

Some applications of the Kronecker product in Hubbard representation

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Abstract. The properties of the Kronecker product are revisited in terms of Hubbard operators. The simplest representation of a Hubbard operator $X_n^{i,j}$ is a square matrix of size n with an entry equal to 1 and zero elsewhere. This framework simplifies the calculation of the Kronecker product of arbitrary matrices no matter the size or the number of the involved factors. Some applications are presented, these include the algebra of permutation matrices, the Hadamard matrix, the XXX Heisenberg model and the interaction of an atom with radiation fields.

1. Introduction

The Kronecker (tensor or direct) product ‘ \otimes ’ is widely used in several areas of mathematics and theoretical physics [1–9]. This is particularly useful in the description of multipartite quantum systems $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2 + \dots + \mathcal{S}_N$, the pure states of which are represented by vectors in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_N$, with \mathcal{H}_k the pure state space of the k th subsystem \mathcal{S}_k [10–12]. For finite-dimensional Hilbert spaces ($\dim \mathcal{H}_k = n_k < \infty$) one has $\dim \mathcal{H} = n_1 \dots n_N$, so that any operator A_k defined to act on \mathcal{H}_k can be represented by a square matrix of order n_k^2 , this can be also promoted to act on the entire space \mathcal{H} as follows

$$\mathbf{A}_k \leftrightarrow \mathbb{I}_1 \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_{k-1} \otimes A_k \otimes \mathbb{I}_{k+1} \otimes \dots \otimes \mathbb{I}_N,$$

where \mathbb{I}_k is the identity operator in \mathcal{H}_k . Thus, \mathbf{A}_k must be represented by a square matrix of order $(n_1 \dots n_N)^2$. As the Kronecker product of two matrices of any order is the block matrix $A \otimes B = [a_{i,j}B]$, we realize that the promoted operator \mathbf{A}_k has at most $n_1 n_2 \dots n_{k-1} n_k^2 n_{k+1} \dots n_N < (n_1 \dots n_N)^2$ entries different from zero. This last fact suggests the looking for simple algorithms to calculate the Kronecker product of operators.

In a recent work we have applied the Hubbard operators as the building-blocks of the Kronecker product $A \otimes B \otimes C \otimes \dots$ [13]. These are two-indexed operators $X^{i,j}$ fulfilling the properties [14–16]:

- i) $X^{i,j} X^{k,m} = \delta_{jk} X^{i,m}$ (multiplication rule)
- ii) $\sum_k X^{k,k} = \mathbb{I}$ (completeness)
- iii) $(X^{i,j})^\dagger = X^{j,i}$ (non-hermiticity)
- iv) $[X^{i,j}, X^{k,m}]_\pm = \delta_{jk} X^{i,m} \pm \delta_{mi} X^{k,j}$ (commutation rules)



Let $\{|\psi_k\rangle, k \in \mathcal{I} \subseteq \mathbb{Z}^+\}$ be an orthonormal basis of the pure state space \mathcal{H} with $|\psi_k\rangle$ the column vector having 1 in the k th row and zero elsewhere. The Hubbard operators $X^{i,j}$ cause the transition from the state $|\psi_j\rangle$ to the state $|\psi_i\rangle$ and can be expressed as the outer products

$$X^{i,j} = |\psi_i\rangle\langle\psi_j|, \quad i, j \in \mathcal{I}.$$

That is, in such a representation the Hubbard operator $X^{i,j}$ corresponds to an square matrix that has all the entries equal to zero except the one at the i th row and j th column, where it takes the value 1. In this context any linear operator $O : \mathcal{H} \rightarrow \mathcal{H}$ admits the X -representation

$$O = \sum_{i,j \in \mathcal{I}} o_{i,j} X_n^{i,j}, \quad o_{i,j} = \langle\psi_i|O|\psi_j\rangle.$$

Using the X -representation the algebra of square matrices can be operated in compact form, no matter the size or the number of the factors.

In this contribution we follow the approach introduced in [13] to summarize some of the Kronecker product properties in terms of Hubbard operators. In X -representation the algebraic properties associated to multipartite systems can be analyzed in simple form since even complicated calculations involving large matrices are reduced to simple relations of subscripts. Our interest is to address some applications of this representation in the study of permutation matrices, quantum logic gates, and some interaction models as the Heisenberg or the Jaynes-Cummings ones. With this aim we introduce the notation and basic definitions in Section 2. The properties of the Kronecker product in X -representation are revisited in Section 3 and some of the applications are reported in Section 4. Some final remarks are presented in Section 5.

2. Notation and basic definitions

Let $\mathbb{K}^n = \text{sp}\{|e_i^n\rangle\}_{i=1}^n$ be a vector space of dimension n defined on the field \mathbb{K} . The basis vectors $|e_k^n\rangle$ are orthonormal n -tuples with 1 in the position k and zero elsewhere. Any element $|x\rangle \in \mathbb{K}^n$ is represented by a column vector

$$|x\rangle = \sum_{k=1}^n x_k |e_k^n\rangle \equiv (x_1, x_2, \dots, x_n)^T, \quad x_\ell = \langle e_\ell^n | x \rangle \in \mathbb{K}.$$

The Hermitian transpose of the ket-vector $|x\rangle$ is represented by the bra-vector

$$|x\rangle^\dagger := \langle x| = (x_1^\dagger, x_2^\dagger, \dots, x_n^\dagger) = \sum_{k=1}^n x_k^\dagger \langle e_k^n|, \quad x_\ell^\dagger = \langle x | e_\ell^n \rangle \in \mathbb{K},$$

with $x_k^\dagger = x_k$ if $\mathbb{K} = \mathbb{R}$ and $x_k^\dagger = \bar{x}_k$ for $\mathbb{K} = \mathbb{C}$. Here \bar{z} stands for the complex conjugate of $z \in \mathbb{C}$. The inner product of two vectors $|x\rangle$ and $|y\rangle$ yields

$$\langle x | y \rangle = \sum_{k,\ell=1}^n x_k^\dagger y_\ell \langle e_k^n | e_\ell^n \rangle = \sum_{k=1}^n x_k^\dagger y_k.$$

Therefore $(\langle x | y \rangle)^\dagger = \langle y | x \rangle$, so that the vector space \mathbb{K}^n is Euclidean (Hermitian) with linear (sesquilinear) metric if $\mathbb{K} = \mathbb{R}$ ($\mathbb{K} = \mathbb{C}$) [17].

