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Hadronic Shear Viscosity: A Comparison between the Green-Kubo and Chapmann Enskog Methods

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Abstract. A detailed comparison between two methods to calculate shear viscosity is presented. We choose two systems in this comparison which are massless particles with current algebra cross section and a mixture comprised of pions with rho resonances. The two methods involved are the Green-Kubo method, applied using the Ultrarelativistic Quantum Molecular Dynamics (UrQMD) model to simulate the hadronic medium, and the Chapmann-Enskog method.

1. Introduction

Ultrarelativistic heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) are thought to have created a Quark Gluon Plasma (QGP) with characteristics of an ideal fluid [2, 25]. As such, the characterization of transport coefficients of hot, dense QCD matter is of significant interest. In particular, much attention has been given to the shear and bulk viscosity coefficients η and ζ, respectively, and the corresponding viscosity to entropy density ratios $\eta/s, \zeta/s$. Although an ideal fluid traditionally has been defined as having zero viscosities, semiclassical arguments have been presented that suggest a lower limit to the shear viscosity to entropy density ratio of a fluid [9], and there has been a conjecture by Kovtun, Son, and Starinets (KSS) based upon a calculation using the AdS/CFT correspondence that $\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$, where $\hbar$ is Planck's constant and k_B is Boltzmann's constant [21], and a modified bound of $\frac{\eta}{s} \geq \frac{4\hbar}{25\pi k_B}$ in order to avoid acausalities in gravity duals [5]. Ideal hydrodynamics calculations assuming no shear viscosity have been very successful in reproducing the elliptic flow (v_2) data from RHIC, yet the deconfined phase of the reaction cannot have zero shear viscosity. In the quest by the RHIC community to determine the viscosity of the deconfined phase, several groups have implemented viscous relativistic hydrodynamics simulations and examined what range of η/s would be permissible to reproduce the RHIC v_2 data [30, 15, 19]. Viscous hydrodynamical analyses [23, 29] finds that η/s should lie within a range of 0.08-0.24 depending upon the choice of initial conditions and equation of state. This finding is supported by lattice calculations of η/s for pure gluonic QCD, which yield values close to the KSS bound, and indirect estimates of η/s from calculations of the diffusion of heavy quarks, elliptic flow measurements, and transverse momentum correlations that arrive at roughly comparable values [24, 22, 14, 17, 1]. However,

the aforementioned viscous hydrodynamics calculations assume a fixed value of η/s throughout the entire evolution of the reaction and hence neglect its temperature dependence. One should note that the shear viscosity, especially in the context of a relativistic heavy ion collision, is a *time-dependent* quantity. While the partonic phase of such a collision is expected to have a very low value of η/s, after hadronization occurs η/s is expected to rapidly increase [12]. In order to make precise quantitative statements about the viscosity of the deconfined phase of a relativistic heavy ion collision, it is necessary to separately determine the viscosity of the hadronic phase in order to quantitatively constrain the viscosity of the deconfined phase.

There have been a number of analytic calculations of η, ζ and the corresponding viscosity to entropy density ratios for a number of different hadronic systems, including pure pion gases ([16, 13, 28, 7, 8]) and binary hadronic mixtures ([28, 6, 20]). In the aforementioned analytical calculations, the linearized Boltzmann equation was solved, and the cross sections in the collision integral were treated using a variety of methods, such as chiral perturbation theory, effective NN theory, and phenomenological amplitudes [7, 6, 20]. Another analytical treatment used was the relaxation time approximation [16, 28]. There also exist numerical calculations of shear viscosity using semiclassical microscopic transport approaches of mesonic matter [26], hadronic matter at finite baryon number density [27], and for a gas of hard spheres in the van der Waals excluded volume hadron-resonance gas model [18]. However, while several calculations for the hadronic shear viscosity exist, in order to validate a method and test the limits to which a calculation can be reliable, it is necessary to perform a systematic comparison between two different methods and see where the differences exist [32] , and that is the goal of this paper.

In this paper we compare two methods of shear viscosity calculations which are the Chapmann-Enskogg approximation and the Green-Kubo Method. We chose the following two systems for the comparison: (i) massless pions with an energy-dependent current algebra cross section (ii) massive pions with both elastic scattering and inelastic scattering enabled through the rho resonance as an intermediate channel. In this work, we want to observe the origin of the difference in the result of shear viscosity calculations.

