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Bounded motion for classical systems with position-dependent mass

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Abstract. In this work the dynamical equations for a system with position-dependent mass are considered. The phase space trajectories are constructed by means of the factorization method for classical systems. To illustrate how this formalism works the phase space trajectories for position-dependent mass oscillator, Scarf and Pöschl-Teller potentials with Gaussian and singular masses are presented.

1. Introduction

The problem of describing the dynamics of systems with position-dependent mass has gain much interest in the last decades. Yet, it appears in many physical areas such as semiconductor theory [1–3], molecular dynamics [4], geometric optics [5] and astrophysics [6] among others. Of particular importance are its applications in the study of the properties of semiconductors. In this context one can mention, *e. g.*, the evaluation of linear and nonlinear optical properties and the binding energies of square, parabolic, V-shaped and Pöschl-Teller-type quantum dots and quantum wells, in the effective mass approximation. A wide range of mass functions have been used including constant, exponential and inverse of quadratic polynomials depending on the structure and chemical composition of each system [7–12]. Some other important applications include the description of the dynamics of systems in curved spaces [13–15] as well as nonlinear oscillators [16, 17]. In these cases some masses varying as the inverse of quadratic functions evolving under the influence of quadratic potentials have been considered. Further, in the generation of inversion potentials of molecules in density theory, some masses with singularities are used as the reduced mass of the molecules as functions of the inversion coordinate [4], and some power law masses model the galactic mass loss [6] and the deploying of a cable on a reel [18].

The position-dependent mass concept is, by itself, a fundamental problem which is far from being understood. On this matter, many contributions have been developed all over the years from different approaches [19–29]. In particular, the factorization method [30–32] has been applied in constructing exactly solvable position-dependent mass potentials [21–28]. From the classical point of view, the factorization of the Hamiltonian in terms of two functions that together with the Hamiltonian induce a particular Poisson algebra has been discussed in [27,33].

In this regime, the position-dependent mass functions lead to dynamical equations including terms quadratic in the velocity, a problem which have been continuously discussed [18, 34–39]. In this work we consider the dynamical equations for classical systems with position-dependent mass in the Newton, Lagrange and Hamilton approaches. The factorization method applied to an *energy invariant* is used to construct, in an algebraic form, the phase space trajectories for systems under the effect of harmonic oscillator, Scarf and Pöschl-Teller potentials. The formalism works equally well for regular and singular masses. To illustrate this fact, the cases of a Gaussian mass and a mass with a quadratic singularity are considered in order to connect our approach to some of the applications mentioned above.

The paper is organized as follows. In section 2 we derive the Lagrangian and Hamiltonian equations of motion for a position-dependent mass system. These equations include a non-inertial force quadratic in the velocity. Since the corresponding Hamiltonian is not conserved, an integral of motion in energy units is constructed that allow to express the dynamical equations in the standard form. In section 3, the factorization method is applied to this *energy invariant* in order to construct the phase space trajectories. In section 4 the underlying Poisson algebras for oscillator, Scarf and Pöschl-Teller position-dependent mass systems are established and the corresponding phase space trajectories for Gaussian and singular masses are presented. The paper close with some concluding remarks.

2. The position-dependent mass problem in the classical framework

Consider a classical system with position-dependent mass $m(x) > 0$. Suppose that a force $F(x, \dot{x})$ is acting on the system. The Newton second law can be then stated as (assuming null velocity for the accreted or ablated mass)

$$F(x, \dot{x}) = \frac{dp}{dt} = m'(x)\dot{x}^2 + m(x)\ddot{x}, \tag{1}$$

where $p = m(x)\dot{x}$ is the linear momentum of the system and $'$ stand for the derivative with respect to the position. A trajectory in the (x, v) plane can be then constructed by solving the nonlinear set of equations

$$\dot{x} = v, \quad \dot{v} = \frac{1}{m(x)}\left[F(x, \dot{x}) - m'(x)\dot{x}^2\right]. \tag{2}$$

Since $\dot{x}^2 > 0$, it is clear that for an increasing mass, *i.e.*, $m'(x) > 0$ the system is decelerated, while it is accelerated for a decreasing mass $m'(x) < 0$.