The outer product of $|x\rangle$ and $|y\rangle$ yields

$$|x\rangle\langle y| = \sum_{i,j=1}^n x_i y_j^\dagger |e_i^n\rangle\langle e_j^n| \equiv \sum_{i,j=1}^n x_i y_j^\dagger X_n^{i,j},$$

where the “dyadic” matrices

$$X_n^{i,j} = |e_i^n\rangle\langle e_j^n|, \quad i, j = 1, 2, \dots, n \quad (1)$$

are Hubbard operators. Indeed, they satisfy (i) the multiplication rule

$$X_n^{i,j} X_n^{k,\ell} = |e_i^n\rangle\langle e_j^n|e_k^n\rangle\langle e_\ell^n| = \delta_{jk} X_n^{i,\ell}, \quad (2)$$

are (ii) complete

$$\mathbb{I}_n = \sum_{i=1}^n X_n^{i,i}, \quad (3)$$

and (iii) non-hermitian since

$$(X_n^{i,j})^T = (|e_i^n\rangle\langle e_j^n|)^T = (\langle e_j^n|)^T (|e_i^n\rangle)^T = |e_j^n\rangle\langle e_i^n| = X_n^{j,i} \quad (4)$$

implies $(X_n^{i,j})^\dagger = (X_n^{i,j})^T = X_n^{j,i}$. Finally, they satisfy the (iv) commutation relationships

$$[X_n^{i,j}, X_n^{k,m}]_\pm = X_n^{i,j} X_n^{k,m} \pm X_n^{k,m} X_n^{i,j} = \delta_{jk} X_n^{i,m} \pm \delta_{mi} X_n^{k,j}. \quad (5)$$

The action of these X operators on the vector space \mathbb{K}^n is defined by the rule

$$X_n^{i,j} |e_k^n\rangle = \delta_{jk} |e_i^n\rangle \quad \Rightarrow \quad X_n^{i,j} |x\rangle = x_j |e_i^n\rangle, \quad (6)$$

so that their matrix elements are easily computed

$$\langle e_i^n | X_n^{k,\ell} | e_j^n \rangle = \delta_{\ell j} \langle e_i^n | e_k^n \rangle = \delta_{\ell j} \delta_{ik}. \quad (7)$$

Using the completeness (3) we can write any n -square matrix A as the appropriate linear combination of Hubbard operators

$$A = \sum_{i,j=1}^n a_{i,j} X_n^{i,j}, \quad a_{i,j} \in \mathbb{K}. \quad (8)$$

Therefore the complex conjugate \bar{A} , the transpose A^T , and the adjoint A^\dagger of A are computed as follows

$$\bar{A} = \sum_{i,j=1}^n \bar{a}_{i,j} X_n^{i,j}, \quad A^T = \sum_{i,j=1}^n a_{i,j} X_n^{j,i}, \quad A^\dagger = \sum_{i,j=1}^n a_{i,j}^\dagger X_n^{j,i}. \quad (9)$$

Considering (8) and (6) one gets

$$A |e_j^n\rangle = \sum_{k=1}^n a_{k,j} |e_k^n\rangle, \quad (10)$$

so that

$$\langle e_i^n | A | e_j^n \rangle = \sum_{k,\ell=1}^n a_{k,\ell} \delta_{i,k} \delta_{\ell j} = a_{i,j} \quad \Rightarrow \quad \text{Tr} A = \sum_{i=1}^n \langle e_i^n | A | e_i^n \rangle = \sum_{i=1}^n a_{i,i}, \quad (11)$$

and

$$A |x\rangle = \sum_{k,j,\ell=1}^n a_{k,j} x_\ell X_n^{k,j} |e_\ell^n\rangle = \sum_{k,\ell=1}^n a_{k,\ell} x_\ell |e_k^n\rangle. \quad (12)$$

Finally, the usual matrix product is closed

$$AB = \left(\sum_{i,j=1}^n a_{i,j} X_n^{i,j} \right) \left(\sum_{k,\ell=1}^n b_{k,\ell} X_n^{k,\ell} \right) = \sum_{i,\ell=1}^n \left(\sum_k a_{i,k} b_{k,\ell} \right) X_n^{i,\ell} = C, \quad (13)$$

with C the n -square matrix

$$C = \sum_{i,\ell=1}^n c_{i,\ell} X_n^{i,\ell}, \quad c_{i,\ell} = \sum_{k=1}^n a_{i,k} b_{k,\ell}, \quad (14)$$

as expected.

3. The X -representation of the Kronecker algebra

The Kronecker product follows from the next definition

Definition 1. Let $A = [a_{i,j}]$ and $B = [b_{r,s}]$ be respectively matrices of order $m \times n$ and $k \times \ell$ over the field \mathbb{K} . The Kronecker product $A \otimes B$ is the matrix of order $mk \times n\ell$ over the field \mathbb{K} defined as $A \otimes B = [a_{i,j} B]$.

Let us consider the vector space $\mathbb{K}^{n_1 n_2}$ on which the matrix $A \otimes B$ acts on. Here A and B are square matrices defined to act on \mathbb{K}^{n_1} and \mathbb{K}^{n_2} respectively. We have a second definition

Definition 2. Let $|e_{i_1}^{n_1}\rangle$ and $|e_{i_2}^{n_2}\rangle$ be basis vectors of \mathbb{K}^{n_1} and \mathbb{K}^{n_2} respectively. The Kronecker product $|e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle$ is the $n_1 n_2$ -tuple having 1 at $(i_1 - 1)n_2 + i_2$, and zero in all other entries.

The vector space $\mathbb{K}^{n_1 n_2}$ is built according to the following proposition which is introduced without a proof.

Proposition 1. Let $\mathbb{K}^{n_1} = \text{Sp} \{ |e_{i_1}^{n_1}\rangle \}_{i_1=1}^{n_1}$ and $\mathbb{K}^{n_2} = \text{Sp} \{ |e_{i_2}^{n_2}\rangle \}_{i_2=1}^{n_2}$ be vector spaces defined on the field \mathbb{K} . The set of all the Kronecker products $|e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle$ is orthonormal and spans a vector space of dimension $n_1 n_2$, written $\mathbb{K}^{n_1 n_2} = \text{Sp} \{ |e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle, i_1 = 1, \dots, n_1; i_2 = 1, \dots, n_2 \}$, with the following axioms ($\alpha, \beta, \gamma, \eta$ are elements of \mathbb{K}):

- (i) $(\alpha |e_{i_1}^{n_1}\rangle) \otimes |e_{i_2}^{n_2}\rangle = \alpha (|e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle) = |e_{i_1}^{n_1}\rangle \otimes (\alpha |e_{i_2}^{n_2}\rangle)$
- (ii) $(\alpha |e_{i_1}^{n_1}\rangle + \beta |e_{i_1}^{n_1}\rangle) \otimes |e_{i_2}^{n_2}\rangle = \alpha (|e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle) + \beta (|e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle)$
- (iii) $|e_{i_1}^{n_1}\rangle \otimes (\gamma |e_{i_2}^{n_2}\rangle + \eta |e_{i_2}^{n_2}\rangle) = \gamma (|e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle) + \eta (|e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle)$