The organization of this paper is as follows. In Sec. II, the formalism and working formulae of the Chapman- Enskog and Green-Kubo methods are summarized. In Sec. III, the results of the calculations of both methods and their comparisons are given, and a summary is given in Sec. IV.

2. Formalism

In the section, we will elaborate the working expression for the shear viscosity calculations from both methods. It must be stressed that the formalisms used in this work are not new, but the application of these formalisms to test cases is new to the extent that a detailed comparison between two commonly used methods is provided. For the sake of clarity and completeness, the formalisms used in this work are summarized below, along with working formulae.

2.1. Calculating Shear Viscosity: The Chapmann Enskogg Method

In this approximation, the system, which is slightly disturbed from its equilibrium, will have a perturbed distribution function which can be expressed in term of hydrodynamic variables [11]. The perturbed distribution function contains all the information of the system including the information about the transport coefficients (e.g. shear & bulk viscosities). By means of kinetic theory, we can extract shear & bulk viscosities of the system.

For a single component gas, the first approximation to the shear viscosity takes the form of [32]

$$\eta = \frac{1}{10}\, k_B T\, \frac{\gamma_0^2}{c_{00}}\ , \tag{1}$$

where $\gamma_0 = -10\,\hat{h}$, $\hat{h} = K_3(z)/K_2(z)$ and $z = mc^2/k_BT$ is called the relativity parameter. The term c_{00} contains the interaction in the system which is defined as

$$c_{00} = 16\left(w_2^{(2)} - \frac{1}{z}w_1^{(2)} + \frac{1}{3\,z^2}w_0^{(2)}\right). \tag{2}$$

The quantity $w_i^{(s)}$ is so called the relativistic omega integral given by

$$w_i^{(s)} = \frac{2\pi z^3 c}{K_2(z)^2}\int d\psi\,\sinh^7\psi\,\cosh^i\psi\,K_j(2z\cosh\psi)\times\int_0^\pi d\Theta\,\sin\Theta\,\sigma(\psi,\Theta)\,(1-\cos^s\Theta), \tag{3}$$

where $j = 5/2 - 1/2(-1)^i$, $\sinh\psi = g/mc$ and $\cosh\psi = P/2mc$. The quantities g and P are the relative and center of mass momenta. The interaction in the system is hidden in the differential cross section, $\sigma(\psi,\Theta)$.

2.2. Calculating Shear Viscosity: The Green Kubo Method

The second method which we use to calculate the shear viscosity coefficient is the Green-Kubo method. In the Green-Kubo method, linear transport coefficients are cast in terms of the time integral of some sort of a correlation between fluctuations near equilibrium. For the case of the shear viscosity coefficient the formula is

$$\eta = \frac{1}{T}\int d^3r dt\langle\pi^{xy}(0)\,\pi^{xy}(t)\rangle. \tag{4}$$

The brackets refer to taking an average over the ensemble of events generated in the simulation. The

2.3. Equilibriation of Infinite Hadronic Matter

Our hadronic medium is simulated using UrQMD, which is a covariant microscopic transport model based upon the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_r\right)f^{(1)} = \Phi_{coll}, \tag{5}$$

where $f^{(1)}$ is the one-particle phase-space distribution function for a given species, and Φ_{coll} is the collision integral. UrQMD is composed of purely hadronic degrees of freedom, and does not contain information about any possible crossover or phase transition into partonic degrees of freedom. Interactions in our simulation are based only upon scattering, and we neglect any interparticle potential. The criterion for a collision to occur is based upon the geometric interpretation of the cross section:

$$d_{min} \leq \sqrt{\frac{\sigma_{tot}}{\pi}}, \tag{6}$$

where d_{min} is the transverse distance at closest approach between two particles. UrQMD is described in great detail in [3]. Our version of UrQMD includes 55 baryon and baryon resonance species and 32 meson and meson resonance species, (and their antiparticles). These species include strange particles, but no particles containing heavy quarks (c or b quarks).

In order to use the Kubo formalism to extract the shear viscosity coefficient, it is necessary to establish that our system reaches a state of thermal equilibrium. To force our system into equilibrium, we confine our hadronic medium to a cubic box with periodic boundary conditions in real space. This technique was also used in [4, 26, 27]. The input parameters of our system are as follows: volume of the box, initial particle species, along with their chemical potentials.

