

The structure of Ph^* , generalized de Rham, and entropy

O A Laudal

Matematisk institutt, University of Oslo, Pb. 1053, Blindern, N-0316 Oslo, Norway

Abstract. In this note I shall continue the study of the non-commutative phase space functor, $Ph(A)$, defined for any associative algebra A , and its derived differential co-simplicial algebra, $Ph^*(A)$. The main focus will be on its relationship to the classical de Rham complex, to the dynamics of finite dimensional $Ph^\infty(A)$ -modules, and to the notion of Entropy. These subjects are treated within the set-up of my book [2011 *Geometry of Time-Spaces* (World Scientific)].

Keywords: Phase spaces, Dirac derivations, de Rham complexes, connections, entropy.

1. Introduction

If we want to study the *dynamics*, of some mathematical system, say composed of representations of an algebra A , we should, according to [5], introduce the notion of a (non-commutative) *Phase Space*-functor, $Ph(-) : Alg_k \rightarrow Alg_k$, k being a field.

This functor extends to the category of schemes, and its infinite iteration $Ph^\infty(-)$, is outfitted with a universal *Dirac derivation*, $\delta \in Der_k(Ph^\infty(-), Ph^\infty(-))$, with seemingly useful applications in mathematical models of quantum theory, see [5].

The Ph functor extends also, by iteration, in a natural way, to a functor of algebras into differential co-simplicial algebras, such that for any associative k -algebra A , the (graded) family of algebras, $Ph^n(A)$, $n \geq 0$, is outfitted with a differential of degree 1, and a nicely related co-simplicial structure, see [3, 4] and also [5].

The main purpose of this note is, to show the relationship between this differential co-simplicial algebra $Ph^*(A)$ and the different versions of the classical de Rham complex to be found in the literature.

This will force us to take a new look at the non-commutative versions of *connections* on modules of associative k -algebras, A , their reformulations as representations of a *dynamical system*, $A(\sigma_g)$, induced by the choice of a *metric* $g \in Ph(A)$, in the commutative case, see [5].

Extending classical notions of *Modular Suites*, see [6], to the case of representations of $Ph^\infty(A)$, and by considering finite dimensional *approximations* to vector fields, (as representations of the corresponding $Ph^\infty(A)$) we come close to a reasonable mathematical formulation of *Information*, *Entropy*, and classical thermodynamics, in terms of deformation theory.

2. Phase spaces

From now on, we shall be led to talk about an algebra, as a *space*, having in mind the relationship between an associative (geometric), see [5], algebra and the moduli space of finite dimensional simple representations, characterizing it.



Given an associative k -algebra A , denote by $A/k - \underline{alg}$ the category where the objects are homomorphisms of k -algebras $\kappa : A \rightarrow R$, and the morphisms, $\psi : \kappa \rightarrow \kappa'$ are commutative diagrams

$$\begin{array}{ccc} & A & \\ \kappa \swarrow & & \searrow \kappa' \\ R & \xrightarrow{\psi} & R' \end{array}$$

and consider the functor

$$Der_k(A, -) : A/k - \underline{alg} \longrightarrow \underline{Sets}$$

It is, see [4] and [5], representable by a k -algebra-morphism $\iota : A \longrightarrow Ph(A)$ with a *universal family* given by a universal derivation $d : A \longrightarrow Ph(A)$. Clearly we have the identities

$$d_* : Der_k(A, A) = Mor_A(Ph(A), A), \quad d^* : Der_k(A, Ph(A)) = End_A(Ph(A))$$

the last one associating d to the identity endomorphism of Ph .

2.1. First properties

Let now V be a right A -module, with structure morphism $\rho : A \rightarrow End_k(V)$. We obtain a universal derivation

$$c : A \longrightarrow Hom_k(V, V \otimes_A Ph(A))$$

defined by $c(a)(v) = v \otimes d(a)$. Using the long exact sequence of the Hochschild cohomology

$$\begin{aligned} 0 \rightarrow Hom_A(V, V \otimes_A Ph(A)) &\rightarrow Hom_k(V, V \otimes_A Ph(A)) \\ &\xrightarrow{\iota} Der_k(A, Hom_k(V, V \otimes_A Ph(A))) \xrightarrow{\kappa} Ext_A^1(V, V \otimes_A Ph(A)) \rightarrow 0 \end{aligned}$$

we obtain the non-commutative *Kodaira-Spencer class*

$$c(V) := \kappa(c) \in Ext_A^1(V, V \otimes_A Ph(A))$$

inducing, via the identity d_* , the *Kodaira-Spencer morphism*

$$g : \Theta_A := Der_k(A, A) \longrightarrow Ext_A^1(V, V)$$

If $c(V) = 0$, then the exact sequence above proves that there exist an element, $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ such that $c = \iota(\nabla)$. This is just another way of proving that if $c(V) = 0$, then c is given by a connection

$$\nabla : Der_k(A, A) \longrightarrow Hom_k(V, V)$$

In particular, we deduce, from the corresponding long exact sequence

$$0 \rightarrow Hom_A(V, V) \rightarrow Hom_k(V, V) \xrightarrow{\iota} Der_k(A, Hom_k(V, V)) \xrightarrow{\kappa} Ext_A^1(V, V) \rightarrow 0$$

the following elementary

Lemma 2.1. *Let $\rho : A \rightarrow \text{End}_k(V)$, be an A -module, and let $\delta \in \text{Der}_k(A, \text{Hom}_k(V, V))$, map to 0 in $\text{Ext}_A^1(V, V)$, i.e assume $\kappa(\delta) = 0$, then there exist an Hamiltonian-element, $Q_\delta \in \text{Hom}_k(V, V)$, such that*

$$\rho(\delta(a)) = [Q_\delta, \tilde{\rho}(a)], \quad \forall a \in A$$

If V is a simple A -module then $\text{ad}(Q_\delta)$ is unique.

As is well known, in the commutative case, the Kodaira-Spencer class gives rise to a *Chern character* by putting

$$ch^i(V) := 1/i! \ (c(V)^i) \in \text{Ext}_A^i(V, V \otimes_A Ph(A))$$

and if $c(V) = 0$, the curvature $R(V)$ of ∇ , induces a curvature class

$$R_\nabla \in H^2(k, A; \Theta_A, \text{End}_A(V))$$

Any $Ph(A)$ -module W , given by its structure map $\rho_W : Ph(A) \rightarrow \text{End}_k(W)$ corresponds bijectively to an induced A -module structure on W , and a derivation $\delta_\rho \in \text{Der}_k(A, \text{End}_k(W))$, defining an element $[\delta_\rho] \in \text{Ext}_A^1(W, W)$.

Remark 2.2. Since $\text{Ext}_A^1(V, V)$ is the tangent space of the miniversal deformation space of V as an A -module, we see that the non-commutative space $Ph(A)$ also parametrizes the set of *generalized momenta*, i.e the set of pairs of an A - module V , and a tangent vector of the formal moduli of V , at that point.

$Ph(A)$ is relatively easy to compute. In particular, if $A = k[x_1, \dots, x_n] =: k[\underline{x}]$ is the polynomial algebra, we have

$$Ph(A) = k \langle x_1, \dots, x_n, dx_1, \dots, dx_n \rangle / ([x_i, x_j], [x_i, dx_j] + [dx_i, x_j])$$

Notice that, in this case, any rank 1 representation of $Ph(A)$ is represented by a pair, (q, p) , of a closed point, q of $\text{Spec}(k[\underline{x}])$, and a tangent, p at that point.

We shall need the following formulas,

Theorem 2.3. *Given two such points, $(q_i, p_i), i = 1, 2$, we find*

$$\dim_k \text{Ext}_{Ph(A)}^1(k(q_1, p_1), k(q_2, p_2)) = 2n, \quad \text{for } (q_1, p_1) = (q_2, p_2)$$

$$\dim_k \text{Ext}_{Ph(A)}^1(k(q_1, p_1), k(q_2, p_2)) = n, \quad \text{for } q_1 = q_2, p_1 \neq p_2$$

$$\dim_k \text{Ext}_{Ph(A)}^1(k(q_1, p_1), k(q_2, p_2)) = 1, \quad \text{for } q_1 \neq q_2$$

Moreover, there is, in the last case, a generator of

$$\text{Ext}_{Ph(A)}^1(k(q_1, p_1), k(q_2, p_2)) = \text{Der}_k(Ph(A), \text{Hom}_k(k(q_1, p_1), k(q_2, p_2))) / \text{Triv}$$

uniquely characterized by the tangent line defined by the vector $\overline{q_1 q_2}$.