In general one has two different forms to write the vector $|x\rangle \in \mathbb{K}^{n_1 n_2}$; this can be done by using either two independent indices i_1, i_2 ,

$$|x\rangle = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} x_{i_1, i_2} |e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle,$$

or a single index k ,

$$|x\rangle = \sum_{k=1}^{n_1 n_2} \tilde{x}_k |e_k^{n_1 n_2}\rangle, \quad k = (i_1 - 1)n_2 + i_2.$$

Notice that the latter expression considers the construction of the Kronecker products $|e_{k=(i_1-1)n_2+i_2}^{n_1 n_2}\rangle = |e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle$ as this is indicated in Definition 2. In correspondence, there are $n_1 n_2$ different coefficients x_{i_1, i_2} that can be mapped into the \tilde{x}_k ones and vice versa. The generalization of Definition 1 and Proposition 1 to the case of N factors is straightforward. In that case the state space $\mathbb{K}^{n_1 n_2 \dots n_N}$ is spanned by the basis vectors $|e_{k'}^{n_1 n_2 \dots n_N}\rangle = |e_{i_1}^{n_1}\rangle \otimes |e_{i_2}^{n_2}\rangle \otimes \dots \otimes |e_{i_N}^{n_N}\rangle$, with k' properly defined.

Proposition 2. Let $X_m^{i,j}$ and $X_n^{k,\ell}$ be two Hubbard operators of order n and m respectively. Then the \otimes -product

$$X_m^{i,j} \otimes X_n^{k,\ell} = X_{mn}^{n(i-1)+k, n(j-1)+\ell}, \quad (15)$$

is a Hubbard operator of order mn .

Proof. The proof follows from Definition 1, explicitly

$$\begin{aligned} & X_m^{i,j} \otimes X_n^{k,\ell} \\ &= \begin{pmatrix} & & 0 & & & \\ & 0_{(i-1) \times (j-1)} & \vdots & & 0_{(i-1) \times (n-j)} & \\ & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & 0_{(n-i) \times (j-1)} & \vdots & & 0_{(n-i) \times (n-j)} & & \\ & & 0 & & & & \end{pmatrix} \otimes X_n^{k,\ell} \\ &= \begin{pmatrix} & & 0 & & & \\ & 0_{[n(i-1)+k-1] \times [n(j-1)+\ell-1]} & \vdots & & 0_{[n(i-1)+k-1] \times [mn-n(j-1)-\ell]} & \\ & & 0 & & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & 0_{[mn-n(i-1)-k] \times [n(j-1)+\ell-1]} & \vdots & & 0_{[mn-n(i-1)-k] \times [n(j-1)+\ell-1]} & & \\ & & 0 & & & & \end{pmatrix} \\ &= X_{mn}^{n(i-1)+k, n(j-1)+\ell}. \quad \square \end{aligned}$$

This last result shows that the tensor product is closed on the set of Hubbard operators. The following properties are also fulfilled

Proposition 3. Let $X_\alpha^{\beta,\gamma}$ be Hubbard operators of order α and take $\lambda \in \mathbb{K}$. Then

- i) $X_m^{i,j} \otimes X_n^{k,\ell} \neq X_n^{k,\ell} \otimes X_m^{i,j}$ in general.
- ii) $(X_m^{i,j} \otimes X_n^{k,\ell})^T = (X_m^{i,j})^T \otimes (X_n^{k,\ell})^T$.
- iii) $(\lambda X_m^{i,j}) \otimes X_n^{k,\ell} = \lambda (X_m^{i,j} \otimes X_n^{k,\ell}) = X_m^{i,j} \otimes (\lambda X_n^{k,\ell})$.
- iv) $(X_m^{i,j} + X_m^{r,s}) \otimes X_n^{k,\ell} = X_m^{i,j} \otimes X_n^{k,\ell} + X_m^{r,s} \otimes X_n^{k,\ell}$.
- v) $X_n^{k,\ell} \otimes (X_m^{i,j} + X_m^{r,s}) = X_n^{k,\ell} \otimes X_m^{i,j} + X_n^{k,\ell} \otimes X_m^{r,s}$.
- vi) $(X_m^{i,j} \otimes X_n^{k,\ell}) \otimes X_p^{r,s} = X_m^{i,j} \otimes (X_n^{k,\ell} \otimes X_p^{r,s})$.

Proof. See Ref. [13], pp. 230.

Thus, the product \otimes is distributive over ordinary matrix addition (iv), (v), associative (vi), compatible with ordinary matrix transposition (ii) as well as with matrix multiplication by an scalar (iii) and, in general, non-abelian (i). We can use now the X -representation of n -square matrices (8), and the properties included in the Propositions 1–3, to define the \otimes -product of an arbitrary number of operators A_k in terms of simple relations of subscripts. We have the next proposition.

Proposition 4. Let $A_r = [a_{i,j}^{(r)}]$ be a square matrix of order n_r . The Kronecker product $A_1 \otimes A_2 \otimes \cdots \otimes A_{k+1}$ is the square matrix of order $n^{(k)} = n_1 n_2 \cdots n_{k+1}$, expressed as the following linear combination of Hubbard operators

$$A = A_1 \otimes A_2 \otimes \cdots \otimes A_{k+1} = \sum_{p,q=1}^{n^{(k)}} \tilde{a}_{p,q}^{(k)} X_{n^{(k)}}^{p,q}, \quad k \geq 1, \quad (16)$$

where

$$\tilde{a}_{p,q}^{(k)} = a_{p_k, q_k}^{(1)} \prod_{s=0}^{k-1} a_{p_s + n_{k-s+1} - n_{k-s+1} p_{s+1}, q_s + n_{k-s+1} - n_{k-s+1} q_{s+1}}^{(k-s+1)}, \quad (17)$$

and

$$p_s = \left\lceil \frac{p}{\prod_{\ell=1}^s n_{k-\ell+2}} \right\rceil, \quad q_s = \left\lceil \frac{q}{\prod_{\ell=1}^s n_{k-\ell+2}} \right\rceil. \quad (18)$$

Here

$$\lceil x \rceil = \min\{z \in \mathbb{Z} : x \leq z\} \quad (19)$$

is the ceiling function of $x \in \mathbb{R}$ (this yields the smallest integer greater than or equal to x).

Proof. See Ref. [13], pp. 238-39.

