint of the vector field associated with the Hamiltonian is available (see Sec. 5.6).

Note that the existence of the HJ transformation is assured by the *Straightening theorem*, the first step in the induction proof of Frobenius Theorem (see, for instance, [4]). This result states that any vector field can be written, with a suitable choice of local coordinates, as the partial derivative with respect to the first coordinate ($\frac{\partial}{\partial\tau}$ in our case). Applying this to the Cartan formulation of Mechanics, the quotient of the contact manifold by the distribution generated by the vector field associated with Hamilton's EoM leads to a symplectic manifold parametrized by the initial constants (constants of the motion). By an ulterior use of Darboux theorem, a suitable family of constants can be found which are canonical coordinates, thus completing the HJ transformation. This provides an alternative to solving the Hamilton-Jacobi equation for Hamilton's principal function S, which is a type-2 generating function for a canonical transformation where the new coordinates and momenta are constants.

The main purpose of lifting symmetries from the solution manifold is that it provides a simple way of computing all contact symmetries, and these are necessary and sufficient to generate the solution manifold (i.e. expand the tangent space at each point). This is required, for instance, to quantize properly the system. It should be stressed that point symmetries do not provide, except in simple cases like linear systems, enough symmetries to generate the solution manifold (see Sec. 6.3).

Working with contact symmetries release us from the requirement of fixing a given Lagrangian (and considering just variational symmetries of this concrete Lagrangian), since it accounts for all symmetries of all Lagrangian providing the same EoM. Of course, to define a given system, we shall use a particular Lagrangian, but we shall not focus on this Lagrangian and will consider variational and non-variational symmetries of it.

In this paper we shall not restrict ourselves to symmetries generated by jet-prolonged vector fields on the configuration space, but we shall consider in general vector fields on the phase space. We shall not consider other kinds of "exotic" symmetries, such as *non-local* symmetries (see [5]). We shall restrict ourselves to dynamical systems with EoM given by the Euler-Lagrange (EL) equations derived from a (regular) Lagrangian.

The usual notion of symmetry for a dynamical system is that of transforming solutions of the EoM into solutions. More precisely, if M denotes the m-dimensional configuration space, then $\Phi : \mathbb{R} \times M \to \mathbb{R} \times M$ is a symmetry if given a curve $\gamma : I \subset \mathbb{R} \to U \subset M$ solution of the EoM, $\Phi(t, \gamma(t)) = (\tilde{t}, \tilde{\gamma}(\tilde{t}))$ defines a curve $\tilde{\gamma} : \tilde{I} \subset \mathbb{R} \to \tilde{U} \subset M$ which is also a solution of the EoM.

The paper is organized as follows. In Section 2 the point symmetries of the EoM are characterized. In Section 3 variational point symmetries are addressed. Generalized symmetries, in particular contact symmetries are discussed in Section 4. In Section 5 symmetries are lifted from the solution manifold. Some examples are given in Section 6 and finally in Section 7 some outlook is provided.

2. Point Symmetries of the equations of motion

The most elementary notion of symmetry is that which transforms just the variables[1] (t, q). The subsequent transformation on the derivatives q', q'', etc. is the one derived from the transformation of (t, q) (i.e. the derivatives are not considered as independent quantities). These symmetries are usually denoted as *point* or *geometrical*. When the transformation Φ depends smoothly on some parameter ϵ, it can be expanded around the identity up to first order:

$$\tilde{t} = t + \epsilon\, \xi(t, q) + O(\epsilon^2)\,, \qquad \tilde{q}_i = q_i + \epsilon\, \eta_i(t, q) + O(\epsilon^2)\,. \tag{1}$$

Deriving the previous equation with respect to ϵ at $\epsilon = 0$ we obtain the coefficients of the infinitesimal generator of the transformation:

$$X = \xi(t, q)\frac{\partial}{\partial t} + \eta_i(t, q)\frac{\partial}{\partial q_i}\,. \tag{2}$$

Infinitesimal generators of symmetries are usually denoted Lie symmetries, since the Lie bracket of two of them is also an infinitesimal generator of a symmetry (thus closing a Lie algebra).