Proof. Assume for convenience that $n = 3$. Put

$$x_j(q_i, p_i) := q_{i,j}, \quad dx_j((q_i, p_i)) := p_{i,j}, \quad \alpha_j = q_{1,j} - q_{2,j}, \quad \beta_j = p_{1,j} - p_{2,j}$$

See that for every element $\alpha \in \text{Hom}_k(k(q_1, p_1), k(q_2, p_2))$ we have

$$x_j \alpha = q_{1,j} \alpha, \quad \alpha x_j = q_{2,j} \alpha, \quad dx_j \alpha = p_{1,j} \alpha, \quad \alpha dx_j = p_{2,j} \alpha$$

with the obvious identification. Every derivation

$$\delta \in \text{Der}_k(\text{Ph}(A), \text{Hom}_k(k(q_1, p_1), k(q_2, p_2)))$$

must satisfy the relations

$$\delta([x_i, x_j]) = [\delta(x_i), x_j] + [x_i, \delta(x_j)] = 0$$

$$\delta([dx_i, x_j] + [x_i, dx_j]) = [\delta(dx_i), x_j] + [dx_i, \delta(x_j)] + [\delta(x_i), dx_j] + [x_i, \delta(dx_j)] = 0$$

Using the above left-right action-rules, the result follows from the long exact sequence computing $\text{Ext}_{\text{Ph}A}^1$. The two families of relations above give us two systems of linear equations.

The first one, in the variables $\delta(x_1), \delta(x_2), \delta(x_3)$, with the matrix

$$\begin{pmatrix} -\alpha_2 & \alpha_1 & 0 \\ -\alpha_3 & 0 & \alpha_1 \\ 0 & -\alpha_3 & \alpha_2 \end{pmatrix}$$

and the second one, in the variables $\delta(x_1), \delta(x_2), \delta(x_3), \delta(dx_1), \delta(dx_2), \delta(dx_3)$, with the matrix

$$\begin{pmatrix} -\beta_2 & \beta_1 & 0 & -\alpha_2 & \alpha_1 & 0 \\ -\beta_3 & 0 & \beta_1 & -\alpha_3 & 0 & \alpha_1 \\ 0 & -\beta_3 & \beta_2 & 0 & -\alpha_3 & \alpha_2 \end{pmatrix}$$

In particular, we see that the *trivial* derivation given by

$$\delta(x_i) = \alpha_i, \quad \delta(dx_j) = \beta_j$$

satisfies the relations, and the generator of $\text{Ext}_{\text{Ph}A}^1(k(q_1, p_1), k(q_2, p_2))$ is represented by

$$\delta(x_i) = 0, \quad \delta(dx_j) = \alpha_j$$

This is, in an obvious sense, the "tangent vector" $-\overline{q_1 q_2}$.

It is easy to extend this result from dimension 3 to any dimension n . □

Example 2.4. Let $A = M_2(k)$ and compute $\text{Ph}(A)$. Clearly the existence of the canonical homomorphism, $i : M_2(k) \rightarrow \text{Ph}(M_2(k))$ shows that $\text{Ph}(M_2(k))$ must be a matrix ring, generated, as an algebra, over $M_2(k)$ by $d\epsilon_{i,j}$, $i, j = 1, 2$, where $\epsilon_{i,j}$ is the elementary matrix. A little computation shows that we have the following relations

$$\begin{aligned} d\epsilon_{1,1} &= \begin{pmatrix} 0 & (d\epsilon_{1,1})_{1,2} = -(d\epsilon_{2,2})_{1,2} \\ (d\epsilon_{1,1})_{2,1} = -(d\epsilon_{2,2})_{2,1} & 0 \end{pmatrix} \\ d\epsilon_{2,2} &= \begin{pmatrix} 0 & (d\epsilon_{2,2})_{1,2} = -(d\epsilon_{1,1})_{1,2} \\ (d\epsilon_{2,2})_{2,1} = -(d\epsilon_{1,1})_{2,1} & 0 \end{pmatrix} \\ \epsilon_{1,2} &= \begin{pmatrix} \epsilon_{1,2}(d\epsilon_{2,2})_{2,1} & (d\epsilon_{1,2})_{1,2} = -(d\epsilon_{2,1})_{2,1} \\ 0 & -(d\epsilon_{2,2})_{2,1}\epsilon_{1,2} \end{pmatrix} \\ \epsilon_{2,1} &= \begin{pmatrix} (d\epsilon_{2,2})_{1,2}\epsilon_{2,1} & 0 \\ (d\epsilon_{2,1})_{2,1} = -(d\epsilon_{1,2})_{1,2} & \epsilon_{2,1}(d\epsilon_{1,1})_{1,2} \end{pmatrix} \end{aligned}$$

From this follows that every cosection $\rho : \text{Ph}(M_2(k)) \rightarrow M_2(k)$ of $i : M_2(k) \rightarrow \text{Ph}(M_2(k))$ is given in terms of an element $\phi \in M_2(k)$, such that $\rho(da) = [\phi, a]$.

2.2. The structure of $Ph^*(-)$ and the de Rham complex

The phase-space construction may, of course, be iterated. Given the k -algebra A we may form the sequence, $\{Ph^n(A)\}_{0 \leq n}$, defined inductively by

$$Ph^0(A) = A, \quad Ph^1(A) = Ph(A), \dots, \quad Ph^{n+1}(A) := Ph(Ph^n(A))$$

Let $i_0^n : Ph^n(A) \rightarrow Ph^{n+1}(A)$ be the canonical imbedding, and let $d_n : Ph^n(A) \rightarrow Ph^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $Ph^n(A) \rightarrow Ph^{n+2}(A)$, there exist by universality a homomorphism $i_1^{n+1} : Ph^{n+1}(A) \rightarrow Ph^{n+2}(A)$, such that

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}$$

Notice that we here compose functions and functors from left to right. Clearly we may continue this process constructing new homomorphisms

$$\{i_j^n : Ph^n(A) \rightarrow Ph^{n+1}(A)\}_{0 \leq j \leq n}$$

with the property

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}$$

We find, see [4], the following identities

$$\begin{aligned} i_p^n i_q^{n+1} &= i_{q-1}^n i_p^{n+1}, & p < q \\ i_p^n i_p^{n+1} &= i_p^n i_{p+1}^{n+1} \\ i_p^n i_q^{n+1} &= i_q^n i_{p+1}^{n+1}, & q < p \end{aligned}$$

To see this, compose i_0^{n-1} with d_{n-1} and use induction. Thus, the $Ph^*(A)$ is a semi-cosimplicial k -algebra with a cosection onto A . Therefore, for any object

$$\kappa : A \rightarrow R \in A/k - \underline{alg}$$

the semi-cosimplicial algebra above induces a semi-simplicial k -vectorspace $Der_k(Ph^*(A), R)$ and one should be interested in its homology.