2.1. The Lagrangian and Hamiltonian approaches

For external conservative forces $F(x) = -V'(x)$ and applying de D'Alembert principle we obtain that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F(x) + \tilde{R}(x, \dot{x}), \tag{3}$$

where the kinetic energy T and the reacting thrust $\tilde{R}$ [33] are given by

$$T(x, \dot{x}) = \frac{1}{2}m(x)\dot{x}^2, \quad \tilde{R}(x, \dot{x}) = -\frac{1}{2}m'(x)\dot{x}^2. \tag{4}$$

Since the force is velocity independent the Lagrangian form of the Newton second law (1) reads

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = \tilde{R}(x, \dot{x}), \quad \mathcal{L}(x, \dot{x}) = T(x, \dot{x}) - V(x). \tag{5}$$

The canonical momentum obtained from the Lagrangian $\mathcal{L}$ turn out to be the linear momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m(x)\dot{x}. \tag{6}$$

Now the position-dependent mass Hamiltonian $\mathcal{H}$ is obtained from the Legendre transformation

$$\mathcal{H}(x, p) = p\dot{x} - \mathcal{L}(x, \dot{x}) = \frac{p^2}{2m(x)} + V(x). \tag{7}$$

Note that this Hamiltonian is not time invariant since

$$\frac{d}{dt}\mathcal{H} = -\frac{1}{2}m'(x)\dot{x}^3 \neq 0. \tag{8}$$

The dynamical equation (1) can be expressed in canonical form as

$$\dot{x} = \{x, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = \{p, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial x} + R, \tag{9}$$

with

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} \tag{10}$$

the Poisson bracket and

$$R(x, p) = \tilde{R}\left(x, \dot{x}(x, p)\right) = -\frac{m'(x)}{m(x)}\left(\frac{p^2}{2m(x)}\right) \tag{11}$$

the thrust in the phase space.

So far, we have constructed the Lagrangian and Hamiltonian description of the dynamics of a position-dependent mass system. We have shown that, even when the Lagrangian and Hamiltonian functions are of the standard form, the Lagrange and Hamilton equations are not. Additionally, we have found that the Hamiltonian $\mathcal{H}$ is not time independent, its time rate of change is cubic in the velocity leading to non standard canonical equations and to the problem of dissipative systems [36, 40]. It turns out, however, that the dynamical equations can be expressed in the conventional form by means of an *energy constant of motion*.

2.2. Energy invariant and phase space trajectories

In this Section we construct an *energy invariant* and show that the corresponding dynamical equations have the form of the standard Hamilton ones. Let the function $I(x, v)$ be such an invariant, then, it must satisfy

$$\frac{d}{dt}I(x, v) = \dot{x}\frac{\partial}{\partial x}I(x, v) + \dot{v}\frac{\partial}{\partial v}I(x, v) = 0. \tag{12}$$

Substituting $\dot{x}$ and $\dot{v}$ from (2), this equation can be written as

$$v\frac{\partial I}{\partial x} - \frac{1}{m(x)}\left[V'(x) + m'(x)\dot{x}^2\right]\frac{\partial I}{\partial v} = 0. \tag{13}$$

A simple calculation leads to the function [33]

$$I = \frac{m^2(x)v^2}{2m_0} + \int_{x_0}^{x} \frac{m(s)}{m_0} V'(s) ds = \frac{p^2}{2m_0} + \int_{x_0}^{x} \frac{m(s)}{m_0} V'(s) ds, \tag{14}$$

with m_0 and x_0 two constants with units of mass and position respectively. This invariant can be considered as the conservative Hamiltonian of the system. It is important to stress that it is reduced to the constant mass Hamiltonian $H = \frac{p^2}{2m_0} + V$ in the case that $m = m_0$. Now defining new variables

$$P = \frac{m(x)}{m_0} p = \frac{m^2(x)}{m_0} v = M(x)v, \quad M(x) = \frac{m^2(x)}{m_0}, \tag{15}$$

the energy invariant in the (x, P) phase space is

$$H = \frac{P^2}{2M} + V_{\text{eff}}(x), \tag{16}$$

where

$$V_{\text{eff}}(x) = \int_{x_0}^{x} \frac{m(s)}{m_0} V'(s) ds \tag{17}$$

is the effective potential experienced by the position-dependent mass system.

