

Exactly solvable pairing models in nuclear and mesoscopic physics

J. Dukelsky

Instituto de Estructura de la Materia, C.S.I.C., Serrano 123, 28006 Madrid, Spain

E-mail: j.dukelsky@csic.es

Abstract. The exact solution of the BCS pairing Hamiltonian dates back to the work of Richardson in 1963. Little attention was paid to this exactly solvable model for almost 40 years. However, at the beginning of the 21th century, there was a burst of work focusing on its applications in different areas of quantum physics. In this contribution we introduce the generalized integrable Richardson-Gaudin models, and discuss a recent application of the hyperbolic model to heavy nuclei.

1. Introduction

After a long struggle to explain from a microscopic description the phenomenon of superconductivity, Bardeen, Cooper and Schrieffer published in 1957 [1] a successful theory for treating variationally a pairing Hamiltonian. The BCS theory was able to explain quantitatively most of the superconducting properties from the associated BCS wave function, leading to complete microscopic explanation of superconductivity.

The success of the BCS theory quickly spread to other quantum many-body systems, including the atomic nucleus. Soon thereafter Bohr, Mottelson and Pines published a paper [2] suggesting that the gaps observed in even-even nuclei could be due to superconducting correlations. They noted, however, that these effects should be strongly influenced by the finite size of the nucleus. Since then, and up to the present, number projection and in general symmetry restoration in the BCS and Hartree-Fock-Bogoliubov approximations have been important issues in nuclear structure.

At the beginning of the sixties, while several groups were developing numerical techniques for number-projected BCS calculations [3, 4], Richardson provided an exact solution for the reduced BCS Hamiltonian [5, 6]. In spite of the importance of his exact solution, this work did not have much impact in nuclear physics with just a few exceptions. Later on, his exact solution was rediscovered in the framework of ultrasmall superconducting grains [7] where BCS and number-projected BCS were unable to describe appropriately the crossover from superconductivity to a normal metal as a function of the grain size. Since then, there has been a flurry of work extending the Richardson exact solution to families of exactly-solvable models, now called the Richardson-Gaudin (RG) models [8, 9], and applying these models to different areas of quantum many-body physics including mesoscopic systems, condensed matter, quantum optics, cold atomic gases, quantum dots and nuclear structure [10]. In this contribution, we introduce the generalized RG integrable models and a recent application to nuclear physics.



2. Richardson-Gaudin integrable models

The fully integrable and exactly solvable RG models are based on the $SU(2)$ algebra. We first introduce the generators of $SU(2)$, using a basis familiar to nuclear structure,

$$K_j^0 = \frac{1}{2} \left(\sum_m a_{jm}^\dagger a_{jm} - \Omega_j \right), \quad K_j^+ = \sum_m a_{jm}^\dagger a_{j\bar{m}}^\dagger, \quad K_j^- = (K_j^+)^\dagger. \quad (1)$$

Here a_{jm}^\dagger creates a fermion in single-particle state jm , $j\bar{m}$ denotes the time reverse of jm , and $\Omega_j = j + \frac{1}{2}$ is the pair degeneracy of orbit j . These operators fulfill the $SU(2)$ algebra $[K_j^+, K_j^-] = 2\delta_{jj'} K_j^0$, $[K_j^0, K_j^\pm] = \pm\delta_{jj'} K_j^\pm$.

We now consider a general set of L Hermitian and number-conserving operators that can be built up from the generators of $SU(2)$ with linear and quadratic terms,

$$R_i = K_i^0 + 2g \sum_{j(\neq i)} \left[\frac{X_{ij}}{2} (K_i^+ K_j^- + K_i^- K_j^+) + Y_{ij} K_i^0 K_j^0 \right]. \quad (2)$$

Following Gaudin[11], we then look for the conditions that the matrices X and Y must satisfy in order that the R operators commute with one another. It turns out that there are essentially two families of solutions, referred to as the rational and hyperbolic families, respectively.

i. The rational family

$$X_{ij} = Y_{ij} = \frac{1}{\eta_i - \eta_j} \quad (3)$$

ii. The hyperbolic family

$$X_{ij} = 2 \frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Y_{ij} = \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \quad (4)$$

Here the set of L parameters η_i are free real numbers.

The reduced BCS Hamiltonian with a constant pairing interaction is an example of the rational family. It can be obtained as a linear combination of the integrals of motion, $H_P = \sum_j \varepsilon_j R_j(\eta_j)$, with $\eta_j = \varepsilon_j$.

The complete set of eigenstates of the rational and hyperbolic integrals of motion can be found in Ref. [8, 9, 10].

The key point of the RG models is that any Hamiltonian that can be expressed as a linear combination of the R operators can be treated exactly using this method. The most general exactly solvable Hamiltonian has $2L + 1$ free parameters: a set of L internal parameters η_i , a set of L free coefficients ε_i defining the linear combination of integrals of motion, and the pairing strength g . As a result, the RG Hamiltonians have an enormous flexibility to model physical situations.