Consider now the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) & \xrightarrow{i_p^3} & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & m_1^1 & \xrightarrow{i_p^1} & m_2^1 & \xrightarrow{i_p^2} & m_3^1 & \xrightarrow{i_p^3} & \cdots \end{array}$$

where, for each integer n , the symbol i_p^n for $p = 0, 1, \dots, n$ signifies the family of A -morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above, and where m_n^1 is the ideal of $Ph^n(A)$ generated by $im(d)$, which is the same as the ideal generated by the family

$$\{i_p^{n-1}(i_p^{n-2}(\dots(i_p^1(d(A))\dots))\}$$

for all possible p . And, inductively, let $\mathfrak{m}_n^m$ be the ideal generated by $\mathfrak{m}_n^1 \mathfrak{m}_n^{m-1}$. Then we find an extended diagram

$$\begin{array}{ccccccc}
 A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) \xrightarrow{i_p^3} \dots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{i_0^0} & A & \xrightarrow{i_p^1} & A & \xrightarrow{i_p^2} & A \xrightarrow{i_p^3} \dots \\
 & \searrow d_0 & & \searrow d_1 & & \searrow d_2 & \searrow d_3 \\
 & & \mathfrak{m}_1^1/\mathfrak{m}_1^2 & \xrightarrow{i_p^1} & \mathfrak{m}_2^1/\mathfrak{m}_2^2 & \xrightarrow{i_p^2} & \mathfrak{m}_3^1/\mathfrak{m}_3^2 \xrightarrow{i_p^3} \dots \\
 & & & \searrow d_1 & & \searrow d_2 & \searrow d_3 \\
 & & & & \mathfrak{m}_2^2/\mathfrak{m}_1^3 & \xrightarrow{i_p^1} & \mathfrak{m}_2^2/\mathfrak{m}_2^3 \xrightarrow{i_p^2} \dots \\
 & & & & & \searrow d_1 & \searrow d_2 \\
 & & & & & & \mathfrak{m}_3^2/\mathfrak{m}_1^3 \xrightarrow{i_p^1} \dots
 \end{array}$$

The diagonals are not necessarily complexes, but it suffices to kill d^2 , to kill all d^n , $n \geq 2$, and for this it suffices to kill $d_1 d_0$, as one easily see, operating with the edge homomorphisms on the elements, $d_1(d_0(a))$ for $a \in A$. Therefore we shall, in this general situation, make the following definition.

Definition 2.5. The curvature $R(A)$ of the associative k algebra A is the k -linear map composition of d_0 and d_1

$$R(A) = d_0 d_1 : A \rightarrow \mathfrak{m}_2^2/\mathfrak{m}_2^3$$

Now, kill the curvature $R(A)$, and all the terms under the first diagonal, beginning with $\mathfrak{m}_1^2/\mathfrak{m}_1^3$, together with all terms generated by the actions of the edge homomorphisms on these terms, and let, Ω_n^m be the resulting quotient of $\mathfrak{m}_n^m/\mathfrak{m}_n^{m+1}$, for $n \geq 0$. Clearly, $\Omega_n^0 = A$ for all $n \geq 0$, and we have got a graded semi co-simplicial A -module, with a k -differential d , such that $d^2 = 0$, looking like

$$\begin{array}{ccccccc}
 A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) \xrightarrow{i_p^3} \dots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{i_0^0} & A & \xrightarrow{i_p^1} & A & \xrightarrow{i_p^2} & A \xrightarrow{i_p^3} \dots \\
 & \searrow d & & \searrow d & & \searrow d & \searrow d \\
 & & \Omega_1^1 & \xrightarrow{i_p^1} & \Omega_2^1 & \xrightarrow{i_p^2} & \Omega_3^1 \xrightarrow{i_p^3} \dots \\
 & & & \searrow d & & \searrow d & \searrow d \\
 & & & & \Omega_2^2 & \xrightarrow{i_p^2} & \Omega_3^2 \xrightarrow{i_p^3} \dots
 \end{array}$$

It is a graded complex, in two ways. First as a complex induced from the semi-cosimplicial structure, with differential of bidegree (1,0), and second, as complex with differential d , of bidegree (1,1).

Lemma 2.6. Suppose A is commutative, then there is a natural morphism of complexes of A -modules $\Omega_A^* \subset \Omega_\star^*$ with $\Omega_A^n := \wedge^r \Omega_A \simeq \Omega_\star^n$.

Proof. Let $a_i \in A, i = 1, \dots, r$, and compute in $\Omega_\star^r$ the value of, $d^r(a_1 a_2 \dots a_r)$. It is clear that this gives the formula

$$\sum d_{i_1}(a_1) d_{i_2}(a_2) \dots d_{i_r}(a_r) = 0$$

where the sum being over all permutation $(i_1, i_2, \dots, i_r)$ of $(0, 1, \dots, r-1)$. Here we consider A as a subalgebra of $Ph^n(A)$ via the unique compositions of the $i_0^s : Ph^s(A) \subset Ph^{s+1}(A)$. In particular, we have

$$d_0(a_1) d_1(a_2) + d_1(a_1) d_0(a_2) = 0$$

for all $a_1, a_2 \in A$. This relation and the relation $d_0(a_2) d_1(a_1) = d_1(a_1) d_0(a_2)$, which follows from commutativity, $d(a_2) a_1 = a_1 d(a_2)$, forcing the left and right A -action on Ω_A to be equal, immediately give us, $d_0(a_1) d_1(a_2) = -d_0(a_2) d_1(a_1)$.

Consider now the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_0^1} & Ph^2(A) & \xrightarrow{i_0^2} & Ph^3(A) \xrightarrow{i_0^3} \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{i_0^0} & A \oplus \Omega_A^1 & \xrightarrow{i_0^1} & A \oplus \Omega_A^1 \oplus \Omega_A^2 & \xrightarrow{i_0^2} & A \oplus \Omega_A^1 \oplus \Omega_A^2 \oplus \Omega_A^3 \longrightarrow \dots \end{array}$$

where the bottom line is a sequence of Nagata-extensions of the k -algebra A , and the vertical homomorphisms correspond to the natural derivations among these, defined by the derivations of the de Rham complex $\Omega^\star$ of A .

The universality of the two systems proves that there is a surjective map

$$\alpha : \Omega_n^n \rightarrow \Omega_A^n := \wedge^n \Omega_A$$

The map that sends the element $da_1 \wedge da_2 \wedge \dots \wedge da_n \in \Omega_A^n$ to $d_0(a_1) d_1(a_2) \dots d_{n-1}(a_n) \in \Omega_n^n$ is an inverse, proving that α is an isomorphism. $\square$

Let now V be a right A -module and assume $c(V) = 0$, such that there exist an element, $\nabla' \in Hom_k(V, V \otimes_A Ph(A))$ with $c = \iota(\nabla')$. This implies that for $a \in A$ and $v \in V$ we have $\nabla'(va) = \nabla'(v)a + v \otimes d_0(a)$. Composing ∇' with the projection, $o : Ph(A) \rightarrow A$, corresponding to the 0-derivation of A , we therefore obtain an A -linear homomorphism $P : V \rightarrow V$, a *potential*. Since $i_0^0 : A \rightarrow Ph(A)$ is a section of o , we find a k -linear map

$$\nabla_0 := \nabla' - P : V \rightarrow V \otimes \mathfrak{m}_1^1$$

Using the property

$$d_n \circ i_{j+1}^{n+1} = i_j^n d_{n+1}$$

we find well defined k -linear maps

$$\nabla_1 : V \rightarrow V \otimes \Omega_2^1, \quad \nabla_2 : V \rightarrow V \otimes \Omega_3^1 \quad \dots \quad \nabla_n : V \rightarrow V \otimes \Omega_{n+1}^1, \quad \forall n \geq 0$$

given by

$$\nabla_{n+1} := \nabla_n \circ i_1^{n+1}, \quad n \geq 0$$

such that, for all, $v \in V, \omega \in \Omega_p^n$, the formula,

$$\nabla_n(v \otimes \omega) = \nabla_n(v)\omega + v \otimes d_n(\omega)$$

makes sense, and defines a sequence of *derivations*

$$\nabla_n : V \otimes \Omega_p^n \rightarrow V \otimes \Omega_p^{n+1}$$

sometimes just denoted d_n , and called a *connection* ∇ on the A -module V . We obtain a situation just like above,

$$\begin{array}{ccccccc} V & \xrightarrow{i_0^0} & V \otimes Ph(A) & \xrightarrow{i_p^1} & V \otimes Ph^2(A) & \xrightarrow{i_p^2} & V \otimes Ph^3(A) \xrightarrow{i_p^3} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ V & \xrightarrow{i_0^0} & V & \xrightarrow{i_p^1} & V & \xrightarrow{i_p^2} & V \xrightarrow{i_p^3} \\ & \searrow d_0 & & \searrow d_1 & & \searrow d_2 & & \searrow d_3 \\ & & V \otimes \Omega_1^1 & \xrightarrow{i_p^1} & V \otimes \Omega_2^1 & \xrightarrow{i_p^2} & V \otimes \Omega_3^1 \xrightarrow{i_p^3} \\ & & & \searrow d_1 & & \searrow d_2 & & \searrow d_3 \\ & & & & V \otimes \Omega_2^2 & \xrightarrow{i_p^2} & V \otimes \Omega_3^2 \xrightarrow{i_p^3} \end{array}$$

In general, there are no reasons for these derivations $d_n := \nabla_n$, $n \geq 0$ to define complexes and we make the following definition.