The results reported in the previous propositions allow to recover all the known properties of the Kronecker product of square matrices. Most of them are straightforwardly extended to matrices of arbitrary order though some caution is necessary (see the discussion on the matter in Section 2.2 of [13]). In this context we present the next theorems and propositions without a proof (the proofs in X -representation can be found in our review [13], for conventional approaches see e.g. [7–9]).

Theorem 1. Let A , B and C be n -square matrices and $\lambda \in \mathbb{K}$. Then

- i) In general, $A \otimes B \neq B \otimes A$.
- ii) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ and $(A \otimes B)^T = A^T \otimes B^T$.
- iii) $(\lambda A) \otimes B = \lambda(A \otimes B) = A \otimes (\lambda B)$.
- iv) $(A + B) \otimes C = A \otimes C + B \otimes C$.
- v) $A \otimes (B + C) = A \otimes B + A \otimes C$.
- vi) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.

Theorem 2. Let $A = [a_{i,j}]$, $C = [c_{p,q}]$, and $B = [b_{k,\ell}]$, $D = [d_{r,s}]$, be pairs of n and m -square matrices respectively. The usual matrix product of the nm -square matrices $A \otimes B$ and $C \otimes D$ fulfills

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \quad (20)$$

That is, the Kronecker product of square matrices is compatible with ordinary matrix multiplication.

Proposition 5. Let $A = [a_{i,j}]$ and $B = [b_{k,\ell}]$ be two square matrices of order n and m respectively. Then

$$\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B) = \text{Tr}(B \otimes A).$$

Proposition 6. Let $A = [a_{i,j}]$ and $B = [b_{k,\ell}]$ be two n -square matrices. Then

$$\text{Det}(A \otimes B) = (\text{Det } A)^n (\text{Det } B)^n. \quad (21)$$

Additional properties (and proofs) of the Kronecker product of matrices can be found in the books [5–7]. The most recent summary of the properties of the \otimes operation has been reported in [9] (see also [8]).

4. Applications

4.1. Permutation matrices

As a first example consider a permutation defined by the bijection π of the set of natural numbers $S = \{1, \dots, n\}$ onto itself. In the Cauchy's two-line notation this map reads as

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

The set of all $n!$ permutations of S forms the symmetric (or permutation) group S_n . The identity element π_e is defined as $\pi_e(k) = k$ for all k in S . A linear representation of S_n is obtained by assigning a matrix P_π per each permutation π ; this is a square matrix of order n that has only one entry 1 per row and column, and is zero elsewhere. In X-representation we have

$$P_\pi = \sum_{j=1}^n X_n^{j, \pi(j)}. \quad (22)$$

Then the product of P_σ and P_π , two permutation matrices of order n , yields

$$P_\sigma P_\pi = \sum_{k, \ell=1}^n X_n^{k, \sigma(k)} X_n^{\ell, \pi(\ell)} = \sum_{k, \ell=1}^n \delta_{\sigma(k), \ell} X_n^{k, \pi(\ell)} = \sum_{k=1}^n X_n^{k, \pi(\sigma(k))} = P_{\pi \circ \sigma}. \quad (23)$$

This shows that the composition $\pi \circ \sigma$ of two permutations π and σ is obtained from the product of the corresponding matrices. It also follows that in general the product of permutation matrices is non-commutative as $P_\pi P_\sigma = P_{\sigma \circ \pi}$ and $\sigma \circ \pi \neq \pi \circ \sigma$. The inverse of P_π is the matrix P_π^T , indeed

$$P_\pi P_\pi^T = \sum_{k, \ell=1}^n \delta_{\pi(k), \pi(\ell)} X_n^{k, \ell} = \sum_{k=1}^n X_n^{k, k} = \mathbb{I}_n. \quad (24)$$

In similar form, $P_\pi^T P_\pi = \mathbb{I}_n$, so that $(P_\sigma P_\pi)^{-1} = P_\pi^{-1} P_\sigma^{-1}$. On the other hand, the action of a permutation matrix P_π on a vector $|x\rangle \in \mathbb{K}^n$ reads

$$P_\pi |x\rangle = \sum_{j, k=1}^n x_k X_n^{j, \pi(j)} |e_k^n\rangle = \sum_{j, k=1}^n x_k \delta_{\pi(j), k} |e_j^n\rangle = \sum_{j=1}^n x_{\pi(j)} |e_j^n\rangle. \quad (25)$$

Now we consider the application of the Kronecker product to construct permutation matrices.

Theorem 3. The square matrix

$$\Pi = \sum_{i, j=1}^n X_n^{i, j} \otimes X_n^{j, i} \quad (26)$$

is a permutation matrix of order n^2 , defined by the rule

$$\pi(p) = n(p + n - 1) - (n^2 - 1)p', \quad p = 1, 2, \dots, n^2, \quad (27)$$

with

$$p' = \left\lceil \frac{p}{n} \right\rceil \quad (28)$$

the ceiling function applied on $\frac{p}{n}$.

Proof. See Ref. [13], pp 231.

To get some insight on the meaning of the permutation matrix (26) let us consider an arbitrary contravariant tensor of rank 2:

$$|x_1\rangle \otimes |x_2\rangle = \sum_{i_1, i_2=1}^n x_{i_1} x_{i_2} |e_{i_1}^n\rangle \otimes |e_{i_2}^n\rangle = \sum_{i_1, i_2=1}^n x_{i_1} x_{i_2} |e_{(i_1-1)n+i_2}^{n^2}\rangle. \quad (29)$$

The action of Π on this last vector reads $\Pi(|x_1\rangle \otimes |x_2\rangle) = |x_2\rangle \otimes |x_1\rangle$. Thus, relative to the indices labeling the contravariant tensor (29), the operator Π corresponds to the bijection $\pi_2 : (1, 2) \rightarrow (2, 1)$. Hence $\Pi \equiv P_{\pi_2} \in S_2$. Indeed, there are only $2! = 2$ different permutations on the set $\{1, 2\}$, these are the identity $\pi_e \equiv \pi_1$ and π_2 . In Hubbard representation we have

$$P_{\pi_1} = \sum_{i=1}^n X_n^{i,i} \otimes X_n^{i,i} \equiv \sum_{i,j=1}^n \delta_{ij} X_n^{i,j} \otimes X_n^{j,i}, \quad P_{\pi_2} = \Pi. \quad (30)$$