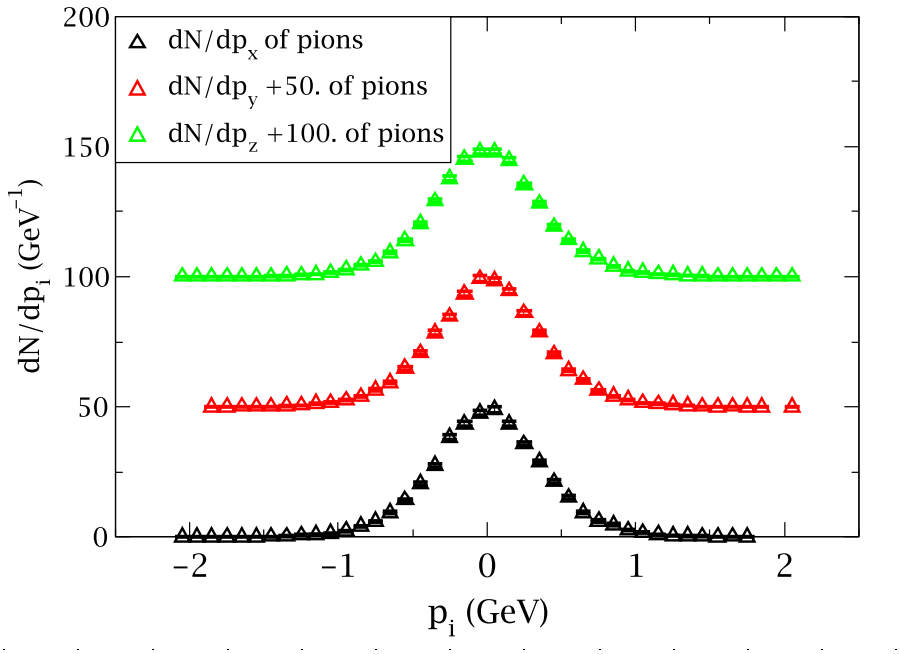

Figure 1. Momentum distributions in the x, y, and z directions, indicating the momentum distribution is indeed isotropic.

We initialize our systems to be in thermal and chemical equilibrium. For the test case involving a gas of massless pions with energy dependent cross sections, the initial yields corresponding to a given temperature and chemical potential are calculated using the Boltzmann distribution function. For the test case involving a pion rho mixture, the initial particle yields are calculated using the Statistical Hadronization with Resonances (SHARE) model [31], with the resonance and decay tables modified to represent the degrees of freedom of the system.

2.4. Thermal and Kinetic Equilibriation

As mentioned above, in order to apply the Green-Kubo method to calculate the shear viscosity coefficient, it is necessary to establish that kinetic equilibrium has been achieved. A system which has attained thermal equilibrium should have its momenta distributions following a Boltzmann distribution:

$$\frac{d^3N_i}{d^3p} \propto exp\left(-\frac{E_i}{T}\right) \tag{7}$$

for a given species N_i. We know that for a system realizing kinetic equilibrium, the momenta distributions should be isotropic, enabling us to write $d^3p = 4\pi p^2 dp$, where p is the three-momentum magnitude. Using the relativistic dispersion relation $E_i = \sqrt{p_i^2 + m_i^2}$ enables us to use the alternate observable $\frac{1}{pE}\frac{dN_i}{dE}$. Figure 1 shows the momenta distribution in the x, y, and z directions evaluated at chemical equilibriation time for the case for a chemically weighted (by the pion/rho densities) temperature of $T = 126$ MeV for a $\pi\rho$ mixture. Figure 2 shows the observable $\frac{1}{pE}\frac{dN}{dE}$ versus E for the same case.