Definition 2.7. The curvature $R(V, \nabla)$ of the connection ∇ , defined on the right A -module V , is the k -linear map, composition of d_0 and d_1 ,

$$R(V) = d_0 d_1 : V \rightarrow V \otimes \Omega_2^2$$

The following Lemma is then easily proved.

Lemma 2.8. Suppose A is commutative, and assume $c(V) = 0$. Let $\nabla : \Theta_A \rightarrow \text{End}_k(V)$ be the classical connection corresponding to ∇_0 . Suppose moreover that the curvature R of ∇ is 0, then $R(V) = 0$, implying that $d^2 = 0$, and so the diagonals in the diagram above, are all complexes.

Proof. We may put

$$\nabla(v_i) = \sum_{j,k} a_{i,j}^k v_j d_0(x_k)$$

and obtain

$$\nabla_1(\nabla_0(v_i)) = \sum_{j,k,l} \frac{\partial a_{i,j}^k}{\partial x_l} v_j d_1(x_l) d_0(x_k) + \sum_{j,k,l,m} a_{i,j}^k a_{j,m}^l v_m d_1(x_l) d_0(x_k).$$

Now the classical of curvature of ∇ may be defined as

$$R_{k,l}^i = \sum_j \frac{\partial a_{i,j}^k}{\partial x_l} v_j + \sum_{j,m} a_{i,j}^k a_{j,m}^l v_m - \sum_j \frac{\partial a_{i,j}^l}{\partial x_k} v_j - \sum_{j,m} a_{i,j}^l a_{j,m}^k v_m$$

so if $R = 0$ and $d_1(x_l) d_0(x_k) = -d_1(x_k) d_0(x_l)$, it follows that $\nabla_1(\nabla_0(v_i)) = 0$, from which it follows that $d^2 = 0$. $\square$

2.3. Dirac derivation

The system of k -algebras and homomorphisms of k -algebras $\{Ph^n(A), i_j^n\}_{n, 0 \leq j \leq n}$ has an inductive (direct) limit, $Ph^\infty A$, together with homomorphisms, $i_n : Ph^n(A) \longrightarrow Ph^\infty(A)$ satisfying, see [5],

$$i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \dots, n$$

Moreover, the family of derivations $\{d_n\}_{0 \leq n}$ defines a unique derivation, the *Dirac derivation*

$$\delta : Ph^\infty(A) \longrightarrow Ph^\infty(A)$$

such that

$$i_n \circ \delta = d_n \circ i_{n+1}$$

and it is easy to see that this is a universal construction, i.e for any pair of a morphism $i : A \longrightarrow B$ and a derivation $\xi \in Der_k(B)$, $\iota \circ \xi$ factorizes via $Ph^\infty(A)$ and δ . Put

$$Ph^{(n)}(A) := im \ i_n \subseteq Ph^\infty(A).$$

The k -algebra $Ph^\infty(A)$ has a descending filtration of two-sided ideals, $\{\mathbf{F}_n\}_{0 \leq n}$ given inductively by

$$\mathbf{F}_1 = Ph^\infty(A) \cdot im(\delta) \cdot Ph^\infty(A)$$

and

$$\delta(\mathbf{F}_n) \subseteq \mathbf{F}_{n+1}, \quad \mathbf{F}_{n_1} \mathbf{F}_{n_2} \dots \mathbf{F}_{n_r} \subseteq \mathbf{F}_n, \quad \forall n_1 + \dots + n_r = n$$

such that the derivation δ induces derivations $\delta_n : \mathbf{F}_n \longrightarrow \mathbf{F}_{n+1}$. Using the canonical homomorphism $i_n : Ph^n(A) \longrightarrow Ph^\infty(A)$ we pull the filtration $\{\mathbf{F}_p\}_{0 \leq p}$ back to $Ph^n(A)$, not bothering to change the notation.

Definition 2.9. Let

$$\mathbf{D}(A) := \varprojlim_{n \geq 1} Ph^\infty(A)/\mathbf{F}_n$$

be the completion of $Ph^\infty(A)$ in the topology given by the filtration $\{\mathbf{F}_n\}_{0 \leq n}$. The k -algebra $Ph^\infty(A)$ will be referred to as the k -algebra of higher differentials, and $\mathbf{D}(A)$ will be called the k -algebra of formalized higher differentials. Put

$$\mathbf{D}_n := \mathbf{D}_n(A) := Ph^\infty(A)/\mathbf{F}_{n+1}$$

Clearly δ defines a derivation on $\mathbf{D}(A)$, and an homomorphism of k -algebras,

$$\epsilon := \exp(\delta) : \mathbf{D}(A) \rightarrow \mathbf{D}(A)$$

and in particular, an algebra homomorphism

$$\tilde{\eta} := \exp(\delta) : A \rightarrow \mathbf{D}(A)$$

Recall now that for any associative k -algebra A , and for any right A -module V , any derivation $\delta \in Der_k(A, End_k(V))$ defines a class, $\xi(v) \in Ext_A^1(V, V)$, i.e a *tangent vector* of the *formal moduli* of the representation V , at the unique point.

Remark 2.10. The above implies that any representation $\rho : Ph^\infty(A) \rightarrow End_k(V)$ corresponds to a family of $Ph^n(A)$ -module-structures on V for $n \geq 1$, i.e to an A -module $V_0 := V$, an element $\xi_0 \in Ext_A^1(V, V)$, i.e a tangent of the deformation functor of $V_0 := V$, as A -module, an element $\xi_1 \in Ext_{Ph(A)}^1(V, V)$, i.e a tangent of the deformation functor of $V_1 := V$ as $Ph(A)$ -module, an element $\xi_2 \in Ext_{Ph^2(A)}^1(V, V)$, i.e a tangent of the deformation functor of $V_2 := V$ as $Ph^2(A)$ -module, etc.

All this is just V , considered as an A -module, together with a sequence $\{\xi_n\}, 0 \leq n$, of a tangent, or a *momentum* ξ_0 , an acceleration vector ξ_1 , and any number of higher order *momenta* ξ_n . Thus, specifying a $Ph^\infty(A)$ -representation V , implies specifying a *formal curve* through v_0 , the base-point of the *miniversal deformation space* of the A -module V . Formally, this curve is given by the composition of the homomorphism $\epsilon(\tau) := exp(\tau\delta)$ and ρ . This is the *Time Evolution* of a representation ρ , to be treated later on.

3. Non-commutative algebraic geometry

The basic notions of affine non-commutative algebraic geometry related to a (not necessarily commutative) associative k -algebra, for k an arbitrary field, have been treated by many authors in several texts, see e.g [7, 8, 1, 2]. Given a finitely generated algebra A , we prove the existence of a non-commutative scheme-structure on the set of isomorphism classes of simple finite dimensional representations, i.e right modules, $Simp_{<\infty}(A)$.

We showed, in [2], that any finitely generated *geometric* k -algebra A , k algebraically closed, see definition in [5], including the commutative algebras, may be recovered from the (non-commutative) structure of $Simp_{<\infty}(A)$, and that there is an underlying quasi-affine (commutative) scheme-structure on each component $Simp_n(A) \subset Simp_{<\infty}(A)$, parametrizing the simple representations of dimension n . In fact, we have the following.

Theorem 3.1. *There is a commutative k -algebra $C(n)$ with an open subvariety $U(n) \subseteq Simp_1(C(n))$, an étale covering of $Simp_n(A)$, over which there exists a versal representation $\tilde{V} \simeq C(n) \otimes_k V$, a vector bundle of rank n defined on $Simp_1(C(n))$, and a versal family, i.e a morphism of algebras*

$$\tilde{\rho} : A \longrightarrow End_{C(n)}(\tilde{V}) \rightarrow End_{U(n)}(\tilde{V})$$

inducing all isoclasses of simple n -dimensional A -modules.