2.1. Characterization of Lie symmetries

If the EoM are of the form $F_i(t, q, q', q'', \ldots, q^{(n)}) = 0$, then the infinitesimal generator X of the symmetry verifies:

$$X^{[n]}(F_i) = 0\,, \tag{3}$$

on solutions $q(t)$ of the equations of motion, where $X^{[n]}$ is the n-th jet prolongation of X, given by [6]:

$$X^{[n]} = X + X_i^1\frac{\partial}{\partial q_i'} + X_i^2\frac{\partial}{\partial q_i''} + \cdots X_i^n\frac{\partial}{\partial q_i^{(n)}}\,, \tag{4}$$

with

$$X_i^1 = \frac{d\eta_i}{dt} - q_i'\frac{d\xi}{dt}\,, \qquad X_i^k = \frac{dX_i^{k-1}}{dt} - q_i^{(k)}\frac{d\xi}{dt} = \frac{d^k}{dt^k}(\eta_i - q_i'\xi) + q_i^{(k+1)}\xi\,. \tag{5}$$

2.2. Characterization in terms of the total derivative vector field

If the EoM can be written as $q_i^{(n)} = f_i(t, q, q', \ldots, q^{(n-1)})$, then an alternative description can be given in terms of the total derivative (along trajectories) vector field, defined as:

$$D_t = \frac{\partial}{\partial t} + q_i'\frac{\partial}{\partial q_i} + q_i''\frac{\partial}{\partial q_i'} + \cdots + f_i\frac{\partial}{\partial q_i^{(n-1)}}\,. \tag{6}$$

Then

$$X^{[n]}(F) = 0 \quad \Leftrightarrow \quad X^{[n-1]}f_i = X_i^n \quad \Leftrightarrow \quad [X^{[n-1]}, D_t] = -(D_t\xi)D_t\,. \tag{7}$$

on solutions $q(t)$ of the EoM.

[1] To avoid cluttering the notation, we shall denote the vector $(q_1, q_2, \ldots, q_m)$ by q.

2.3. First Integrals
A function $I(t, q, q', \ldots, q^{(n-1)})$ satisfying $D_t I = 0$ is named a first integral of the EoM. First integrals are quantities constant along trajectories which are solutions of the EoM. They take the same value along a single trajectory, although it can take different values in different trajectories. The knowledge of $(n-1)m$ independent first integrals solves completely the EoM.

3. Point Symmetries of the Lagrangian
When the EoM result from the EL equations of a Lagrangian L, determining the extremals of the action functional $\mathcal{L} = \int dt L(t, q, q', \ldots, q^{(n-1)})$, it is natural to wonder about the transformations preserving this action functional (the importance of this is related to Noether theorem and conserved quantities). These symmetries are usually called *variational* symmetries.

The infinitesimal generators $X = \xi(t, q)\frac{\partial}{\partial t} + \eta_i(t, q)\frac{\partial}{\partial q_i}$ of variational symmetries verify:

$$X^{[n-1]}(L) + L D_t \xi = D_t B\,, \tag{8}$$

for some function B. Note that if $B \neq 0$, although the action functional $\mathcal{L}$ is invariant, the Lagrangian is invariant up to the total derivative $D_t B$.

3.1. Noether theorem
By Noether theorem [6, 7, 8], with each variational symmetry a conserved quantity (first integral) can be associated.

For a first-order Lagrangian $L(t, q, q')$, the conserved quantity associated with the variational symmetry $X = \xi(t, q)\frac{\partial}{\partial t} + \eta_i(t, q)\frac{\partial}{\partial q_i}$ is:

$$N = (\eta_i - \xi q_i')\frac{\partial L}{\partial q_i'} + \xi L - B\,. \tag{9}$$

4. Generalized Symmetries and contact structure
A generalized vector field on $\mathbb{R} \times M$ is a vector field of the form:

$$X = \xi(t, q, q', \ldots)\frac{\partial}{\partial t} + \eta_i(t, q, q', \ldots)\frac{\partial}{\partial q_i}\,. \tag{10}$$

A generalized vector field is not a proper vector field in $\mathbb{R} \times M$ in the sense that it does not generate a flow on it (however it is a proper vector field in a suitable jet space). But further computations can be carried out without inconsistencies as though it were a true vector field in $\mathbb{R} \times M$. In particular, formulas for prolongations, invariance of the EoM and of the Lagrangian apply without changes.

4.1. Evolutionary vector fields
Given $f_i(t, q, q', \ldots)$, an evolutionary vector field with characteristic f is defined as:

$$X_f = f_i \frac{\partial}{\partial q_i}\,. \tag{11}$$

To any generalized vector field $X = \xi(t, q, q', \ldots)\frac{\partial}{\partial t} + \eta_i(t, q, q', \ldots)\frac{\partial}{\partial q_i}$, there corresponds an evolutionary vector field X_f with characteristics $f_i = \eta_i - \xi q_i'$, in the sense that they generate the same symmetry transformation. In fact, $X^{[1]} = (X_f)^{[1]} + \xi D_t$, and therefore, according to (7), $X^{[1]}$ is a symmetry iff $(X_f)^{[1]}$ also is, and both vector fields coincide on solutions of the EoM.