Therefore one arrives at the symmetrization operator

$$\mathcal{S}_{(2)} = \frac{1}{2}(P_{\pi_1} + P_{\pi_2}) = \frac{1}{2} \sum_{i,j=1}^n (1 + \delta_{ij}) X_n^{i,j} \otimes X_n^{j,i}, \quad (31)$$

together with the anti-symmetrization operator

$$\mathcal{A}_{(2)} = \frac{1}{2} [\chi(\pi_1)P_{\pi_1} + \chi(\pi_2)P_{\pi_2}] = \frac{1}{2} \sum_{i,j=1}^n [\chi(\pi_1) + \chi(\pi_2)\delta_{ij}] X_n^{i,j} \otimes X_n^{j,i}, \quad (32)$$

where $\chi(\pi)$ is the parity of the bijection π [17]. The generalization of the above results to tensors of arbitrary rank is straightforward:

Proposition 7. The operators

$$\mathcal{S}_{(p)} = \frac{1}{p!} \sum_{\ell=1}^p P_{\pi_\ell} \quad \text{and} \quad \mathcal{A}_{(p)} = \frac{1}{p!} \sum_{\ell=1}^p \chi(\pi_\ell) P_{\pi_\ell} \quad (33)$$

with $\pi_\ell \in S_p$, $\pi_1 \equiv \pi_e$, and P_{π_ℓ} a definite linear combination of the Kronecker products

$$X_n^{i_1, j_1} \otimes X_n^{i_2, j_2} \otimes \dots \otimes X_n^{i_p, j_p}, \quad i_k, j_k \in \{1, \dots, n\},$$

produce respectively the symmetrization and anti-symmetrization of the contravariant tensors of rank p .

The Kronecker product of permutation matrices is also compatible with the composition of permutations.

Theorem 4. Let $P_\pi(n)$ and $P_\sigma(m)$ be the n and m -permutation matrices defined by the rules π and σ respectively. The Kronecker product $P_\pi(n) \otimes P_\sigma(m)$ is the nm -permutation matrix $P_\alpha(n, m)$ defined by the rule

$$\alpha(p) = m[\pi(p') - 1] + \sigma(p - mp' + m), \quad (34)$$

with $p' = \lceil \frac{p}{m} \rceil$.

Proof. See Ref. [13], pp 233.

Although the product $X_n^{i,j} \otimes X_m^{k,\ell}$ is non-abelian in general (Proposition 3i), it is possible to arrive at $X_m^{k,\ell} \otimes X_n^{i,j}$ by applying the appropriate permutation of rows and columns in $X_n^{i,j} \otimes X_m^{k,\ell}$. That is, these last operators must be permutation equivalent.

Proposition 8. The Kronecker product $X_n^{i,j} \otimes X_m^{k,\ell}$ is permutation equivalent to $X_m^{k,\ell} \otimes X_n^{i,j}$. That is, there exist P_π , a permutation matrix of order nm , such that

$$P_\pi^T \left(X_n^{i,j} \otimes X_m^{k,\ell} \right) P_\pi = X_m^{k,\ell} \otimes X_n^{i,j}. \quad (35)$$

Proof. See Ref. [13], pp 233.

In general, for any square matrices A and B we have

Theorem 5. Let $A = [a_{i,j}]$ and $B = [b_{k,\ell}]$ be two square matrices of order n and m respectively. The Kronecker product $A \otimes B$ is permutation equivalent to $B \otimes A$.

Proof. From Proposition 8 we know that there exists a permutation matrix P such that (35) is true. Then, by linearity in the conventional matrix product we have

$$\begin{aligned} P^T(A \otimes B)P &= \sum_{i,j}^n \sum_{k,\ell}^m a_{i,j} b_{k,\ell} \left[P^T \left(X_n^{i,j} \otimes X_m^{k,\ell} \right) P \right] \\ &= \sum_{i,j}^n \sum_{k,\ell}^m b_{k,\ell} a_{i,j} X_m^{k,\ell} \otimes X_n^{i,j} = B \otimes A. \quad \square \end{aligned}$$

Permutation matrices play an important role in combinatorics and quantum mechanics (see e.g. Ch. 13 of Ref. [18] and Ref. [17]). For example, the Schrödinger equation is invariant under the permutation of electrons since the physical equivalence of all these particles.

4.2. The Haddamard matrix

The n -square matrices with ± 1 entries and having pairwise orthogonal rows are named after Hadamard [19]. In the simplest case ($n = 2$) one has

$$H = \frac{1}{\sqrt{2}} \sum_{i,j=1}^2 (-1)^{(i-1)(j-1)} X_2^{i,j} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (36)$$

This operator is unitary and its action on a vector $|x\rangle \in \mathbb{K}^2$ reads

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{i,k=1}^2 (-1)^{(i-1)(k-1)} x_k |e_i^2\rangle = \frac{1}{\sqrt{2}} [(x_1 + x_2)|e_1^2\rangle + (x_1 - x_2)|e_2^2\rangle]. \quad (37)$$

The multiplication of H with itself yields

$$H^2 = HH = \sum_{i,\ell=1}^2 c_{i,\ell} X_2^{i,\ell} = \mathbb{I}_2, \quad c_{i,\ell} = \frac{1 + (-1)^{i+\ell}}{2} = \delta_{i,\ell}. \quad (38)$$

The Kronecker powers of Hadamard matrices $H^{\otimes k+1}$ are of interest in quantum computing algorithms. Let us give a concrete realization of such powers in terms of Hubbard operators

Proposition 9. Let H be the Hadamard matrix (36), then

$$H^{\otimes k+1} = \frac{1}{\sqrt{2^{k+1}}} \sum_{p,q=1}^{2^{k+1}} (-1)^{\vec{p} \cdot \vec{q}} X_{2^{k+1}}^{p,q}, \quad k \geq 1, \quad (39)$$

with $p_s = \lceil \frac{p}{2^s} \rceil$, $q_s = \lceil \frac{q}{2^s} \rceil$, and

$$\vec{p} \cdot \vec{q} := \sum_{s=0}^k (p_s - 1)(q_s - 1). \quad (40)$$

Proof. See Ref. [13], pp 236.