$End_{C(n)}(\tilde{V})$ induces also a bundle, of *operators*, on the étale covering $U(n)$ of $Simp_n(A)$. Assume given a derivation, $\gamma \in Der_k(A)$. Pick any $v \in Simp_n(A)$ corresponding to the right A -module V , with structure homomorphism $\rho_v : A \rightarrow End_k(V)$, then γ composed with ρ_v , gives us an element $\gamma_v \in Ext_A^1(V, V)$.

Therefore, γ defines a unique one-dimensional distribution in $\Theta_{Simp_n(A)}$, which, once we have fixed a versal family, defines a vector field $[\gamma] \in \Theta_{Simp_n(A)}$ and, in good cases, a (rational) derivation $[\gamma] \in Der_k(C(n))$. Notice also that we have the canonical isomorphism

$$Der_k(A, A) \simeq Mor_A(Ph(A), A)$$

Therefore the derivation γ , and the A -module V , correspond to a $Ph(A)$ -module V_γ .

3.1. Dynamical structures and time

As we have seen, in the Remark (2.10), the dynamics of the space of representations of our algebra A , i.e the dynamics of the *space of measurements* of the family of *observables* that A is assumed to represent, can be encoded in the category of representations of the k -algebra $Ph^\infty(A)$, with the Dirac derivation as an infinitesimal time parameter. We would therefore like to use the tools developed above, for the k -algebra $Ph^\infty(A)$.

However, $Ph^\infty(A)$ is rarely of finite type, and so the space of simple modules does not have a classical algebraic geometric structure. We shall therefore introduce the notion of *dynamical structure*, to reduce the problem to a situation we can handle. This is also what physicists do. They invoke a parsimony principle, or an action principle, originally proposed by Fermat, and later by Maupertuis, with the purpose of reducing the preparation needed, to be able to see ahead.

Definition 3.2. A *dynamical structure*, σ , is a two-sided δ -stable ideal $(\sigma) \subset Ph^\infty(A)$, such that $A(\sigma) := Ph^\infty(A)/(\sigma)$ the corresponding, *dynamical system*, is of finite type. A dynamical structure, or system, is of order n if the canonical morphism $\sigma : Ph^{(n-1)}(A) \rightarrow A(\sigma)$ is surjective. If A is generated by the *coordinate functions*, $\{t_i\}_{i=1,2,\dots,d}$ a dynamical system of order n may be defined by a *force law*, i.e by a system of equations

$$\delta^n t_p = \Gamma^p(t_i, \underline{dt}_j, \underline{d}^2 t_k, \dots, \underline{d}^{n-1} t_l), \quad p = 1, 2, \dots, d$$

Put

$$A(\sigma) := Ph^\infty(A)/(\delta^n t_p - \Gamma^p)$$

where $\sigma := (\delta^n t_p - \Gamma^p)$ is the two-sided δ -ideal generated by the defining equations of σ . Obviously δ induces a derivation $\delta_\sigma \in Der_k(A(\sigma), A(\sigma))$, also called the Dirac derivation, and usually just denoted δ .

Notice that if σ_i , $i = 1, 2$, are two different order n dynamical systems, then we may well have

$$A(\sigma_1) \simeq A(\sigma_2) \simeq Ph^{(n-1)}(A)/(\sigma_*)$$

as k -algebras.

Assuming that the k -algebra A is finitely generated, and that the dynamical structure σ is such that also $A(\sigma)$ is finitely generated, we can now use the machinery above, with $\gamma = \delta$, the Dirac derivation. The following extension of Theorem (3.2), proved in [5], is important for the philosophy of this paper.

Theorem 3.3. Formally, at any point $v \in U(n)$, with local ring $\hat{C}(n)_v$, there is a derivation $[\delta] \in Der_k(\hat{C}(n)_v)$, and a Hamiltonian $Q \in End_{\hat{C}(n)_v}(\tilde{V}_v)$, such that, as operators on $\tilde{V}_v$, we have

$$\delta = [\delta] + [Q, -]$$

This means that for every $a \in A(\sigma)$, considered as an operator $\tilde{\rho}(a) \in End_{C(n)_v}(\tilde{V}_v) \simeq M_n(C(n)_v)$, $\delta(a)$ acts formally on $\tilde{V}_v$ as

$$\tilde{\rho}(\delta(a)) = [\delta](\tilde{\rho}(a)) + [Q, \tilde{\rho}(a)]$$

Notice that, to make visible the action of $[\delta]$ on $\rho(a)$, we have included the obvious isomorphism, $End_{C(n)}(\tilde{V}) \simeq M_n(C(n))$, which will, in the sequel, be used without further warning.

There are local (and even global) extensions of this result, where $[\gamma]$ and Q may be assumed to be defined (rationally) on $C(n)$, see [5].

In line with our general philosophy, we shall therefore consider $[\delta]$ as measuring *time* in $Simp_n A(\sigma)$, respectively in $Spec(C(n))$. Assume for a while that $k = \mathbf{R}$, the real numbers, and that our constructions go through, as if k were algebraically closed. Let $v \in Simp_n(A(\sigma))$

be an element, an *event*. Suppose there exist an integral curve $\mathbf{c}$ of $[\delta]$ through some element $v(\tau_0) \in \text{Simp}_1(C(n))$, equivalent to v , ending at $v(\tau_1) \in \text{Simp}_1(C(n))$, given by the automorphisms $e(\tau) := \exp(\tau[\delta])$, for $\tau \in [\tau_0, \tau_1] \subset \mathbf{R}$. The supremum of τ for which the corresponding point, $v(\tau)$, of $\mathbf{c}$ is in $\text{Simp}_n(A(\sigma))$ should be called the *lifetime* of the *particle* $v(\tau)$. It is relatively easy to compute these lifetimes, and so be able to talk about *decay*, when the fundamental vector field $[\delta]$ has been computed. In [5], we have also proposed a mathematically sound way of treating interaction, purely in terms of non-commutative deformation theory.

Let $\phi(\tau_0) \in \tilde{V}(v_0) \simeq V$ be a (classically considered) state of our *quantum system*, at the time τ_0 , and consider the (uni)versal family $\tilde{\rho} : A(\sigma) \rightarrow \text{End}_{C(n)}(\tilde{V})$ restricted to $U(n) \subseteq \text{Simp}_1(C(n))$, the étale covering of $\text{Simp}_n(A(\sigma))$. We shall consider $A(\sigma)$ as our *ring of observables*. What happens to $\phi(\tau_0) \in V(0)$ when *time* passes from τ_0 to τ , along $\mathbf{c}$? This leads to the problem of finding a solution of the Schrödinger equation, *along* $\mathbf{c}$, given by the next result, proving that ϕ is completely determined, by the value of $\phi(\tau_0)$, for any $\tau_0 \in \mathbf{c}$. Here, we shall not go into the problem of *preparing* $\psi(\tau_0) \in V(\tau_0)$, i.e of how to exactly determin *where we are*, at some chosen clock-time, τ , see [5].

Theorem 3.4. *The evolution operator $u(\tau_0, \tau_1)$ that changes the state $\phi(\tau_0) \in \tilde{V}(v_0)$ into the state $\phi(\tau_1) \in \tilde{V}(v_1)$, where $\tau_1 - \tau_0$ is the length of the integral curve $\mathbf{c}$ connecting the two points v_0 and v_1 , i.e the time passed, is given by*

$$\phi(\tau_1) = u(\tau_0, \tau_1)(\phi(\tau_0)) = \exp \left[\int_{\mathbf{c}} Q(\tau) d\tau \right] (\phi(\tau_0))$$

where $\exp \int_{\mathbf{c}}$ is the non-commutative version of the ordinary action integral, essentially defined by the equation

$$\exp \left[\int_{\mathbf{c}} Q(\tau) d\tau \right] = \exp \left[\int_{\mathbf{c}_2} Q(\tau) d\tau \right] \circ \exp \left[\int_{\mathbf{c}_1} Q(\tau) d\tau \right]$$

where $\mathbf{c}$ is $\mathbf{c}_1$ followed by $\mathbf{c}_2$.