4.2. Contact symmetries and contact structure

Generalized symmetries of first order are also known as contact symmetries since they preserve the contact 1-form, also known as Poincaré-Cartan 1-form,

$$\theta_L = \frac{\partial L}{\partial q'_i}(dq_i - q'_i dt) + L dt\,, \tag{12}$$

in the sense that $\mathbb{L}_X \theta_L = \theta_{L'}$, with $\theta_{L'}$ a contact 1-form for the Lagrangian $L' = X^{[1]}(L) + L D_t \xi$ with the same EL EoM (see [9]).

Note that by (8) (extended to generalized symmetries), if X is a variational (non-point, in general) symmetry, $L' = D_t B$, and $\theta_{L'} = \mathbb{L}_X \theta_L = df$, for some function f.

The Poincaré-Cartan 1-form defines a contact structure on the odd dimensional manifold $\mathbb{R} \times T^*M$, where it is written as

$$\theta_H = p_i dq^i - H\,dt\,, \tag{13}$$

where $H = p_i q^{i\prime} - L$ is the Legendre transform of L and $p_i = \frac{\partial L}{\partial q^{i\prime}}$. θ_H is also known as the Poincaré-Cartan 1-form associated with H.

5. Symmetries from the solution manifold

5.1. Hamiltonian formulation

Given a Poincaré-Cartan 1-form θ_H, its differential $\omega_H = d\theta_H = dp_i \wedge dq^i - dH \wedge dt$, known as the Poincaré-Cartan 2-form, has maximal rank. Its radical is generated, as a module, by a vector field $\bar{X}_H$ such that $i_{\bar{X}_H}\omega_H = 0$. This vector field can be written as:

$$\bar{X}_H = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\,. \tag{14}$$

The flow of $\bar{X}_H$ is given by Hamilton's EoM:

$$\frac{dt}{ds} = 1\,, \qquad \frac{dq^i}{ds} = X_H^{q^i} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{ds} = X_H^{p_i} = -\frac{\partial H}{\partial q^i}\,. \tag{15}$$

The time evolution is given in terms of the Poisson bracket in T^*M:

$$\bar{X}_H F = \frac{d}{dt}F = \frac{\partial F}{\partial t} + \{F, H\}\,. \tag{16}$$

5.2. The solution manifold

Taking quotient on the contact manifold by Hamiltons's equations (i.e. by the distribution generated by $\bar{X}_H$), we obtain a symplectic manifold $\mathbb{S}$ parametrized by, say, the constants of motion, with symplectic form $\omega = \omega_H$ (i.e. ω_H simply “falls down” to the quotient).

On Darboux coordinates, $\omega = dP_i \wedge dQ^i$, where $P_i\,, Q^i$ are canonically conjugated constants of motion, and the Poisson bracket is given by:

$$\{F, G\} = \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial Q^i} - \frac{\partial F}{\partial Q^i}\frac{\partial G}{\partial P_i}\,. \tag{17}$$

Given a function F on $\mathbb{S}$, the Hamiltonian vector field X_F associated with F is given by:

$$X_F = \{F, \cdot\} = \frac{\partial F}{\partial P_i}\frac{\partial}{\partial Q^i} - \frac{\partial F}{\partial Q^i}\frac{\partial}{\partial P_i}\,. \tag{18}$$

5.3. Hamilton-Jacobi transformation

The passage from the contact manifold parametrized by (q^i, p_j, t) to the solution manifold parametrized by (Q^i, P_j, τ) is given by the Hamilton-Jacobi transformation:

$$q_i = q_i(Q, P, \tau)\,, \qquad p^i = p^i(Q, P, \tau)\,, \qquad t = \tau\,. \tag{19}$$

Note that a trivial change in time, $t = \tau$, has been introduced, in order to render the transformation invertible (in the solution manifold the time evolution is *frozen* since it is parametrized by constants of the motion, therefore τ *decouples* everywhere and can be safely removed). In this way the associated change in the partial derivatives, necessary to compute the transformation of vector fields, can be obtained:

$$\begin{aligned} \frac{\partial}{\partial Q^i} &= \frac{\partial q^j}{\partial Q^i}\frac{\partial}{\partial q^j} + \frac{\partial p_j}{\partial Q^i}\frac{\partial}{\partial p_j} \\ \frac{\partial}{\partial P_i} &= \frac{\partial q^j}{\partial P_i}\frac{\partial}{\partial q^j} + \frac{\partial p_j}{\partial P_i}\frac{\partial}{\partial p_j} \\ \frac{\partial}{\partial \tau} &= \frac{\partial}{\partial t} + \frac{\partial q^j}{\partial \tau}\frac{\partial}{\partial q^j} + \frac{\partial p_j}{\partial \tau}\frac{\partial}{\partial p_j} = \frac{d}{dt}\,. \end{aligned} \tag{20}$$

Under the Hamilton-Jacobi transformation the Poincaré-Cartan 1-form is written as

$$\theta_H = p_i dq^i - H dt = P_i dQ^i + d(S - P_i Q^i)\,, \tag{21}$$

where $S = S(q, P, t)$ is Hamilton's Principal function, which coincides with the action integral $\int dt L(t, q, q')$ (on solutions) up to a constant.