The action of $H^{\otimes k+1}$ on the basis vectors of $\mathbb{K}^{2^{k+1}}$ reads

$$H^{\otimes k+1} |e_j^{2^{k+1}}\rangle = \frac{1}{\sqrt{2^{k+1}}} \sum_{p=1}^{2^{k+1}} (-1)^{\sum_{s=0}^k (p_s - 1)(j_s - 1)} |e_p^{2^{k+1}}\rangle, \quad (41)$$

with $y_s = \lceil \frac{y}{2^s} \rceil$ for $y = p, j$. As an example consider the case $k = 1$ for which we have $H_4 = H \otimes H = H^{\otimes 2}$. Explicitly,

$$H_4 = H^{\otimes 2} = \frac{1}{2} \sum_{p,q=1}^4 (-1)^{(p-1)(q-1) + (\lceil \frac{p}{2} \rceil - 1)(\lceil \frac{q}{2} \rceil - 1)} X_4^{p,q} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Hence,

$$2H^{\otimes 2} |e_j^4\rangle \rightarrow \begin{cases} |e_1^4\rangle + |e_2^4\rangle + |e_3^4\rangle + |e_4^4\rangle, & j = 1 \\ |e_1^4\rangle - |e_2^4\rangle + |e_3^4\rangle - |e_4^4\rangle, & j = 2 \\ |e_1^4\rangle + |e_2^4\rangle - |e_3^4\rangle - |e_4^4\rangle, & j = 3 \\ |e_1^4\rangle - |e_2^4\rangle - |e_3^4\rangle + |e_4^4\rangle, & j = 4 \end{cases}$$

In quantum computing it is usual to express the basis vectors $|e_k^4\rangle$ in the binary form

$$|00\rangle \equiv |e_1^4\rangle, \quad |01\rangle = |e_2^4\rangle, \quad |10\rangle = |e_3^4\rangle, \quad |11\rangle = |e_4^4\rangle.$$

To translate the above results into the binary form, we present the following definition

Definition 3 Consider a positive integer $x \leq 2^{k+1}$ with $k \in \mathbb{N}$. The expansion of x in powers of 2 is defined by the binary coefficients $x_s \in \{0, 1\}$, $s = 0, 1, \dots, k+1$, as follows

$$x = \sum_{i=0}^{k+1} x_i 2^i. \quad (42)$$

The coefficients of the linear combination (39) can be written in binary form according to the following proposition

Proposition 10 Let p and q be respectively the i -th and j -th powers of 2 with $i, j = 0, 1, \dots, k+1$, and $k \in \mathbb{N}$. Then

$$(-1)^{\sum_{s=0}^k (\lceil \frac{p}{2^s} \rceil - 1)(\lceil \frac{q}{2^s} \rceil - 1)} = (-1)^{\sum_{s=0}^k (p-1)_s (q-1)_s}, \quad (43)$$

where $(p-1)_s$ and $(q-1)_s$ are the s -th binary coefficients of $p-1$ and $q-1$ respectively.

Proof. See Ref. [13], pp 237.

We finally arrive at

$$2H^{\otimes 2}|e_j^4\rangle \rightarrow \begin{cases} (-1)^{0\cdot 0+0\cdot 0}|e_1^4\rangle + (-1)^{1\cdot 0+0\cdot 0}|e_2^4\rangle + (-1)^{0\cdot 0+0\cdot 1}|e_3^4\rangle + (-1)^{1\cdot 0+1\cdot 0}|e_4^4\rangle, & j=1 \\ (-1)^{0\cdot 1+0\cdot 0}|e_1^4\rangle + (-1)^{1\cdot 1+0\cdot 0}|e_2^4\rangle + (-1)^{0\cdot 1+1\cdot 0}|e_3^4\rangle + (-1)^{1\cdot 1+1\cdot 0}|e_4^4\rangle, & j=2 \\ (-1)^{0\cdot 0+0\cdot 1}|e_1^4\rangle + (-1)^{1\cdot 0+0\cdot 1}|e_2^4\rangle + (-1)^{0\cdot 0+1\cdot 1}|e_3^4\rangle + (-1)^{0\cdot 1+1\cdot 1}|e_4^4\rangle, & j=3 \\ (-1)^{0\cdot 1+0\cdot 1}|e_1^4\rangle + (-1)^{1\cdot 1+0\cdot 1}|e_2^4\rangle + (-1)^{0\cdot 1+1\cdot 1}|e_3^4\rangle + (-1)^{1\cdot 1+1\cdot 1}|e_4^4\rangle, & j=4 \end{cases}$$

On the other hand, in binary form the basis vectors are given by the rule

$$|e_j^{2^{k+1}}\rangle \rightarrow |j-1\rangle_{(k)} := |(j-1)_0, (j-1)_1, \dots, (j-1)_k\rangle,$$

with $(j-1)_s \in \{0, 1\}$ the binary coefficients of $j-1$ up to 2^k . Finally, given a vector $|x\rangle$ written in binary notation, the equation (41) reads

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_z (-1)^{x \cdot z} |z\rangle,$$

where $|z\rangle$ is also in binary form and $n = 2^{k+1}$.

4.3. The Heisenberg XXX model

In this section we analyze the diagonalization of n -level Hamiltonians of the form

$$H = \sum_{p=1}^n \varepsilon_p X_n^{p,p} + \sum_{\substack{p,q=1 \\ p \neq q}}^n V_{p,q} X_n^{p,q}, \quad V_{q,p} = \overline{V_{p,q}}. \quad (44)$$

We use the unitary transformation method introduced in reference [20] (see also [21]). This method is iterative and demands the off-diagonal elements of the transformed Hamiltonian

$$H' = U_{k,m}(\alpha) H U_{k,m}^\dagger(\alpha) = \sum_{p=1}^n \varepsilon'_p X_n^{p,p} + \sum_{\substack{p,q=1 \\ p \neq q}}^n V'_{p,q} X_n^{p,q}, \quad (45)$$

be equal to zero. Here the set of operators

$$U_{k,m}(\alpha) = \exp \left(\alpha X_n^{k,m} - \overline{\alpha} X_n^{m,k} \right), \quad m > k = 1, 2, \dots, n, \quad \alpha = |\alpha| e^{i\mu}, \quad (46)$$

are unitary and α is a complex parameter. Moreover, the coefficients in the transformed Hamiltonian (45) are given by

$$\begin{aligned} \varepsilon'_k &= \frac{1}{2} [\varepsilon_k + \varepsilon_m + (\varepsilon_k - \varepsilon_m) \cos 2|\alpha| + 2\Re(V_{k,m} e^{-i\mu}) \sin 2|\alpha|], \\ \varepsilon'_m &= \frac{1}{2} [\varepsilon_k + \varepsilon_m - (\varepsilon_k - \varepsilon_m) \cos 2|\alpha| - 2\Re(V_{k,m} e^{-i\mu}) \sin 2|\alpha|], \\ V'_{k,m} e^{i\mu} &= \frac{1}{2} \left[\frac{1}{2} (\varepsilon_m - \varepsilon_k) \sin 2|\alpha| + V_{k,m} e^{-i\mu} \cos^2 |\alpha| - \overline{V}_{k,m} e^{i\mu} \sin^2 |\alpha| \right], \\ V'_{k,p} &= V_{k,p} \cos |\alpha| + V_{m,p} e^{i\mu} \sin |\alpha|, \\ V'_{m,p} &= V_{m,p} \cos |\alpha| - V_{p,k} e^{i\mu} \sin |\alpha|, \\ \varepsilon'_p &= \varepsilon_p, \quad V'_{p,q} = V_{p,q} \quad p, q \neq k, m. \end{aligned} \quad (47)$$