It is not difficult to show that the dynamical equation (1) can be written in the form

$$\frac{\partial H}{\partial P} = \{x, H\}_{x,P} = \dot{x}, \quad \frac{\partial H}{\partial x} = \{P, H\}_{x,P} = -\dot{P}. \tag{18}$$

Additionally, the time evolution of an observable $O(x, P; t)$ is given by

$$\frac{d}{dt} O(x, P; t) = \{O, H\}_{x,P} + \frac{\partial O}{\partial t}. \tag{19}$$

This approach shows that, in order to preserve the canonical equations, the mass m, the linear momentum p and the potential V must be transformed into M, P and V_{eff} by (15) and (17). The phase space trajectories of a particular system, for an arbitrary mass function $m(x)$, can be now constructed by solving the set of equations (18).

3. Factorization method and phase space trajectories for position-dependent mass systems

Once we have the Hamiltonian description for the position-dependent mass systems, it is worthwhile to mention that one can reach to the solution in two different approaches. In the first one it is assumed that the potential $V(x)$ and the mass function $m(x)$ are known and the phase space trajectories must be determined by integrating the dynamical equations (18). In the second one it is considered that the underlying algebra associated to the dynamics of the system and its mass are given and the potentials and phase space trajectories must be determined. In this work we will adopt the second approach. Thus, we will suppose that the algebraic structure of the system is known and the corresponding position-dependent mass potential and phase space trajectories must be determined. The problem will be solved by means of the factorization method in the (x, P) plane. A complete discussion of the factorization method for one dimensional classical systems can be found in [41].

Suppose that the Hamiltonian

$$H = \frac{P^2}{2M} + V_{\text{eff}}(x) \tag{20}$$

can be factorized as

$$H = A^+ A^- + \epsilon, \tag{21}$$

with

$$A^\pm = \mp i f(x) \frac{P}{\sqrt{2M(x)}} + \sqrt{\gamma H} g(x) + \varphi(x), \tag{22}$$

where $f(x)$, $g(x)$, $\varphi(x)$ are functions to be determined and ϵ is a constant. Here we consider the possibility of bound states for negative energies, then γ take the values $+1$ and -1 for $H > 0$ and $H < 0$ respectively. The factorization approach (20-22) leads to the relation

$$(f^2 + \gamma g^2 - 1)H - f^2 V_{\text{eff}} + 2\varphi g \sqrt{\gamma H} + \varphi^2 + \epsilon = 0, \tag{23}$$

from which it is clear that $g(x)$ and $\varphi(x)$ can not be both different from zero.

Next, we demand that the functions H and $A^\pm$ define the Poisson algebra (for the sake of simplicity, from now on, we drop the subindex x, P from the Poisson brackets)

$$i\left\{H, A^\pm\right\} = \pm\alpha(H) A^\pm \tag{24}$$

$$i\left\{A^-, A^+\right\} = \beta(H) \tag{25}$$

with $\alpha(H)$ and $\beta(H)$ functions of $\sqrt{\gamma H}$ to be fixed. Note that (24) leads to

$$i\left\{H, A^+ A^-\right\} = 0,$$

meaning that A^+A^- is a function of H, consistently with (21). Additionally, the Jacobi identity

$$\left\{H, i\left\{A^-, A^+\right\}\right\} + \left\{A^-, i\left\{A^+, H\right\}\right\} + \left\{A^+, i\left\{H, A^-\right\}\right\} = 0$$

implies that

$$\left\{H, i\left\{A^-, A^+\right\}\right\} = 0,$$

meaning that the Poisson bracket $i\{A^-, A^+\}$ can be written as (25).

Furthermore,

$$i\left\{H, A^\pm\right\} = \pm \frac{2}{\sqrt{2M}} f\left(\sqrt{H} g' + \varphi'\right) A^\pm, \tag{26}$$

which, together with (23) lead to the set of equations

$$f^2 V_{\text{eff}} = \left(f^2 + \gamma g^2 - 1\right) H + \varphi^2 + \epsilon, \tag{27}$$

$$\alpha(H) = \frac{2}{\sqrt{2M}} f\left(\sqrt{\gamma H} g' + \varphi'\right) \tag{28}$$

to determine the functions f, g, φ and α.