3.2. The generic dynamical structures associated to a metric

Now, let $C = k[t_1, \dots, t_n]$, $k = \mathbf{R}$, the real numbers, be a commutative polynomial k -algebra, and let

$$g = \frac{1}{2} \sum_{i,j=1,\dots,r} g_{i,j} dt_i dt_j \in Ph(C)$$

be a Riemannian metric. Recall the formula for the Levi-Civita connection

$$\sum_l g_{l,k} \Gamma_{j,i}^l = \frac{1}{2} \left(\frac{\partial g_{k,i}}{\partial t_j} + \frac{\partial g_{j,k}}{\partial t_i} - \frac{\partial g_{i,j}}{\partial t_k} \right)$$

Since in $Ph^\infty(C)$, we have

$$\delta(g) = \frac{1}{2} \sum_{i,j,k=1,\dots,r} \frac{\partial g_{i,j}}{\partial t_k} dt_k dt_i dt_j + \frac{1}{2} \sum_{i,j=1,\dots,r} g_{i,j} (d^2 t_i dt_j + dt_i d^2 t_j)$$

and we may plug in the formula

$$\delta^2 t_l = -\Gamma^l := -\sum \Gamma_{i,j}^l dt_i dt_j$$

on the right hand side, and see that we have got a solution of the Lagrange equation $\delta(g) = 0$ in the the commutativization of $Ph(C)$, i.e in the classical phase space. This solution has the form of a *force law*

$$d^2 t_l = -\Gamma^l := -\sum \Gamma_{i,j}^l dt_i dt_j$$

generating a dynamical structure $(\sigma) := (\sigma(g))$ of order 2. The dynamical system is the algebra $\mathbf{C}(\sigma) = k[\underline{t}, \underline{\xi}]$, where ξ_j is the class of dt_j . The Dirac derivation now takes the form

$$\delta = \sum_l \left(\xi_l \frac{\partial}{\partial t_l} - \Gamma^l \frac{\partial}{\partial \xi_l} \right)$$

coinciding with the fundamental vector field $[\delta]$ in $Simp_1(\mathbf{C}(\sigma)) = Spec(k[t_i, \xi_j])$.

The equation $[\delta](g) = 0$ imply that g is constant along the integral curves of $[\delta]$ in $Simp_1(Ph(C))$, and these integral curves projects into $Simp_1(C)$ to give the geodesics of the metric g with equations

$$\ddot{t}_l = -\sum_{i,j} \Gamma_{i,j}^l \dot{t}_i \dot{t}_j$$

We may also look at this from another point of view. Suppose given any dynamical structure with Dirac derivation δ on $Ph(C)$. Consider $Simp_1(Ph(C))$. It is obviously represented by $C(1) := k[\underline{t}, \underline{\xi}]$, and the Dirac derivation induces a derivation $[\delta] \in Der_k(C(1))$, and the Hamiltonian must vanish. Therefore we have two options for *the same notion of time* in the picture, g and $[\delta]$. The last derivation should therefore be a *Killing vector field*, i.e we must have a solution of Lagrange equation $[\delta](g) = 0$ and we are left with the above solution for δ .

3.3. Connections as representations

As a contrast to this 1-dimensional representation case, let us consider infinite dimensional representations, for which the Dirac derivation $[\delta]$ vanishes, and the notion of time is taken care of by the Hamiltonian Q .

A non-degenerate metric $g = 1/2 \sum_{i=1}^d g_{i,j} dt_i dt_j \in Ph(C)$ induces an isomorphism of C -modules $\Theta_C = Hom_C(\Omega_C, C) \simeq \Omega_C$. Consider the bilateral ideal (σ_g) of $Ph(C)$ generated by

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j})$$

and put $C(\sigma_g) := Ph(C)/(\sigma_g)$. Let, moreover,

$$T := -\frac{1}{2} \left(\sum_{k,l} \Gamma_{k,l}^k dt_l + \sum_{k,p,q} g^{k,q} \Gamma_{k,q}^p g_{p,l} dt_l \right)$$

and consider the inner derivation of $C(\sigma_g)$, defined by $\delta := ad(g - T)$. After a dull computation, we obtain, see [5], that in $C(\sigma_g)$, $\delta(t_i) = dt_i, i = 1, \dots, d$. Therefore we have a well-defined dynamical structure (σ_g) , with Dirac derivation, $\delta = ad(g - T)$. In partiucular, there is a unique homomorphism, $Ph^\infty(C) \rightarrow C(\sigma_g)$, consistent with the Dirac derivation δ and $ad(g - T)$. It is also easy to see that (σ_g) is invariant w.r.t. isometries.

Example 3.5. The trivial metric, $g_0 = 1/2 \sum_{i=1}^d dt_i^2$ will give us the dynamic structure, $[dt_j, t_i] = \delta_{i,j}$, with the dynamic system $C(\sigma_{g_0}) = W(d)$, the Weyl space. Clearly $[g_0, t_i] = dt_i$, and, of course, $d^2 t_i = [g_0, dt_i] = 0$ for all $i = 1, \dots, d$, so that this gives us the classical set-up for Quantum Mechanics.

Any connection $\nabla : \Theta_C \rightarrow \text{End}_k(E)$, on a free C -module E is given in terms of the operators,

$$\delta_{t_i} := \frac{\partial}{\partial t_i}, \quad \nabla_{\delta_{t_i}} = \delta_{t_i} + \nabla_i$$

where $\nabla_i \in \text{End}_C(E)$ is now a representation

$$\rho_{\nabla} : C(\sigma_g) \rightarrow \text{End}_k(E)$$

mapping t_i to the obvious operator and mapping dt_i to $\sum_{j=1}^d g^{i,j} \nabla_{\delta_j}$. Put

$$\xi_i := \delta^i = \sum_{j=1}^d g^{i,j} \delta_j.$$

Conversaly, every C -representation $\rho : C(\sigma_g) \rightarrow \text{End}_k(E)$ will map the element, dt_i to a differential operator ξ_i , with $[\xi_i, t_j] = g^{i,j}$, i.e we have a connection on E .

Notice that the representation $\rho = \rho_{\Theta}$ of $C(\sigma_g)$ defined on Θ_C by the Levi-Civita connection has an Hamiltonian

$$Q := \rho(g - T) = \frac{1}{2} \sum_{i,j} g^{ij} \nabla_{\delta_i} \nabla_{\delta_j}$$

i.e the generalized Laplace-Beltrami operator, which is also invariant w.r.t. isometries, although the proof demands some algebra.

Notice also that for the Levi-Civita connection, we shall use the notations

$$D_{t_i} := \nabla_{\delta_{t_i}}, \quad D(\xi) := \nabla_{\xi} \quad \text{for } \delta_{t_i}, \quad \xi \in \Theta_{k[t]}$$

As above, we now have two options for the notion of time, and time-development. In line with classical Quantum Theory, we would, for a *state* $\xi \in \Theta_C$, consider the Schrödinger equation

$$\frac{d\xi}{d\tau} = Q(\xi)$$

where τ would be an ad hoc chosen time parameter. In our case there are just two natural choices for τ , namely ξ itself, or the metric g , measuring the time t , see the discussion above. Since we have

$$D_{\xi}(\xi) = \mu \frac{d\xi}{dt}$$

where $\mu = g(\xi, \xi)^{1/2}$, it seems reasonable to replace the classical Schrödinger equation, in our situation, by the following one:

$$\frac{d\xi}{dt} = Q(\xi), \quad \xi \in \Theta_C$$

Notice that the left hand side of the Furniture Equation is

$$\frac{1}{\mu} D_{\xi}(\xi) = \frac{1}{\mu} \sum_{i,j} \xi_i \xi_j \Gamma_{i,j}^k \delta_l + \frac{1}{\mu} \sum_{i,l} \xi_l \frac{\partial \xi_l}{\partial t_i} \delta_l$$

If we go to our Toy Model, see [5], then $1/\mu \xi_i \xi_j$ would, in the physical interpretation of that model, have been written as $T^{i,j} = 1/E p_i p_j$, where $E := \mu$ is the energy, and (p_i) is the momentum of the *particles* that populate the universe. Classically T is called it the *Mass-stress Tensor*, T_{part} , in the classical relativity theory.