Note that we recover that $\omega_H = d\theta_H = dP_i \wedge dQ^i = \omega$, therefore the Poincaré-Cartan 2-form "falls down" to the Solution Manifold.

It should be stressed that the trivial change of variables $\tau = t$ in eq. (19) and the fact that the constants of motion (Q^i, P_j) are canonically conjugated variables have been chosen for simplicity. In general we can take $\tau = \tau(t, q, p)$ and (Q^i, P_j) not canonically conjugated, but in this case the HJ transformation does not coincide with the one generated by Hamilton's Principal function S. The expressions appearing in Secs. 5.1, 5.2 and 5.3 should be modified accordingly, but the main conclusions still hold (see [10] for details)[2].

5.4. Noether theorem in the Solution Manifold

All Hamiltonian vector fields X_F generate symmetries since:

$$i_{X_F}\omega = -dF \quad \Rightarrow \quad \mathbb{L}_{X_F}\omega = 0 \quad \Rightarrow \quad \mathbb{L}_{X_F}\omega_H = 0\,, \tag{22}$$

and this implies $\mathbb{L}_{X_F}\theta_H = df$, therefore X_F (when lifted to the contact manifold) is a variational contact symmetry, and it might be (the evolutionary form of) a point symmetry.

In particular $\frac{\partial}{\partial Q^i}$ and $\frac{\partial}{\partial P_i}$, when lifted by the inverse HJ transformation, i.e. the expressions (20), are (basic) variational symmetries.

It should be stressed that in the expression of X_F there is no "dynamical" information. It is only when lifted by the inverse HJ transformation that the information on the Hamiltonian or the Lagrangian appears.

[2] This freedom, far from being a drawback of the method, is an important advantage, allowing to exploit the natural (and non-canonical) symplectic structure of the Solution Manifold when it is a co-adjoint orbit of a Lie group.

5.5. non-Noether symmetries in the Solution Manifold
If $X \neq X_F$ in $\mathbb{S}$, then

$$\mathbb{L}_X\omega = i_X d\omega + d(i_X\omega) = d(i_X\omega) = \omega' \,, \tag{23}$$

with ω' a closed 2-form: $\omega' = d\theta'$ with $\theta' = i_X\omega$. If $\theta' = \theta_{H'}$ with H' the Legendre transform of L' providing the same EL EoM as L, then X, when lifted by the inverse HJ transformation, is a (non-Noether) contact symmetry.

Note that in one dimension, since ω is a volume form, $\omega' = f\omega$ for some function f on $\mathbb{S}$ (in fact $f = div(X)$).

5.6. Lifting symmetries from the solution manifold
In the solution manifold $\mathbb{S}$ there are, obviously, infinitely many symmetries. The problem is lifting these symmetries to the contact manifold by the inverse HJ transformation.

If the system is conservative[3] ($L \neq L(t)$, i.e $H \neq H(t)$), a formal solution can be obtained through the exponential map:

$$\begin{aligned} q &= e^{\tau X_H} Q \\ p &= e^{\tau X_H} P \\ t &= \tau \,. \end{aligned} \tag{24}$$

The partial derivatives necessary to compute (20) can be obtained through ($Y = Q^i,\ P_i$):

$$\frac{\partial}{\partial Y} e^{\tau X_H} = e^{\tau X_H} e^{-\tau ad_{X_H}}(\frac{\partial}{\partial Y}) = e^{\tau X_H} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \tau^n ad^n_{X_H}(\frac{\partial}{\partial Y}) \,. \tag{25}$$

In the previous formulas

$$\begin{aligned} ad_{X_H}(\frac{\partial}{\partial Y}) &= [X_H, \frac{\partial}{\partial Y}] \\ ad^n_{X_H}(\frac{\partial}{\partial Y}) &= [X_H, ad^{n-1}_{X_H}(\frac{\partial}{\partial Y})] \,. \end{aligned} \tag{26}$$

This provides a series expansion in powers of τ for $q^i(Q, P, \tau), p_j(Q, P, \tau)$ and their partial derivatives with respect to Q^i, P_j.