On the other hand, the algebraic relations (24) allow us to write two non autonomous integrals of motion

$$Q^\pm(x, P; t) = A^\pm e^{\mp i\alpha(H)t} \tag{29}$$

satisfying

$$\left|Q^\pm\right|^2 = Q^+ \, Q^- = A^+ A^- = H - \epsilon. \tag{30}$$

If $q^\pm$ denote the values of $Q^\pm$, then

$$q^\pm = \sqrt{E - \epsilon} e^{\pm i\theta_0}, \quad \theta_0 \in \mathbb{R}, \quad E - \epsilon > 0.$$

Now the set of equations

$$q^{\pm} = Q^{\pm}\left(x\left(t\right), P\left(t\right); t\right) = \sqrt{E-\epsilon}\, e^{\pm i\theta_0} \tag{31}$$

can be used to determine algebraically the phase trajectories in terms of two constants of motion (E, θ_0) fixed by the initial conditions

$$\sqrt{\gamma E} g\left(x(t)\right) + \varphi\left(x(t)\right) = \sqrt{E-\epsilon}\, \cos\left(\theta_0 + \alpha(E)t\right), \tag{32}$$

$$P(t) = -\sqrt{\frac{2(E-\epsilon)M\left(x(t)\right)}{f^2\left(x(t)\right)}}\, \sin\left(\theta_0 + \alpha(E)t\right). \tag{33}$$

4. Position-dependent mass oscillator, Scarf and Pöschl-Teller potentials

4.1. The harmonic oscillator

Consider again the set of equations (27-28). If we assume that $g(x) = 0$ and $\gamma = 1$, then $\alpha(H) = \beta(H) = \alpha =$ constant and

$$\varphi(x) = \sqrt{\frac{m_0\alpha^2}{2}} \int_c^x J(s)ds, \quad J(x) = \sqrt{\frac{M(x)}{m_0}}. \tag{34}$$

Without loss of generality we may set $f(x) = f_0 = 1$. The potential is then defined by (27) as

$$V_{\text{eff}} = \frac{m_0\alpha^2}{2}\left(\int_c^x J(s)ds\right)^2 + \epsilon. \tag{35}$$

Hence, given the mass $M(x)$, a potential of the harmonic oscillator form (35) is such that the algebraic structure becomes the constant mass oscillator algebra [41]

$$i\left\{A^-, A^+\right\} = \alpha, \quad i\left\{H, A^{\pm}\right\} = \pm\alpha A^{\pm}. \tag{36}$$

4.1.1. Gaussian mass-function Consider the mass function

$$M(x) = m_0 e^{-2\lambda^2 x^2}, \tag{37}$$

with $m_0 > 0$ and $\lambda > 0$. This function is defined on the whole real line, as well as the corresponding harmonic oscillator ($c = 0$)

$$V_{\text{eff}}(x) = \frac{\pi m_0\alpha^2}{8\lambda^2}\left(\text{Erf}\left(\lambda x\right)\right)^2. \tag{38}$$

The phase space trajectories read

$$x(t) = \frac{1}{\lambda}\text{Erf}^{-1}\left[\sqrt{\frac{8\lambda^2}{\pi m_0\alpha^2}}(E-\epsilon)\cos\left(\theta_0 + \alpha t\right)\right], \tag{39}$$

$$P(t) = -\sqrt{2m_0}e^{-\lambda^2 x^2(t)}\sqrt{E}\, \sin\left(\theta_0 + \alpha t\right). \tag{40}$$

The potential, phase space trajectories $(x(t), P(t))$ and the 3D phase space surface $(x(t), P(t), E)$ are shown in Figure 1 for different values of the parameters. The mass takes its maximum value at $x = 0$, as well as the canonical momentum P. As $|x| \to \infty$, the system losses mass and the momentum decrease until the particle reaches the turning points, where P changes sign.

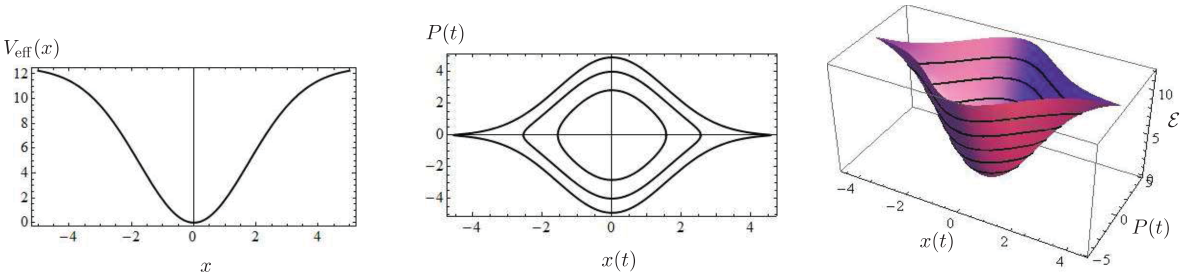

Figure 1. The potential (left), the phase space trajectories (middle) and the 3-D phase space (right) for the position-dependent mass harmonic oscillator with a Gaussian mass. In this plots $\alpha = 2$, $m_0 = 1$, $\lambda = 0.5$ and $\epsilon = 0$.