We shall return to this last equation, in the next section, where we, sometimes, will refer to it as the *Furniture Equation*.

But, first, notice the generic equation of formal motion for representations of $C(\sigma_g)$.

Corollary 3.6 (the Generic Equation of Motion). *Consider a $C(\sigma_g)$ -representation, $\rho : C(\sigma_g) \rightarrow \text{End}_k(V)$. Then the time development is, formally, given in terms of a parameter τ as*

$$\tilde{\rho}(\tau) = \rho(\epsilon(\tau)) : C(\sigma_g) \rightarrow \text{End}_k(V)$$

where

$$\epsilon(\tau) = \exp(\tau \cdot \text{ad}(g - T))$$

Now, to be able to handle this time development, we need to know formulas, in $C(\sigma_g)$, for $d^l t_i$, $l \geq 1$, $i = 1, \dots, n$. To this end, put

$$\bar{\Gamma}_{p,q}^i := \sum_{l,r} g^{r,i} \Gamma_{r,p}^l g_{l,q}, \quad \bar{\nabla}_l := (\bar{\Gamma}_{i,l}^j), \quad T = \sum_l T_l dt_l$$

Then

$$\begin{aligned} T_l &= -1/2 \left(\sum_j (\Gamma_{j,l}^j + \bar{\Gamma}_{j,l}^j) \right) = -1/2 (\text{tr } \nabla_l + \text{tr } \bar{\nabla}_l), \\ \delta^2 t_i &= [g - T, dt_i] = -1/2 \sum_{p,q} (\bar{\Gamma}_{p,q}^i + \bar{\Gamma}_{q,p}^i) dt_p dt_q + 1/2 \sum_{p,q} g_{p,q} (R_{p,i} dt_q + dt_p R_{q,i}) + [dt_i, T] \end{aligned}$$

where, as above, $R_{i,j} = [dt_i, dt_j]$. Put

$$\Gamma_p^{j,i} = \sum_k g^{j,k} \Gamma_{k,p}^i, \quad F_{i,j} := R_{i,j} - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j}) dt_p$$

and recall (see [5], p. 82) that for any connection ρ on a C -bundle E , the corresponding representation of $C(\sigma_g)$ will map $F_{i,j}$ to the ordinary curvature of the connection. In fact,

$$\rho(F_{i,j}) = [\rho(dt_i), \rho(dt_j)] - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j}) \rho(dt_p) = [\nabla_{\xi_i}, \nabla_{\xi_j}] - \nabla_{[\xi_i, \xi_j]}$$

Put, for short,

$$F(\xi_i, \xi_j) = F_{i,j} := \rho(F_{i,j}) \in \text{End}_C(E)$$

Computing, we find, see [5], for a proof, the following.

Theorem 3.7 (the Generic Force Law). *In $C(\sigma_g)$ we have the following force laws*

$$\begin{aligned} (1) \quad d^2 t_i &= -1/2 \sum_{p,q} (\bar{\Gamma}_{p,q}^i + \bar{\Gamma}_{q,p}^i) dt_p dt_q + 1/2 \sum_{p,q} g_{p,q} (R_{p,i} dt_q + dt_p R_{q,i}) + [dt_i, T] \\ (2) \quad d^2 t_i &= - \sum_{p,q} \Gamma_{p,q}^i dt_p dt_q - 1/2 \sum_{p,q} g_{p,q} (F_{i,p} dt_q + dt_p F_{i,q}) \\ &\quad + 1/2 \sum_{l,p,q} g_{p,q} [dt_p, (\Gamma_l^{i,q} - \Gamma_l^{q,i})] dt_l + [dt_i, T] \end{aligned}$$

Notice that considering the representation, ρ_Θ , corresponding to the Levi-Civita connection, the above translate into

$$\rho_\Theta(dt_i) = [Q, t_i], \quad \rho(d^2 t_i) = \sum_{j=1}^d [Q, \rho(dt_i)]$$

where Q is the Laplace-Beltrami operator.

Given any *observable* $f \in Ph(C)$, we would expect that the dynamics of the future *values* of f , measured via some representation, E , to be the spectrum of the operator

$$F(\tau) := \exp(\tau \cdot ad(Q))(f)$$

Notice again that any representation E of $C(\sigma_g)$, is of course a representation of $Ph^\infty(C)$. However, $Ph^\infty(C)$ is now infinitely generated, and the representation E of $C(\sigma_g)$, is never of finite dimension so we may not use the nice non-commutative algebraic geometry, and its quantum theoretical look-alike without some kind of adaptation, or at least, some kind of approximation, to which we shall return.

Remark 3.8. We shall consider the above formulas as general Force Laws, in $Ph(C)$, induced by the metric g . This means the following: Let $\mathfrak{c}$ be the δ -stable ideal generated by any one of these equation in $Ph^\infty(C)$. Since the force laws above holds in the dynamical system defined by (σ_g) , we obviously have $\mathfrak{c} \subset (\sigma_g)$, and we might hope these new dynamical systems might lead to new Quantum Field Theories, as defined above, with equally new and interesting properties.

For a connection ∇ on a free C -module E , the second Force Law above will now take the form, in $End_C(E)$,

$$\begin{aligned} \rho_E(d^2 t_i) + \sum_{p,q} \Gamma_{p,q}^i \nabla_{\xi_p} \nabla_{\xi_q} &= 1/2 \sum_p F_{p,i} \nabla_{\delta_p} + 1/2 \sum_p \nabla_{\delta_p} F_{p,i} \\ &+ 1/2 \sum_{l,q} \delta_q (\Gamma_l^{i,q} - \Gamma_l^{q,i}) \nabla_{\xi_l} + [\nabla_{\xi_i}, \rho_E(T)] \end{aligned}$$

where $\rho(dt_i) = \sum_j g^{i,j} \nabla_{\delta_j} =: \xi_i$.

4. Entropy

Consider an algebraic geometric object X , and let $aut(X)$ be the Lie algebra of automorphisms of X . The sub-Lie algebra $aut_0(X)$ that lifts to automorphisms of the formal moduli of X , is a Lie ideal. Put $\mathfrak{a}(X) := aut(X)/aut_0(X)$, then if $X(t)$ is a deformation of X along a parameter t , we find $dim_k \mathfrak{a}(X(t)) \leq dim_k \mathfrak{a}(X)$. One may phrase this saying that an object X can never gain *information* when deformed. Moreover, deformation is, obviously, not a reversible process, so information can get lost.

4.1. The classical commutative case

In [6], studying moduli problems of singularities in (classical) algebraic geometry, we were led to consider the notion of *Modular Suite*. This is a canonical partition $\{\mathbf{M}_\alpha\}$, of the versal base space, $\mathbf{M}$, of the deformation functor of an algebraic object, X . The different *rooms*, $\mathbf{M}_\alpha$, correspond to the subsets of equivalence classes of deformations in $\mathbf{M}$, along which the Lie algebra $\mathfrak{a} := aut/aut_0$ deforms as Lie-algebras, and therefore conserves its dimension. Working with Thermodynamics, see [Laudal O A *Cosmos and its Furniture* (Works Sci.) to appear], it occurred to me that the notion of entropy has an interesting parallel in deformation theory. In fact I have proposed the following,

Definition 4.1. Let $X(\underline{t})$ corresponds to the point $\underline{t} \in \mathbf{M}_\alpha$, then we shall term *Entropy* of the state $\underline{t}$, the integer $S(\underline{t}) := dim_k(\mathbf{M}_\alpha)$.

In this classical situation, assuming that the field is algebraically closed, and that $\mathbf{M}$ is of finite Krull dimension, the modular suite $\{\mathbf{M}_\alpha\}$ is finite, with an inner room, the *modular substratum* and an *ambient* (open) *maximal entropy* stratum. But the structure of the modular suite may

be very complex, even for simple singularities X , see the example of the quasi homogenous plane curve singularity $x_1^5 + x_2^{11}$, in [6]. It is also clear that for any *algebraic dynamics* in $\mathbf{M}$, the entropy will always stay or grow, see again [6]. To be able to construct situations where the entropy is lowered, or the information goes up, we must leave classical algebraic geometry, and venture into non-commutative algebraic geometry. Here is where non-commutative deformation theory comes into play.