3. The hyperbolic model

The rational model was extensively used to model different mesoscopic systems like superconducting grains [12], cold atoms [13], quantum dots [14], nuclei [15], etc. The hyperbolic family of models did not find a physical realization until very recently when it was shown that they could model a p -wave pairing Hamiltonian in a 2-dimensional lattice [16], such that it was possible to study with the exact solution an exotic phase diagram having a non-trivial topological phase and a third-order quantum phase transition [17]. A slightly modified version of the p -wave pairing Hamiltonian gives rise to a separable pairing Hamiltonian with 2 extra free parameters that can be adjusted to reproduce the properties of heavy nuclei as described by a Gogny HFB treatment [18]. Both applications are based on a simple linear combination of hyperbolic

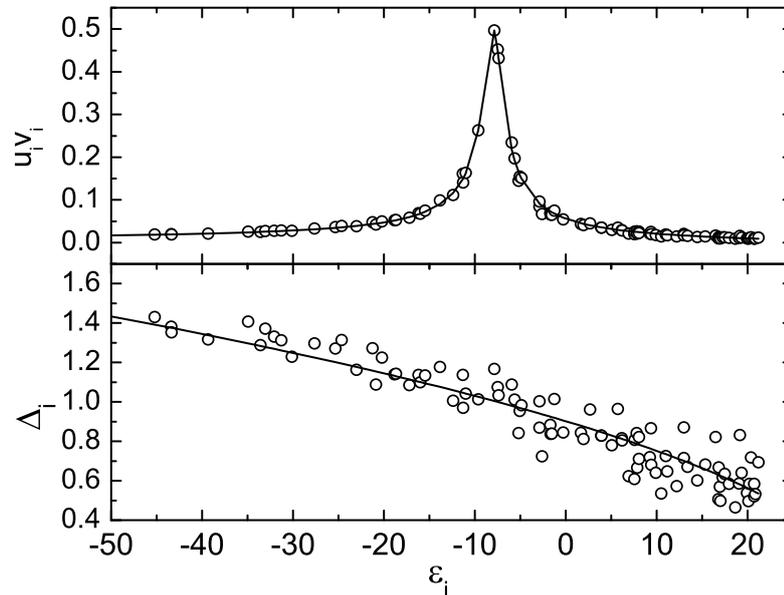


Figure 1. Gaps Δ_i and pairing tensor $u_i v_i$ for protons in ^{154}Sm . Open circles are Gogny HFB results in MeV . Solid lines are BCS results of the hyperbolic Hamiltonian in MeV .

integrals of motion. In the nuclear application the exactly solvable pairing Hamiltonian reduces to

$$H = \sum_i \varepsilon_i \left(c_i^\dagger c_i + c_i^\dagger c_i^\dagger \right) - 2G \sum_{ii'} \sqrt{(\alpha - \varepsilon_i)(\alpha - \varepsilon_{i'})} c_i^\dagger c_i^\dagger c_{i'} c_{i'}, \quad (5)$$

where the free parameter α plays the role of an energy cutoff and ε_i is the single-particle energy of level i . The complete set of eigenvalues are $E = 2\alpha M + \sum_i \varepsilon_i \nu_i + \sum_\beta E_\beta$. Where the pair energies E_β correspond to a particular solution of the set of non-linear Richardson equations

$$\frac{1}{2} \sum_i \frac{1}{\eta_i - E_\beta} - \sum_{\beta' (\neq \beta)} \frac{1}{E_{\beta'} - E_\beta} = \frac{Q}{E_\beta}, \quad (6)$$

with $Q = \frac{1}{2G} - \frac{L}{2} + M - 1$. Each particular solution of Eq. (6) defines a unique eigenstate.

Due to the separable character of the hyperbolic Hamiltonian, in BCS approximation the gaps $\Delta_i = 2G\sqrt{\alpha - \varepsilon_i} \sum_{i'} \sqrt{\alpha - \varepsilon_{i'}} u_{i'} v_{i'} = \Delta\sqrt{\alpha - \varepsilon_i}$ and the pairing tensor $u_i v_i = \frac{\Delta\sqrt{\alpha - \varepsilon_i}}{2\sqrt{(\varepsilon_i - \mu)^2 + (\alpha - \varepsilon_i)\Delta^2}}$ have a very restricted form. In order to test the validity of the exactly solvable Hamiltonian (5) we take the single-particle energies ε_i from the HF energies of a Gogny HFB calculation and we fit the parameters α and G to the gaps and pairing tensor in the HF basis. Figure 1 shows the comparison for protons in ^{154}Sm between the Gogny HFB results in the HF basis and the BCS approximation of the hyperbolic model. As it can be seen, there is a remarkable agreement between the Gogny force and the hyperbolic Hamiltonian for the pairing tensor. The Gogny gaps exhibit large fluctuations due to the details of the two-body Gogny force. However, the general trend of the gaps is very well described by the square root $\sqrt{\alpha - \varepsilon_i}$ of the hyperbolic model. From these results we extracted the values $\alpha = 32.7 \text{ MeV}$ and $G = 2.24 \times 10^{-3} \text{ MeV}$. The valence space determined by the cutoff α corresponds to $L = 91$

levels with $M = 31$ proton pairs. The size of the Hamiltonian matrix in this space is 1.98×10^{24} , well beyond the limits of exact diagonalization. However, the integrability of the hyperbolic model provides an exact solution by solving a set of 31 non-linear coupled equations. Moreover, the exact solution shows a gain in correlation of ~ 2 MeV suggesting the importance of taking into account correlations beyond mean-field.

4. Conclusions

The key feature of the RG integrable models, is that they transform the diagonalization of the hamiltonian matrix, whose dimension grows exponentially with the size of the system, to the solution of a set of M coupled non-linear equations where M is the number of pairs. This makes it possible to treat problems that could otherwise not be treated and in doing so to obtain information that is otherwise inaccessible. The exactly solvable RG Hamiltonians also provide excellent benchmarks for testing approximations beyond HFB in realistic situations. Moreover, the RG integrable models have been extended to larger rank algebra making possible the exact solution of a proton-neutron pairing Hamiltonian with isospin $T = 1$ [19] or $T = 0$ and $T = 1$ [20].

Acknowledgments

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