4.1.2. Singular mass Consider now the mass function

$$M(x) = \frac{m_0}{\lambda^2 x^2}, \tag{41}$$

with $m_0 > 0$ and $\lambda > 0$. This mass is defined in the half real line. The corresponding harmonic oscillator potential ($c = 1$)

$$V_{\text{eff}} = \frac{m_0 \alpha^2}{2\lambda^2} (\ln \lambda x)^2 , \tag{42}$$

is defined in the same domain. The corresponding phase space trajectories have the form

$$x(t) = \frac{1}{\lambda} \exp \left[\sqrt{\frac{2\lambda^2 (E - \epsilon)}{m_0 \alpha^2}} \cos(\theta_0 + \alpha t) \right] \tag{43}$$

$$P(t) = -\sqrt{2m_0(E - \epsilon)} \exp \left[-\sqrt{\frac{2\lambda^2 (E - \epsilon)}{m_0 \alpha^2}} \cos(\theta_0 + \alpha t) \right] \sin(\theta_0 + \alpha t) . \tag{44}$$

Figure 2 shows the potential, the phase space trajectories $(x(t), P(t))$ and the 3D phase space surface $(x(t), P(t), E)$ for different values of the parameters. In this case the behavior of the system is different from the previous one. The mass function has a singularity at $x = 0$. As the particle approaches this point the mass increases as well as the momentum P. When the particle reaches the turning point $x = 0$ the momentum change sign and the motion is reverted.

4.2. Scarf and Pöschl-Teller potentials

Concerning equations (27-28), now consider $g(x) \neq 0$ but $\varphi(x) = 0$. We can see that one simple solution to this set of equations can be set on the form

$$f^2 + \gamma g^2 = 1, \quad \alpha(H) = \beta(H) = \alpha\sqrt{\gamma H}, \tag{45}$$

$$g(x) = \begin{cases} \sin \left[\sqrt{\dfrac{m_0 \alpha^2}{2}} \displaystyle\int_c^x J(s) ds \right] & \gamma = 1 \quad H > 0, \\ \sinh \left[\sqrt{\dfrac{m_0 \alpha^2}{2}} \displaystyle\int_c^x J(s) ds \right] & \gamma = -1 \quad H < 0. \end{cases} \tag{46}$$

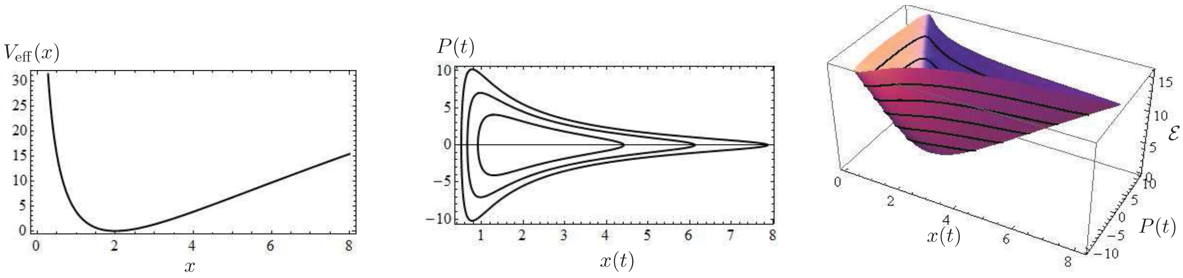

Figure 2. The potential (left), the phase space trajectories (middle) and the 3-D phase space (right) for the position-dependent mass harmonic oscillator with singular mass. In this plots $\alpha = 2$, $m_0 = 1$, $\lambda = 0.5$ and $\epsilon = 0$.

The Scarf and Pöschl-Teller potentials in this case have the form

$$V_{\text{eff}}(x) = \frac{\epsilon}{1-\gamma g^2(x)} = \begin{cases} \dfrac{\epsilon}{\cos^2\sqrt{\frac{m_0\alpha^2}{2}}\int_c^x J(s)ds} & \gamma = 1 \quad H > 0, \\ \dfrac{\epsilon}{\cosh^2\sqrt{\frac{m_0\alpha^2}{2}}\int_c^x J(s)ds} & \gamma = -1 \quad H < 0 \end{cases} \tag{47}$$

respectively.