The condition $V'_{k,m} = 0$ produces

$$\tan 2|\alpha| = \frac{2(-1)^{\kappa+1}|V_{k,m}|}{\varepsilon_m - \varepsilon_k}. \quad (48)$$

Hence

$$\varepsilon'_k = \frac{1}{2} - \sqrt{\frac{1}{4}(\varepsilon_m - \varepsilon_k)^2 + |V_{k,m}|^2}, \quad \varepsilon'_m = \frac{1}{2} + \sqrt{\frac{1}{4}(\varepsilon_m - \varepsilon_k)^2 + |V_{k,m}|^2}. \quad (49)$$

The Hamiltonian (44) is diagonalized by iterating the above procedure as many times as necessary. Now we consider a system of n spin-1/2 particles that interact according to the Hamiltonian

$$H = -\frac{1}{2} \sum_{j=1}^n \left(J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z \right), \quad (50)$$

$$\sigma_j^k = \underbrace{\mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2}_{j-1} \otimes \sigma^k \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2.$$

Here $\sigma_{n+1}^k := \sigma_1^k$, with σ^k , $k = x, y, z$, standing for the Pauli matrices. Such a Hamiltonian is known as the Heisenberg Hamiltonian [21]. We will analyze the case when $J_x = J_y = J_z = J$ (the XXX-model) and $n = 2$

$$H = -\frac{J}{2} (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z). \quad (51)$$

In Hubbard representation, the last Hamiltonian reads

$$H = -\frac{J}{4} \left(X_4^{1,1} - X_4^{2,2} - X_4^{3,3} + X_4^{4,4} \right) - 2J \left(X_4^{2,3} + X_4^{3,2} \right). \quad (52)$$

To diagonalize it we make only a single transformation $U_{2,3}(\alpha)$ and from (47) we get $\cos(2|\alpha|) = 0$, $\sin \mu = 0$, and the system

$$\begin{aligned} \varepsilon'_1 &= \varepsilon_1 = -\frac{J}{4}, & \varepsilon'_2 &= \frac{1}{2}(\varepsilon_2 + \varepsilon_3) - V_{2,3} = \frac{9}{4}J, \\ \varepsilon'_3 &= \frac{1}{2}(\varepsilon_2 + \varepsilon_3) + V_{2,3} = -\frac{7}{4}J, & \varepsilon'_4 &= \varepsilon_4 = -\frac{J}{4}. \end{aligned} \quad (53)$$

Therefore we arrive at the diagonal Hamiltonian

$$\mathcal{H}' = -\frac{J}{4} \left(X_4^{1,1} - 9X_4^{2,2} + 7X_4^{3,3} + X_4^{4,4} \right). \quad (54)$$

4.4. Atom + Field models of interaction

In this section we consider the problem of the interaction between a two-level atom (qubit) and a radiation field from both the semiclassical and quantized approaches. In both cases, the Hubbard operators are useful in the construction of the time evolution operator.

4.4.1. Semiclassical model. The ground and excited states of a qubit will be denoted as $|-\rangle$ and $|+\rangle$, respectively. The corresponding Hamiltonian is given by

$$H_0 = \hbar \frac{\omega_a}{2} \sigma_3, \quad (55)$$

here ω_a is the transition frequency. On the other hand, the interaction Hamiltonian is given through the dipolar approximation

$$H_I = -\vec{p} \cdot \vec{E}(t). \quad (56)$$

Here \vec{p} is the dipolar momentum of the atom and $\vec{E}(t)$ is a classical radiation field of the form

$$\vec{E}(t) = \mathcal{E} (e^{-i\omega_f t} + e^{i\omega_f t}) \hat{e}_f, \quad (57)$$

where \mathcal{E} is the amplitude, ω_f the frequency and \hat{e}_f the polarization vector. Thus, the total Hamiltonian is therefore $H = H_0 + H_I$. In the rotating wave approximation this Hamiltonian H can be written in a free-units system as [22]:

$$H = \frac{\Delta}{2} \sigma_3 + g(\sigma_+ + \sigma_-), \quad \Delta = 1 - \omega_f/\omega_a, \quad (58)$$

with Δ the detuning and g the coupling constant. In Hubbard notation the previous Hamiltonian is written as follows

$$H = H_0 + H_I, \quad H_0 = \frac{\Delta}{2} \sum_{p=1}^2 X_2^{p,p}, \quad H_I = g \sum_{p=1}^2 X_2^{p,3-p}. \quad (59)$$

In the case of exact resonance, the time evolution operator is given by

$$U(t) = e^{iH_I t} = \sum_{p,q=1}^2 u_{p,q}(t) X_2^{p,q}, \quad u_{p,q}(t) = e^{i\frac{\pi}{2}|p-q|} \cos\left(gt - \frac{\pi}{2}|p-q|\right) \quad (60)$$

The last expression (60) is useful in considering the evolution of two non-interacting qubits. The time evolution in this case would read

$$U = U_1(t) \otimes U_2(t), \quad (61)$$

where each one of the $U_i(t)$ are written in terms of the X-operators

$$U_i(t) = \sum_{p,q=1}^2 u_{p,q}(t) X_2^{p,q}, \quad i = 1, 2. \quad (62)$$

According to equation (15), the time evolution operator of the whole system reads

$$U(t) = \sum_{p,q=1}^4 u_{p',q'}(t) u_{p+2-2p',q+2-2q'}(t) X_4^{p,q}. \quad (63)$$

The time evolution of any state in the corresponding Hilbert space is obtained by the action of this last operator.