This way of obtaining symmetries from the solution manifold is, to the best of our knowledge, new, and the formal series appearing in Eqs. (25)-(26) can be easily summed up when the vector fields $(\frac{\partial}{\partial Q^i}, \frac{\partial}{\partial P_j}, X_H)$ close a finite dimensional Lie algebra[4], since the terms in the expansion can be arranged in a finite number of expansions (each one proportional to an element of a basis of the Lie algebra) and these are numerical series that can be easily shown to be absolutely convergent.

If $(\frac{\partial}{\partial Q^i}, \frac{\partial}{\partial P_j}, X_H)$ do not close a finite-dimensional Lie algebra, but the infinite-dimensional Lie algebra generated from them is of *finite (polynomial) growth* [12], in the sense that the successive Lie brackets appearing in eq. (25) are proportional to increasing powers of a small perturbative parameter λ, in such a way that the vector space generated at each order of λ is finite, then the formal series can be arranged in terms proportional to powers of λ (and each of these terms is of the type considered before and therefore absolutely convergent). Note that this is the situation when a conserved Hamiltonian is of the form $H = H_0 + \lambda H_I$, where H_0 represents an exactly solvable system, H_I an interaction Hamiltonian (non-solvable) and λ is a small coupling constant.

[3] For time-dependent Hamiltonians, Dyson or Magnus expansion should be used instead, see [11, 10].
[4] For this purpose a proper choice of the variables (Q^i, P_j) can be crucial, see the comment before Sec. 5.4.

6. Examples

In this section we shall provide some examples to illustrate the construction of symmetries from the solution manifold. In all these cases the corresponding HJ transformation is known, and therefore explicit formulas can be provided. In general, a construction like the one given in Sec. 5.6 should be used to obtain approximate solutions and approximated symmetries. Note however that for many purposes (for instance when the system is a small perturbation of an exactly solvable Hamiltonian, see [10]) this is enough.

6.1. Symmetries of the free one-dimensional particle

For the free one-dimensional particle $\ddot{q}=0$, $L=\frac{1}{2}m\dot{q}^2$ and $H=\frac{p^2}{2m}$, where $p=m\dot{q}$. The HJ transformation is $q=Q+\frac{P}{m}\tau$ and $p=P$, providing the general solution to the EoM. Its points symmetries are:

$$\begin{array}{rcll}
G_1 &=& \frac{\partial}{\partial q} &= \frac{\partial}{\partial Q}\\
G_2 &=& t\frac{\partial}{\partial q}+\frac{\partial}{\partial \dot{q}} &= m\frac{\partial}{\partial P}\\
G_3 &=& \frac{\partial}{\partial t} &= \frac{\partial}{\partial \tau}-\frac{P}{m}\frac{\partial}{\partial Q}\\
G_4 &=& t^2\frac{\partial}{\partial t}+qt\frac{\partial}{\partial q}+(q-\dot{q}t)\frac{\partial}{\partial \dot{q}} &= \tau^2\frac{\partial}{\partial \tau}+mQ\frac{\partial}{\partial P}\\
G_5 &=& t\frac{\partial}{\partial t}+\frac{q}{2}\frac{\partial}{\partial q}-\frac{\dot{q}}{2}\frac{\partial}{\partial \dot{q}} &= \tau\frac{\partial}{\partial \tau}+\frac{Q}{2}\frac{\partial}{\partial Q}-\frac{P}{2}\frac{\partial}{\partial P}\\
G_6 &=& q\frac{\partial}{\partial q}+\dot{q}\frac{\partial}{\partial \dot{q}} &= Q\frac{\partial}{\partial Q}+P\frac{\partial}{\partial P}\\
G_7 &=& qt\frac{\partial}{\partial t}+q^2\frac{\partial}{\partial q}+\dot{q}(q-\dot{q}t)\frac{\partial}{\partial \dot{q}} &= (Q+\frac{P}{m}\tau)\tau\frac{\partial}{\partial \tau}\\
 & & & +Q^2\frac{\partial}{\partial Q}+QP\frac{\partial}{\partial P}\\
G_8 &=& q\frac{\partial}{\partial t}-\dot{q}^2\frac{\partial}{\partial \dot{q}} &= (Q+\frac{P}{m}\tau)\frac{\partial}{\partial \tau}\\
 & & & -\frac{QP}{m}\frac{\partial}{\partial Q}-\frac{P^2}{m}\frac{\partial}{\partial P}\,.
\end{array} \tag{27}$$

In the previous equation $G_1,\ldots,G_5$ are variational point symmetries, whereas G_6,G_7,G_8 are non-variational point symmetries. The first expression corresponds to the first jet prolongation in M, and the second expression corresponds to the solution manifold (extended with τ). Note that in the second expression the terms in $\frac{\partial}{\partial\tau}$ can be safely removed (the evolutionary form is then obtained) since its objective was rendering the transformation geometrical.