Now the functions H, $A^{\pm}$ obey the algebraic structure

$$i\left\{A^-, A^+\right\} = \alpha\sqrt{\gamma H}, \quad i\left\{H, A^{\pm}\right\} = \pm\alpha\sqrt{\gamma H}. \tag{48}$$

By defining new functions

$$a^{\pm} = \frac{1}{\alpha}A^{\pm}, \quad a^0 = \gamma\frac{1}{\alpha}\sqrt{\gamma H}, \tag{49}$$

it is possible to see that a^0, $a^{\pm}$ close the $su(11)$ algebra

$$i\left\{a^-, a^+\right\} = 2\alpha^0, \quad i\left\{a^0, a^{\pm}\right\} = \pm a^{\pm}, \tag{50}$$

for $H > 0$ ($\gamma = 1$) and the $su(2)$ algebra

$$i\left\{a^-, a^+\right\} = -2\alpha^0, \quad i\left\{a^0, a^{\pm}\right\} = \pm a^{\pm} \tag{51}$$

for $H < 0$,($\gamma = 1$).

4.2.1. Gaussian mass Consider again the mass function (37). The Scarf and Pöschl-Teller position-dependent mass potentials in this case are

$$V_{\text{eff}} = \begin{cases} \dfrac{\epsilon}{\cos^2\sqrt{\frac{\pi m_0\alpha^2}{8\lambda^2}}\text{Erf}(\lambda x)} & \gamma = 1 \quad H > 0, \\ \dfrac{\epsilon}{\cosh^2\sqrt{\frac{\pi m_0\alpha^2}{8\lambda^2}}\text{Erf}(\lambda x)} & \gamma = -1 \quad H < 0, \end{cases} \tag{52}$$

defined in the domains

$$\mathcal{D} = \left\{x \in \mathbb{R} \left| -\frac{\pi}{2} < \sqrt{\frac{\pi m_0\alpha^2}{8\lambda^2}}\text{Erf}(\lambda x) < \frac{\pi}{2}\right.\right\}, \tag{53}$$

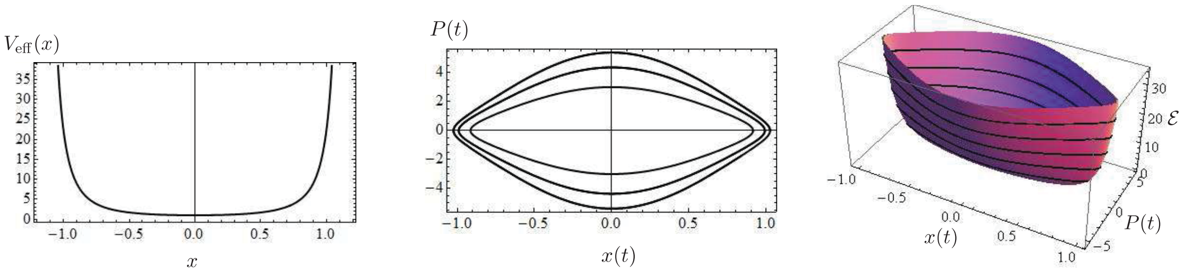

Figure 3. The potential (left), the phase space trajectories (middle) and the 3-D phase space (right) for the position-dependent mass Scarf potential with Gaussian mass. In this plots $\alpha = 2$, $m_0 = 1$, $\lambda = 0.5$ and $\epsilon = 1$.

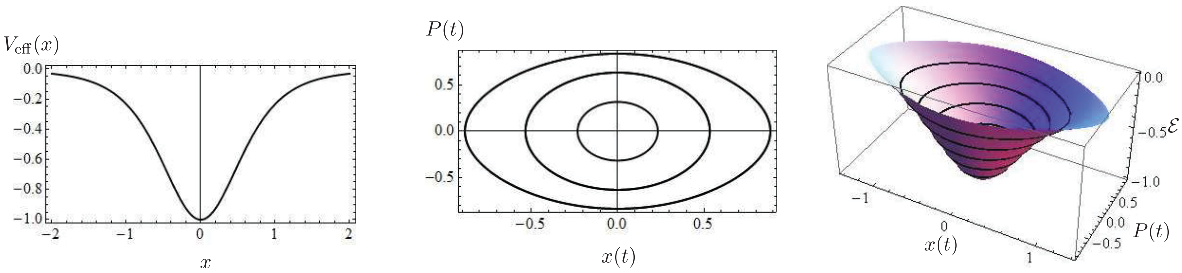

Figure 4. The potential (left), the phase space trajectories (middle) and the 3-D phase space (right) for the position-dependent mass Pöschl-Teller potential with Gaussian mass. In this plots $\alpha = 2$, $m_0 = 1$, $\lambda = 0.5$ and $\epsilon = -1$.