4.4.2. Jaynes-Cummings model. The Jaynes-Cummings model describes the interaction between a single atom and a single mode of the quantized electromagnetic field [23]. This considers the Hilbert space associated to the composite system atom+field, so that the vector states of the entire system are spanned by the Kronecker products of the basis vectors belonging to each of the subsystems. In this case, the Hamiltonian includes three parts: the free atom

Hamiltonian given in equation (55), the field alone H_f and the interaction term, given by the Jaynes Cummings Hamiltonian

$$H_I = \gamma(\sigma^+ a + \sigma^- a^\dagger). \quad (64)$$

where a and a^\dagger are the boson ladder operators. Using the Kronecker algebra of the Hubbard operators it is easy to show that $H_0 + H_f$ and H_I are first integrals of the system [24]. The Hamiltonian can be written in terms of the Hubbard operators

$$H_I = \sum_{p=1}^2 N_p X^{p,3-p} \quad N_p = \sqrt{N+2-p}, \quad (65)$$

where the boson number operator $N = a^\dagger a$ has been promoted to act on the vector space of the entire atom+field system: $N = \mathbb{I}_{at} \otimes N$, with \mathbb{I}_{at} the identity operator in the vector space of the atom. Using the X -operator formalism, it is shown that the time evolution operator reads

$$U(t) = e^{-iH_I t} = \sum_{p,q=1}^2 u_{p,q}(N_p) X^{p,q}, \quad (66)$$

with

$$u_{p,q}(N_p) = e^{i\frac{\pi}{2}|p-q|} \cos\left(\gamma t N_p - \frac{\pi}{2}|p-q|\right). \quad (67)$$

This operator is analogous to the time evolution operator given in Eq. (60). For instance, the dynamics of two non-interacting atoms can be analyzed in terms of the unitary operator

$$U(t) = U_1(t) \otimes U_2(t) \quad (68)$$

where $U_1(t)$ and $U_2(t)$ are expressed in X -operator notation [24]:

$$U_i = \sum_{p,q=1}^2 u_{p,q}(N_{i;p}) X_4^{p,q}, \quad i = 1, 2. \quad (69)$$

The operators $N_{i;p}$ are equivalent to the ones defined in (65) for each mode of the field. By virtue of (15), the time evolution operator of the whole system reads

$$U(t) = \sum_{p,q=1}^4 u_{p',q'}(N_{p'}) u_{p+2-2p',q+2-2q'}(N_{q+2-2q'}) X_4^{p,q}. \quad (70)$$

This last operator is useful in the study of the time evolution of entanglement since the reduced atomic density matrix is straightforwardly computed in this framework [25,26]. To end up this section we remark that our model could include an interaction like the Heisenberg Hamiltonian (52). In such a case, the involved operators are compatible with the Hubbard representation.

5. Conclusions

The Kronecker product algebra has been revisited in terms of Hubbard operators. This representation allows to reduce complicated calculations involving large matrices or a large number of factors into simple relations of subscripts. In particular, we have shown that the construction of permutation matrices and the construction of symmetrization operators is straightforward in X -representation. Besides, the Hadamard matrix and its tensor powers are nicely dealt in the Hubbard framework. On the other hand, we have analyzed the Hamiltonian of the XXX Heisenberg model for two particles. The diagonalization of this Hamiltonian and the calculation of the time evolution operator have been easily done in the X -representation.

Acknowledgments

The support of CONACyT project 152574 and IPN project SIP-SNIC-2011/04 is acknowledged. M.E. gratefully acknowledges the funding received towards his Ph.D. through the CONACyT Scholarship.

References

- [1] Brewer J W 1978 Kronecker products and matrix calculus in system theory *IEEE Trans. Circuits Systems* **25** 772
- [2] Steeb W-H and Hardy Y Entanglement 2012 Kronecker Product, Pauli Spin Matrices and a Nonlinear Eigenvalue Problem *Open Syst. Inf. Dyn.* **19** 1250004
- [3] Black G R E, King R C and Wybourne B G 1983 Kronecker products for compact semisimple Lie groups *J. Phys. A: Math. Gen.* **16** 1555
- [4] Henderson H V, Pukelsheim F and Searle S R 1983 On the history of the Kronecker product *Linear and Multilinear Algebra* **14** 113
- [5] Graham A 1981 *Kronecker Products and Matrix Calculus with Applications* (New York: John Wiley and Sons)
- [6] Horn R A and Johnson C A 1991 *Topics in Matrix Analysis* (New York: Cambridge University Press)
- [7] Steeb W-H and Hardy Y 2011 *Matrix Calculus and Kronecker Product* (Singapore: World Scientific) 2nd ed
- [8] Langville A N and Stewart W J 2004 The Kronecker product and stochastic automata networks *J. Comp. Appl. Math.* **167** 429
- [9] Van Loan C F 2000 The ubiquitous Kronecker product *J. Comp. Appl. Math.* **123** 85
- [10] Peres A 1995 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer Academic Publishers)
- [11] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [12] Mielnik B and Rosas-Ortiz O 2009 Quantum Mechanical Laws *Fundamentals of Physics* ed J L Morán-López, Encyclopedia of Life Support Systems (EOLSS), Developed under the Auspices of the UNESCO (Oxford: Eolss Publishers) 2007/Rev. 2009
- [13] Enríquez M and Rosas-Ortiz O 2013 The Kronecker product in terms of Hubbard operators and the Clebsch–Gordan decomposition of $SU(2) \times SU(2)$ *Annals of Physics* **339** 218
- [14] Hubbard J 1963 Electron correlations in narrow energy bands *Proc. R. Soc. (London)* A **276** 238
- [15] Hubbard J 1964 Electron correlations in narrow energy bands II. The degenerate band case *Proc. R. Soc. (London)* A **277** 237
- [16] Hubbard J 1965 Electron correlations in narrow energy bands IV. The atomic representation *Proc. R. Soc. (London)* A **285** 542
- [17] Bacry H 1977 *Lectures on Group Theory and Particle Theory* (New York: Gordon and Breach)
- [18] Wigner E P 1959 *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* transl J J Griffin (New York: Academic Press)
- [19] Hadamard J 1893 Resolution d'une question relative aux determinants *Bull. Sci. Math.* **17** 240
- [20] Val'kov V V 1985 Unitary transformations of the group $U(N)$ and diagonalization of multilevel Hamiltonians *Theor. Math. Phys.* **76** 766
- [21] Ovchinnikov S G and Val'kov V V 2004 *Hubbard Operators in the Theory of Strongly Correlated Electrons* (London: Imperial College Press)
- [22] Enríquez M, Reyes-García R and Rosas-Ortiz R 2013 Time evolution in classical and quantized fields: geometric representation *preprint* Cinvestav
- [23] Jaynes E T and Cummings F W 1963 Comparison of quantum and semiclassical radiation theories with applications to the beam maser *Proc. IEEE* **51** 89
- [24] Enríquez M and Rosas-Ortiz O 2013 Spectral design to control the time evolution of a two-level atom in single-mode radiation fields *preprint* Cinvestav
- [25] Enríquez M and Rosas-Ortiz O 2013 Dynamics of two atoms in a double Jaynes-Cummings model *preprint* Cinvestav
- [26] Enríquez M, Quintana-Huerta C and Rosas-Ortiz O 2013 On the entanglement of a pair of two-level atoms in quantized radiation fields *preprint* Cinvestav