General contact symmetries are obtained by:

$$\begin{aligned}
X=f(Q,P)\frac{\partial}{\partial Q}+g(Q,P)\frac{\partial}{\partial P} &= \left[f(q-\dot{q}t,\dot{q})+\frac{t}{m}g(q-\dot{q}t,\dot{q})\right]\frac{\partial}{\partial q}\\
&\quad +\frac{1}{m}g(q-\dot{q}t,\dot{q})\frac{\partial}{\partial \dot{q}}\,,
\end{aligned} \tag{28}$$

where this expression is in evolutionary form. The term in $\frac{\partial}{\partial\dot{q}}$ is a prolongation since $D_t(f)=D_t(g)=0$ (f and g are first integrals).

If div $X=0$, then X is a Hamiltonian vector field providing, when lifted to the contact manifold, a (non-geometrical, in general) variational symmetry. This is the case of G_1 ($f=1$, $g=0$), G_2 ($f=0$, $g=m$), G_3 ($f=-P/m$, $g=0$), G_4 ($f=0$, $g=mQ$) and G_5 ($f=Q/2$, $g=-P/2$). In this case these symmetries are rendered geometrical[5] by the addition of a suitable term in $\frac{\partial}{\partial\tau}$.

If div $X\neq 0$, then X is not a Hamiltonian vector field, therefore when lifted to the contact manifold it provides a non-variational (non-geometrical in general) symmetry. This is the case

[5] Note that in general this is not possible and for non-linear systems only a few symmetries turn out to be geometrical.

of G_6 ($f = Q$, $g = P$), G_7 ($f = Q^2$, $g = QP$), and G_8 ($f = -QP/m$, $g = -P^2/m$). Again, these symmetries are rendered geometrical by the addition of a suitable term in $\frac{\partial}{\partial \tau}$.

Equation (27) provides all point symmetries of the free particle (either variational or not), whereas (28) (with the addition of an arbitrary term in $\frac{\partial}{\partial \tau}$) provides all contact symmetries of the free particle, either variational, non variational, geometrical or non geometrical.

6.2. The free relativistic particle

For the free relativistic particle in one dimension, $L = -mc^2\sqrt{1-\frac{\dot{q}^2}{c^2}} \equiv -mc^2/\gamma$, $H = p_0 c$, where $p = m\gamma\dot{q}$ and $p_0 = \sqrt{m^2c^2+p^2} = mc\gamma$. The HJ transformation is: $p = P$, $q = Q + \frac{P}{P_0}ct$, where $P_0 = \sqrt{m^2c^2+P^2}$, providing the general solution to the EoM.

The basic symmetries in this case are:

$$\begin{aligned}
\frac{\partial}{\partial Q} &= \frac{\partial}{\partial q}\,, \\
\frac{\partial}{\partial P} &= \frac{\partial}{\partial p} + \frac{m^2c^3t}{p_0^3}\frac{\partial}{\partial q} = \frac{1}{m\gamma^3}\left(t\frac{\partial}{\partial q} + \frac{\partial}{\partial \dot{q}}\right)\,.
\end{aligned}$$

Note that $\frac{\partial}{\partial Q}$, when lifted to the solution manifold, is a variational point symmetry (reflecting the translation invariance of the Lagrangian). However $\frac{\partial}{\partial P}$ is not a point symmetry, and cannot be rendered a point symmetry by just adding of a term in $\frac{\partial}{\partial \tau}$.

If we consider the vector field $\frac{Po^3}{m^2c^3}\frac{\partial}{\partial P} = t\frac{\partial}{\partial q} + \frac{\partial}{\partial \dot{q}}$, we recover the point (variational) symmetry G_2 of the free non-relativistic particle, which is however a non-variational symmetry for the relativistic particle. A variational symmetry obtained from $\frac{\partial}{\partial P}$ is $\frac{Po^2}{mc^2}\frac{\partial}{\partial P} = \frac{1}{\gamma}\left(t\frac{\partial}{\partial q} + \frac{\partial}{\partial \dot{q}}\right)$, but this is non geometrical.

To obtain a variational point symmetry from $\frac{\partial}{\partial P}$, different expressions for the solutions should be employed, using a non-canonical HJ transformation involving a change in time (i.e. $\tau \neq t$). In this way the expression of the relativistic boost generator can be recovered by lifting $\frac{Po}{c}\frac{\partial}{\partial P}$ (see [10]). Note that this is not possible with the conventional canonical transformation generated by the solution of the Hamilton-Jacobi equation.