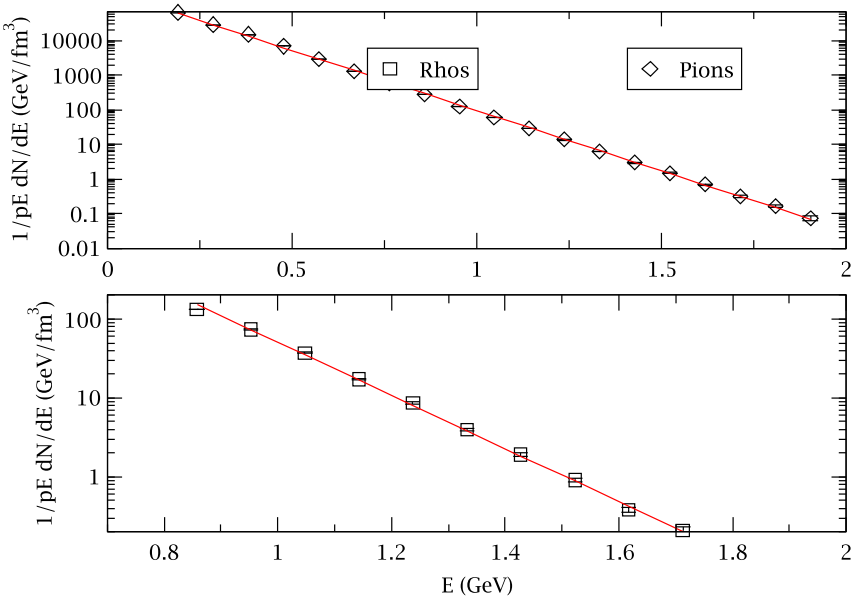

Figure 2. The observable $1/pEdN/dE$ as a function of energy. Both the pions and rhos fit to a Boltzmann distribution. The pion temperature is 125 MeV whereas the rho temperature is 129 MeV.

2.5. Shear Viscosity

Computing the shear viscosity for our system amounts to finding the time integral of the correlations of the shear component of the energy-momentum tensor about the equilibrium state. The energy-momentum tensor is related to the phase-space density of the system through the following equation

$$\pi^{\mu\nu} = \int d^3p \frac{p^\mu p^\nu}{p^0} f(x,p). \tag{8}$$

The phase space density for a system of particles uniformly distributed in phase space is given by

$$f(x,p) = \frac{1}{V}\sum_{j=1}^{N} \delta(\vec{p} - \vec{p_j}). \tag{9}$$

This enables one to calculate pi^{xy} directly

$$\pi^{xy} = \frac{1}{V}\sum_{j=1}^{N} \frac{p^x(j)p^y(j)}{p^0(j)}. \tag{10}$$

Note that since we assumed the particles were uniformly distributed in real space, the Green-Kubo formula for the shear viscosity coefficient becomes

$$\eta = \frac{V}{T}\int dt \langle \pi^{xy}(0)\pi^{xy}(t)\rangle. \tag{11}$$

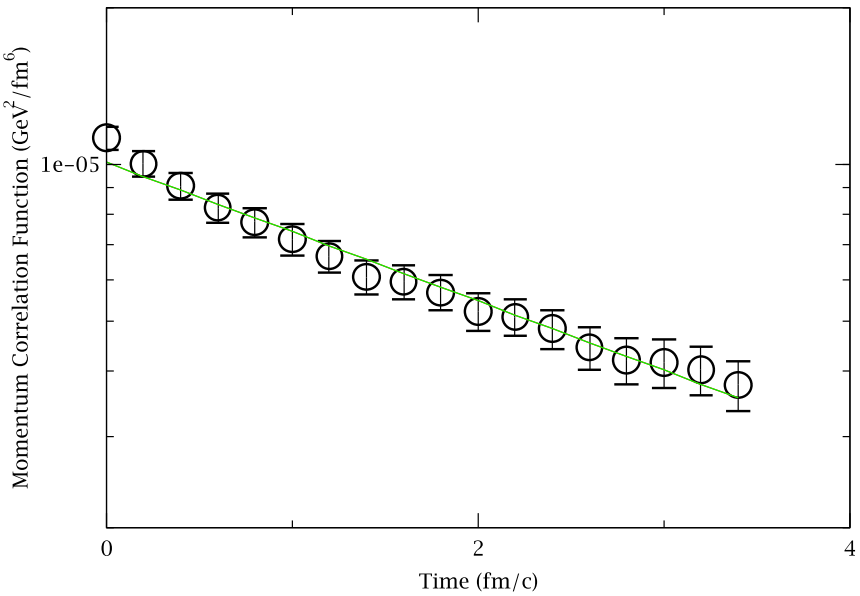

Figure 3. The momentum correlation function as a function of time t. The relaxation time for this case is $\tau_\pi = 3.1$ fm/c.