and $\mathbb{R}$ respectively. The phase space trajectories are given by

$$x(t) = \frac{1}{\lambda}\text{Erf}^{-1}\left[\sqrt{\frac{8\lambda^2}{\pi m_0 \alpha^2}}\arcsin\left(\sqrt{\frac{E-\epsilon}{E}}\cos\left(\theta_0 + \alpha\sqrt{E}t\right)\right)\right], \tag{54}$$

$$P(t) = -\sqrt{\frac{2Em_0}{E-(E-\epsilon)\cos^2\left(\theta_0+\alpha\sqrt{E}t\right)}}e^{-\lambda^2 x^2}\sin\left(\theta_0+\alpha\sqrt{E}t\right), \tag{55}$$

for the position-dependent mass Scarf potential and by

$$x(t) = \frac{1}{\lambda}\text{Erf}^{-1}\left[\sqrt{\frac{8\lambda^2}{\pi m_0 \alpha^2}}\text{arcsinh}\left(\sqrt{\frac{E-\epsilon}{-E}}\cos\left(\theta_0 + \alpha\sqrt{-E}t\right)\right)\right], \tag{56}$$

$$P(t) = -\sqrt{\frac{2Em_0}{E+(E-\epsilon)\cos^2\left(\theta_0+\alpha\sqrt{-E}t\right)}}e^{-\lambda^2 x^2}\sin\left(\theta_0+\alpha\sqrt{-E}t\right). \tag{57}$$

for the position-dependent mass Pöschl-Teller potential.

Figures 3 and 4 shows respectively the potential V_{eff}, the phase space trajectories $(x(x), P(t))$ and the 3D phase space surface $(x(t), P(t), E)$ for the Scarf and Pöschl-Teller potentials. The description of the behavior of the system is similar to that given in Section *4.1.1*.

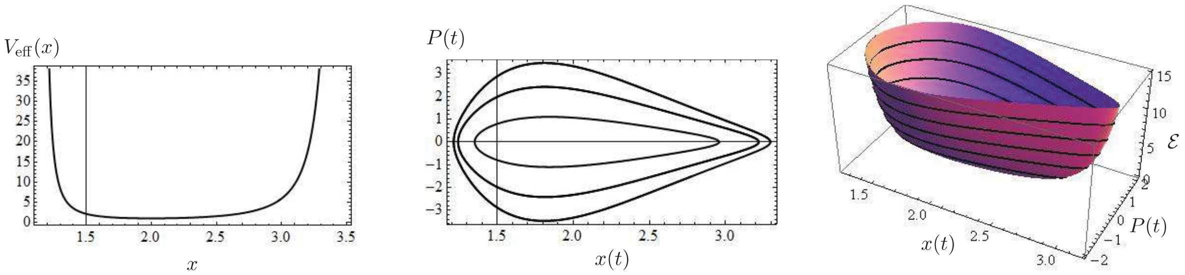

Figure 5. The potential (left), the phase space trajectories (middle) and the 3-D phase space (right) for the position-dependent mass Scarf potential with singular mass. In this plots $\alpha = 2$, $m_0 = 1$, $\lambda = 0.5$ and $\epsilon = 1$.

4.2.2. Singular mass Consider again the mass function (41). The Scarf and Pöschl-Teller position-dependent mass potentials in this case are

$$V_{\text{eff}} = \begin{cases} \dfrac{\epsilon}{\cos^2 \sqrt{\frac{m_0\alpha^2}{2\lambda^2}} \ln(\lambda x)} & \gamma = 1 \quad H > 0, \\ \dfrac{\epsilon}{\cosh^2 \sqrt{\frac{m_0\alpha^2}{2\lambda^2}} \ln(\lambda x)} & \gamma = -1 \quad H < 0, \end{cases} \tag{58}$$

defined in the domains

$$\mathcal{D} = \left\{ x \in \mathbb{R} \middle| -\frac{\pi}{2} < \sqrt{\frac{m_0\alpha^2}{2\lambda^2}} \ln(\lambda x) < \frac{\pi}{2} \right\}, \tag{59}$$