6.3. The Pöschl-Teller potential

As a non-linear example, we shall consider a particle in a Pöschl-Teller potential, with Lagrangian given by $L = \frac{1}{2}m\dot{q}^2 - \frac{D}{\cos(\alpha q)^2}$, and Hamiltonian $H = \frac{p^2}{2m} + \frac{D}{\cos(\alpha q)^2}$. Here D is the energy of the minimum of the potential, and α is related to the width of the potential.

Written in terms of $\xi = \frac{1}{\alpha}\sin(\alpha q)$ and $p_\xi = \frac{\partial L}{\partial \dot{\xi}}$,

$$H = (1-\alpha^2\xi^2)\frac{p_\xi^2}{2m} + \frac{D}{1-\alpha^2\xi^2} = \frac{1}{2}m\dot{\xi}^2 + \frac{1}{2}m\omega(E)^2\xi^2 + D\,, \tag{29}$$

where $\omega(E) = \sqrt{\frac{2E}{D}}\alpha$ is an energy-dependent frequency.

Note that written in this coordinates the Pöschl-Teller potential looks like a harmonic oscillator. However, the frequency of the oscillations depends on the energy E, and therefore on the particular trajectory considered. Therefore they are different systems, although the Pöschl-Teller potential can be interpreted as an harmonic oscillator in the circle (or the n-sphere in higher dimensions).

The solutions of the EoM, providing the HJ transformation, are (see [13]):

$$q = \frac{1}{\alpha}\arcsin\left[\sin(\alpha Q)\cos(\omega(E)t) + \frac{\alpha}{m\omega(E)}P\cos(\alpha Q)\sin(\omega(E)t)\right] \tag{30}$$

$$p = m\dot{q} = \frac{1}{\cos(\alpha q)}\left[\cos(\alpha Q)\cos(\omega(E)t) - \frac{\alpha}{m\omega(E)}P\sin(\alpha Q)\sin(\omega(E)t)\right]. \tag{31}$$

In these solutions, the energy appearing in the frequency should be written in (Q, P) variables. Also, in the denominator of the *rhs* of the second line, q should be substituted by the *rhs* of the first line.

The inverse of the HJ transformation can also be computed, resulting in:

$$Q = \frac{1}{\alpha}\arcsin\left[\sin(\alpha q)\cos(\omega(E)t) - \frac{\alpha}{m\omega(E)}p\cos(\alpha q)\sin(\omega(E)t)\right] \tag{32}$$

$$P = \frac{1}{\cos(\alpha Q)}\left[\cos(\alpha q)\cos(\omega(E)t) + \frac{\alpha}{m\omega(E)}p\sin(\alpha q)\sin(\omega(E)t)\right]. \tag{33}$$

In these expressions, the energy appearing in the frequency should now be expressed in (q, p) variables. As before, in the denominator of the *rhs* of the second line, Q should be substituted by the *rhs* of the first line.

Basic symmetries in the Solution Manifold lifted by the inverse HJ transformation can be computed, although their expressions are rather involved. They can be obtained from the Jacobian appearing in eq. (20), which is given by:

$$\begin{aligned}
\frac{\partial q}{\partial Q} &= (A + Bt)\cos(\omega t) + (C + Dt)\sin(\omega t) \\
A &= \frac{\cos(\alpha Q)}{\cos(\alpha q)} \\
B &= \frac{2\alpha^3 DP\tan(\alpha Q)}{m^2\omega^2\cos(\alpha Q)\cos(\alpha q)} \\
C &= -\frac{4\alpha^3 DP\tan(\alpha Q) + m\alpha P\omega^2\sin(2\alpha Q)}{2m^2\omega^3\cos(\alpha Q)\cos(\alpha q)} \\
D &= -\frac{2\alpha^2 D\sin^2(\alpha Q)}{m\omega\cos(\alpha Q)^3\cos(\alpha q)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p}{\partial Q} &= (A + Bt)\cos(\omega t) + (C + Dt)\sin(\omega t) \\
A &= \frac{\alpha\left(m^2 p\omega^2\tan(\alpha q)\cos(\alpha Q) - m^2 P\omega^2\sin(\alpha Q)\right)}{m^2\omega^2\cos(\alpha q)} \\
B &= \frac{\alpha\left(\alpha^3 DpP\tan(\alpha q)\sin(2\alpha Q) - 2\alpha D m^2\omega^2\sin^2(\alpha Q)\right)}{m^2\omega^2\cos(\alpha q)\cos(\alpha Q)^3} \\
C &= -\frac{2\alpha^2 D\sin^2(\alpha Q) + m\omega^2\cos^4(\alpha Q)}{\omega\cos(\alpha Q)^3\cos(\alpha q)} \\
&\quad + \frac{\alpha^2 pP\tan(\alpha q)\left(\alpha^2 D\sin(2\alpha Q) + m\omega^2\sin(\alpha Q)\cos^3(\alpha Q)\right)}{m^2\omega^3\cos(\alpha Q)^3\cos(\alpha q)} \\
D &= \frac{-\alpha^3 DmP\omega^2\sin(2\alpha Q) - 2\alpha^3 Dmp\omega^2\tan(\alpha q)\sin^2(\alpha Q)}{m^2\omega^3\cos(\alpha Q)^3\cos(\alpha q)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial q}{\partial P} &= (A+Bt)\cos(\omega t)+(C+Dt)\sin(\omega t)\\
A &= 0\\
B &= \frac{m\omega^2\cos^2(\alpha Q)-2\alpha^2 D}{m^2\omega^2\cos(Q\alpha)\cos(\alpha q)}\\
C &= \frac{2\alpha^2 D}{m^2\omega^3\cos(\alpha Q)\cos(\alpha q)}\\
D &= -\frac{\alpha P\tan(\alpha Q)\cos(Q\alpha)}{m^2\omega\cos(\alpha q)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p}{\partial P} &= (A+Bt)\cos(\omega t)+(C+Dt)\sin(\omega t)\\
A &= \frac{\cos(\alpha Q)}{\cos(\alpha q)}\\
B &= \frac{-4\alpha^3 Dp\tan(\alpha q)+2\alpha mp\omega^2\tan(\alpha q)\cos^2(\alpha Q)-\alpha mP\omega^2\sin(2\alpha Q)}{2m^2\omega^2\cos(\alpha Q)\cos(\alpha q)}\\
C &= -\frac{\alpha mP\omega^2\sin(2\alpha Q)-4\alpha^3 Dp\tan(\alpha q)}{2m^2\omega^3\cos(\alpha Q)\cos(\alpha q)}\\
D &= -\frac{-4\alpha^2 Dm\omega^2+2m^2\omega^4\cos^2(\alpha Q)+\alpha^2 pP\omega^2\tan(\alpha q)\sin(2\alpha Q)}{2m^2\omega^3\cos(\alpha Q)\cos(\alpha q)}
\end{aligned}$$

In the previous formulas we have introduced $\omega \equiv \omega(E)$, and at any occurrence of (Q, P) in the *rhs* of an equation, equations (32) and (33) should be used (we write the expressions in such a way in order to make them more compact). As before, the expression of the energy in terms of (q, p) inside ω should be used.

These basic symmetries are non geometrical. It is possible to obtain point symmetries for the Pöschl-Teller potential (apart from the trivial one $\frac{\partial}{\partial t}$) by lifting certain functions on the solution manifold closing an $sl(2,\mathbb{R})$ subalgebra of the Poisson algebra (see [13] for details).

6.4. Other examples

The method of lifting symmetries from the solution manifold can be applied to many other examples, either in exact or in approximate form. Among them, the symmetries of the Kepler problem, the λq^4 potential, the particle in the circle, the Sine-Gordon model or Sigma models can be addressed (see [10]).

7. Comments and outlook

The idea of importing symmetries from the solution manifold *à la Arnold* is, to the best of our knowledge, new, and we feel that deserves further study. A more complete discussion, along with a thoughtful classification of symmetries will be addressed in [10]. Let us comment only on the extension of this classification to the quantum level.

The role of the solution manifold is played by the Hilbert space $\mathcal{H}_0$ of solutions of the time dependent Schrödinger equation, which can be realized by fixing a value of time t_0 (for instance $t_0 = 0$). The role of the Hamilton-Jacobi transformation is played by the inverse of the time evolution operator $\hat{U}(t, t_0)$. The main difference is that, although we have $\hat{Q}$ and $\hat{P}$ operators, wavefunctions depend only on one of the variables Q or P (or a combination thereof). In the context of Geometric Quantization, this means that they are polarized. The EoM is the Schrödinger equation (which is also polarized).

The consequences of all this is that only variational symmetries survive. This can be easily understood since the Schrödinger equation involves the Hamiltonian. Therefore, transformations changing the Lagrangian to an alternative Lagrangian (leading to the same EL equations) are not allowed at the quantum level.

A different approach to the symmetries of the quantum problem is that of Lie symmetries of the Schrödinger equation, see [14, 15].

Acknowledgments
Work partially supported by the Fundación Séneca, Spanish MICINN and Junta de Andalucía under projects 08814/PI/08, FIS2011-29813-C02-01 and FQM219-FQM1951, respectively.

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