In order to simplify evaluating this integral, we assume the correlation function $\langle \pi^{xy}(0)\, \pi^{xy}(t) \rangle$ has an exponential ansatz:

$$\langle \pi^{xy}(0)\, \pi^{xy}(t) \rangle \propto exp\left(-\frac{t}{\tau_\pi}\right) \tag{12}$$

A representative sample of this correlation function is shown in Figure 3. The figure suggests that our assumption is reasonable. This ansatz has also been used in [26]. If we assume the exponential dependence of the stress tensor correlation function on post-equilibriation time t, the Kubo formula reduces to

$$\eta = \frac{V}{T}\tau_\pi \langle \pi^{xy}(0)^2 \rangle. \tag{13}$$

The volume V of the system is an input parameter, the temperature T is extracted from fitting momenta distributions to a Boltzmann distribution after checking that the momenta distributions in the system are isotropic, the correlator relaxation time tau_π is obtained from fitting the momentum correlation function to an exponential, and $\langle \pi^{xy}(0)^2 \rangle$ is obtained from the intercept of the momentum correlation function. Hence, the shear viscosity coefficient can be calculated in this way, and in the next section we present the results of the shear viscosity coefficient for two systems.

3. Results

In this section we present the results for the calculation of the shear viscosity via both the Chapmann-Enskog and Green-Kubo method for two systems. The first system involves a pure chiral pion gas with an energy dependent cross section given by current algebra. The

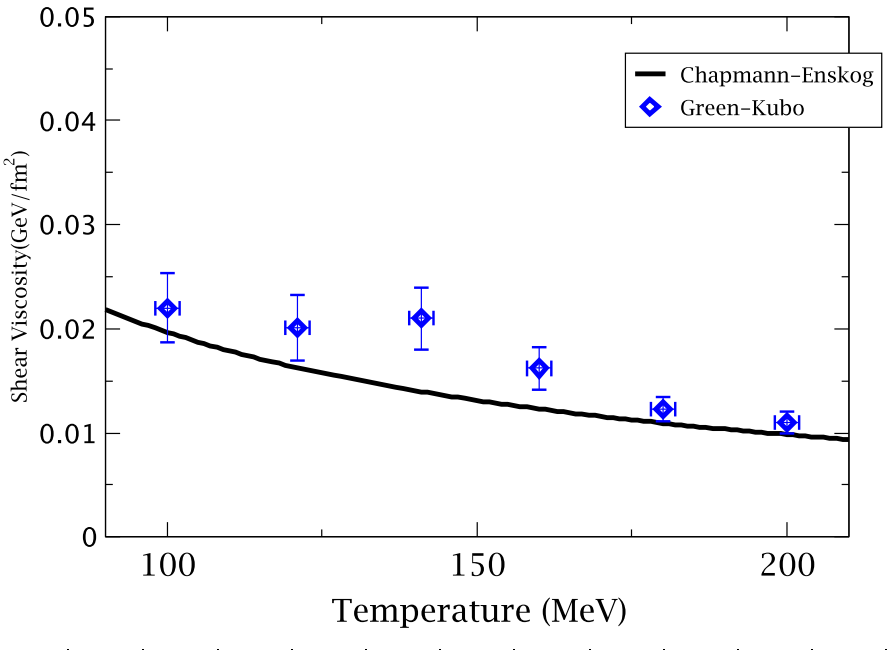

Figure 4. Shear viscosity coefficient as a function of temperature for an gas of chiral pions with energy dependent cross section.

second system is a $\pi\rho$ mixture, where massive pions scatter elastically and inelastically via an intermediate ρ resonance state. More details are given below.

3.1. Massless Pions with Energy Dependent Cross Section (with Current Algebra)
The total (energy-dependent) cross section for the first system is given by

$$\sigma\left(\sqrt{s}\right) = \frac{5s}{72\pi f_\pi^4}, \tag{14}$$

where s is the Mandelstam variable representing the total energy in the center of mass frame and $f_\pi = 93$ MeV is the pion decay constant. The results for the shear viscosity coefficient are shown as a function of temperature in Figure 4, and we find that there is very good agreement between the calculations from the Green-Kubo method and the Chapmann-Enskog method.

3.2. Pion Rho Mixture
The $\pi\rho$ mixture involves the two following processes:

- (a) $\sigma_{\pi\pi\to\pi\pi,elastic} = 17.778$ mb.
- (b) $\pi\pi \to \rho \to \pi\pi$ (inelastic)