and $(0, \infty)$ respectively. The phase space trajectories are given by

$$x(t) = \frac{1}{\lambda} \exp\left[\sqrt{\frac{2\lambda^2}{m_0\alpha^2}} \arcsin\left(\sqrt{\frac{E-\epsilon}{E}} \cos\left(\theta_0 + \alpha\sqrt{E}t\right) \right) \right], \tag{60}$$

$$P(t) = -\sqrt{\frac{2Em_0}{E - (E-\epsilon)\cos^2\left(\theta_0 + \alpha\sqrt{E}t\right)}} \frac{1}{\lambda x(t)} \sin\left(\theta_0 + \alpha\sqrt{E}t\right) \tag{61}$$

for the position-dependent mass Scarf potential and by

$$x(t) = \frac{1}{\lambda} \exp\left[\sqrt{\frac{2\lambda^2}{m_0\alpha^2}} \text{arcsinh}\left(\sqrt{\frac{E-\epsilon}{-E}} \cos\left(\theta_0 + \alpha\sqrt{-E}t\right) \right) \right], \tag{62}$$

$$P(t) - \sqrt{\frac{2Em_0}{E + (E-\epsilon)\cos^2\left(\theta_0 + \alpha\sqrt{-E}t\right)}} \frac{1}{\lambda x(t)} \sin\left(\theta_0 + \alpha\sqrt{-E}t\right). \tag{63}$$

for the position-dependent mass Pöschl-Teller potential.

Figures 5 and 6 shows respectively the potential V_{eff}, the phase space trajectories $(x(t), P(t))$ and the 3D phase space surface $(x(t), P(t), E)$ for the Scarf and Pöschl-Teller potentials. The description of the behavior of the system is similar to that given in Section *4.1.2*.

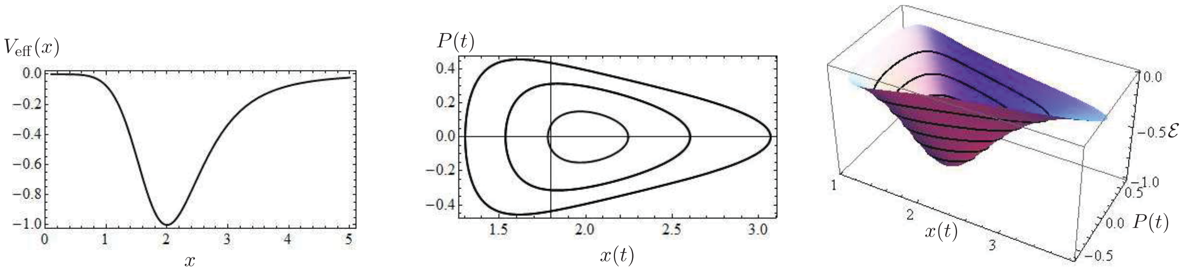

Figure 6. The potential (left), the phase space trajectories (middle) and the 3-D phase space (right) for the position-dependent mass Pöschl-Teller potential. In this plots $\alpha = 2$, $m_0 = 1$, $\lambda = 0.5$ and $\epsilon = -1$.

5. Concluding remarks

The equations of motion for position-dependent mass systems have been considered from the Lagrangian and Hamiltonian approaches. While the Lagrangian and Hamiltonian functions have standard forms, namely

$$\mathcal{L} = \frac{m(x)v^2}{2} + V, \quad \mathcal{H} = \frac{p^2}{2m(x)} + V,$$

a term quadratic in the velocity appears in the dynamical equations as a consequence of the position-dependence of the mass. Since the Hamiltonian is not time invariant, an *energy constant of motion* H was explicitly constructed leading to dynamical equations having the Hamilton form in the phase space $(x(t), P(t))$. From the factorization of this invariant in terms of two functions $A^{\pm}$, and demanding that these functions, together with H close some specific Poisson algebras we have found two integrals of motion $Q^{\pm}$ which allow the construction of the phase space trajectories for bounded motion in different cases. Three kind of potentials were discussed: the harmonic oscillator potentials, connected with the oscillator algebra in terms of Poisson brackets, and the Scarf and Pöschl-Teller potentials related to the $su(1,1)$ and $su(2)$ Poisson algebras respectively. Finally, we have to mention that some other potentials, as singular oscillators, generalized Pöschl-Teller and Morse potentials can be included in this approach for different factorization functions $A^{\pm}$.

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