The value of the cross section in (a) is based upon the additive quark model, which dictates that meson-meson and baryon-baryon cross sections are related via

$$\sigma_{MM} = \left(\frac{2}{3}\right)^2 \sigma_{BB}, \tag{15}$$

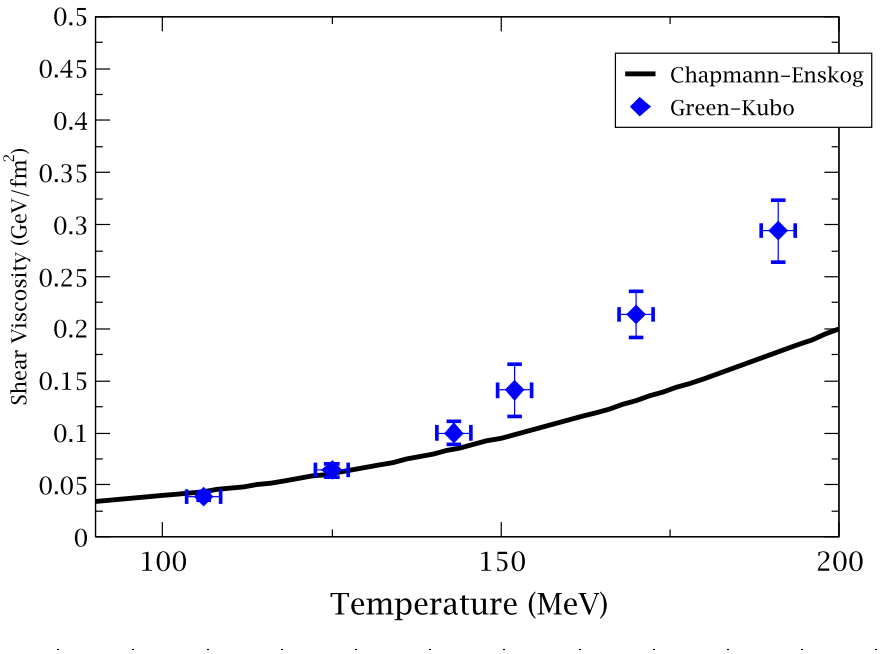

Figure 5. Shear viscosity coefficient versus temperature for a gas of massive pions scattering elastically and inelastically through an intermediate rho resonance.

where M/B stands for meson/baryon, respectively. Since the elastic nucleon-nucleon cross section is known to be approximately 40 mb, this implies that $\sigma_{\pi\pi,elastic} = 17.778$ mb.

The results for the shear viscosity coefficient are shown as a function of temperature in Figure 5. We find that, although there is good agreement between the Green-Kubo and Chapmann-Enskog methods in the temperature range $100 < T < 150$) MeV, the results from the Green-Kubo calculation are systematically higher than those for the Chapmann-Enskog method in the temperature range $150 < T < 200$ MeV. This is a noteworthy result, particularly when one realizes that many more inelastic scattering events through the intermediate resonance channel are expected in the temperature range $T > 150$ MeV than $T < 150$ MeV. In order to fully understand why the Green-Kubo method yields a higher result for the shear viscosity than the Chapmann-Eskog method does, it is necessary to understand the effect of resonance lifetimes in UrQMD on the shear viscosity coefficient. This is a subtle effect and the effect of timescales on transport coefficients in general has been investigated in the past [10]. One of the goals of this paper is to highlight the differences in the calculation of the shear viscosity of the Chapmann-Enskog method versus the Green-Kubo method, and we find that the greatest difference is observed when one examines the results for the shear viscosity coefficient in the temperature range $150 < T < 200$ MeV, where a large number of inelastic scattering events are expected to take place. An investigation of the effect of the resonance lifetime on the shear viscosity coefficient of the system is in progress.

4. Summary

In this section, we summarize the results from both methods to calculate the shear viscosity for two different systems. The systems analyzed included a system of a pure gas of massless

pions interacting via an energy-dependent cross section obtained via chiral current algebra, and a system of massive pions interacting elastically via a constant cross section given by the AQM model, with inelastic scattering enabled through an intermediate ρ resonance channel. The two methods used to calculate the shear viscosity were the Chapmann-Enskog method and the Green-Kubo method using the UrQMD model to simulate the hadronic medium in equilibrium. When the results of the calculation for the two different methods were compared, excellent agreement was found between the Chapmann Enskog and Green Kubo methods for the system of pure massless pions with an energy dependent cross section, whereas a systematic deviation occured between the Chapmann Enskog and Green Kubo methods in the temperature range $150 < T < 200$ MeV, where significant inelastic scattering is expected to occur. The Green-Kubo method systematically yielded a higher result in this temperature range. To fully understand the reason for this, a systematic investigation of the effect of resonance lifetimes on the shear viscosity needs to be performed, and this is in progress. We also plan to extend this comparison to other test systems.

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