4.2. The general case

In the general situation, where our algebras of *observables* are associative but not necessarily commutative, the first interesting cases are deformations of associative algebras, A , or deformations of finite families of representations V_i of an associative algebra A . In [Laudal O A *Cosmos and its Furniture* (World Sci.) to appear] I will touch upon the first case. Here we shall look at the second case, based upon the technique of the previous sections.

There we worked with a polynomial algebra, $C = k[t_1, \dots, t_d]$, with a metric $g \in Ph(C)$, and the Levi-Civita connection, considered as a representation

$$A := Ph(C)/(\sigma_g) \rightarrow End_k(\Theta_C)$$

The Dirac derivation, $[\delta]$, in this case, vanish and the corresponding Hamiltonian turned out to be the Laplace-Beltrami operator, $Q \in End_k(\Theta_C)$. We were then, in analogy with Quantum theory, led to consider, for every *state* $\xi \in \Theta_C$, the time development, or Schrödinger equation, and the corresponding *Furniture Equation* [Laudal O A *Cosmos and its Furniture* (Worlds Sci.) to appear] reads

$$\frac{d\xi}{dt} = Q(\xi)$$

We have shown in [Laudal O A *Cosmos and its Furniture* (Worlds Sci.) to appear] that in the special case of our *toy model*, see [5] and *Cosmos and its Furniture*, this equation amounts to a combined Heat and Navier-Stokes equations. The corresponding notion of entropy, in the above sense, might be defined by the modular suite of the versal base of the deformation functor of the corresponding representation

$$\xi : Ph(C) \rightarrow C = End_C(C)$$

The formal moduli has a tangent space given by $Ext_{Ph(C)}^1(\xi, \xi)$, which is given by the classical long exact sequence

$$0 \rightarrow Hom_{Ph(C)}(\xi, \xi) \rightarrow Hom_C(\xi, \xi) \xrightarrow{\iota} Der_C(Ph(C), Hom_C(\xi, \xi)) \xrightarrow{\kappa} Ext_{Ph(C)}^1(\xi, \xi) \rightarrow 0$$

Here, $\iota = 0$ and $Hom_C(\xi, \xi) = C$, so that

$$Ext_{Ph(C)}^1(\xi, \xi) = C^d \simeq \Theta_C$$

as all relations $[dt_i, t_j] + [t_i, dt_j]$ in $Ph(C)$ are mapped to 0 by any C -derivation into C .

The miniversal base space of ξ , is therefore of infinite dimension, and since the Lie algebra of the automorphism group of C is the Lie algebra $Der_k(C) = \Theta_C$, the Lie algebra of the automorphism group of the representation ξ is

$$aut(\xi) := \{\eta \in \Theta_C \mid [\eta, \xi] = 0\}$$

We should now define the entropy of the state ξ as

$$S(\xi) := \dim\{\eta \mid aut(\xi) \vdash aut(\eta)\}$$

where $aut(\xi) \vdash aut(\eta)$ should mean that η is a deformation of ξ inducing a deformation of the Lie algebras of automorphisms. Obviously this is unrealistic, as most of the terms involved will be of infinite dimension.

We may try to overcome this difficulty by approximating the state ξ , as a representation, by a finite dimensional representation, defined for every point set object $\mathcal{P} = \{P_p\}$ of $Simp_1(C)$ by $\xi_{\mathcal{P}} : Ph(C) \rightarrow End_k(\mathcal{P})$ where

$$\xi_{\mathcal{P}}(t_i) = \alpha_i^0(p), \quad \xi_{\mathcal{P}}(dt_i) = \alpha_i^1(p)$$

where we, in anticipation of the treatment via the technique of the next section, have put

$$\alpha_i^0(p) = P_{p,i}, \quad \alpha_i^1(p) = \xi(P_p)_i$$

We may look at the object $\mathcal{P}$ as the set of molecules in our observatory (or in the Universe), and ξ as the combined state of these, at the outset maybe considered independently. Extending the representation $\xi_{\mathcal{P}}$ to a representation $\xi_{\mathcal{P}} : Ph^{\infty}(C) \rightarrow End_k(\mathcal{P})$, or cutting it down to a representation of some dynamical system,

$$\xi_{\mathcal{P},\sigma} : A(\sigma) := Ph^{\infty}(C)/(\sigma) \rightarrow End_k(\mathcal{P})$$

for some reasonable dynamical structure (σ) of $Ph^{\infty}(C)$, we can now use the technique of section (3), and look at the versal family $\mathbf{M}$ of the representation $\xi_{\mathcal{P},\sigma}$ for some fundamental *state* ξ , of the *Furniture* of the Universe. There will be a canonically defined *Moduli Suite* $\{\mathbf{M}_{\alpha}\}$ and for any deformation η of $\xi_{\mathcal{P},\sigma}$, there will be one α such that $\eta \in \mathbf{M}_{\alpha}$ and the resulting definition of the entropy would be $S(\eta) = \dim(\mathbf{M}_{\alpha})$. The goal is to show that for reasonable classical cases, this should come out close to the Boltzmann's definition

$$S(\eta) := \log Vol(M_{\alpha}), \quad \eta \in M_{\alpha}$$

where M_{α} is the substratum of the corresponding *Coarse Graining* of the classical phase space of the situation, containing the *distribution* η , see again the very readable text of [9].

4.3. Representations of Ph^{∞}

Now let $A = k[t_1, \dots, t_d]$, and consider a representation of $Ph^{\infty}(A)$ as a k -algebra,

$$\rho : Ph^{\infty}(A) \rightarrow End_k(V)$$

where V a k -vector space. Put

$$D_i^0 := \rho(t_i), \quad D_i^p := \rho(d^p t_i), \quad p \geq 1$$

The composition of $\exp(\tau\delta)$ and ρ is a homomorphisms of k -algebras

$$Ph^{\infty}(A) \xrightarrow{\rho[\tau]} End_k(V) \otimes_k k[[\tau]]$$

for which we have

$$X_i := \rho(\tau)(t_i) = \rho(\exp(\tau\delta)) = \sum_{p \geq 0} \tau^p / p! D_i^p$$

Since $[t_i, t_j] = 0$, we must have $[X_i, X_j] = 0$ and, since the relations in $Ph^{\infty}(A)$ are given by

$$\sum_{p+q=n \geq 0} \frac{1}{p!q!} [d^p t_i, d^q t_j] = 0$$

This is the condition:

$$\sum_{p+q=n \geq 0} \frac{1}{p!q!} [D_i^p, D_j^q] = 0$$

for the family of matrices $\{D_i^p, p \geq 0, i = 1, \dots, d\}$ to define a homomorphism ρ of k -algebras.

Clearly if $\dim V = 1$ there are no conditions, and we may pick arbitrarily $D_i^p \in k$, and obtain formal power series,

$$X_i = \sum_n \frac{\tau^n}{n!} D_i^n$$

which, when convergent, gives the dynamics of the point.

Example 4.2. Assume $\dim_k V = 2$ and put

$$\rho(t_i) = D_i^0 = \begin{pmatrix} x_i(1) & 0 \\ 0 & x_i(2) \end{pmatrix} =: \begin{pmatrix} \alpha_i^0(1) & 0 \\ 0 & \alpha_i^0(2) \end{pmatrix}$$

and $\alpha_i^0(r, s) := x_i(r) - x_i(s)$, $r, s = 1, 2$. Let

$$D_i^q = \begin{pmatrix} \alpha_i^q(1) & r_i^q(1, 2) \\ r_i^q(2, 1) & \alpha_i^q(2) \end{pmatrix}, \quad q \geq 0$$

Put

$$\alpha_i^l(r, s) := \alpha_i^l(r) - \alpha_i^l(s), \quad r_i^k(r, s) = \sum_{l=0}^k \binom{k}{l} \sigma_{k-l}(r, s) \alpha_i^l(r, s), \quad r, s = 1, 2$$

where the sequence $\{\sigma_l(r, s)\}$, $l = 0, 1, \dots$ is a sequence of arbitrary *coupling constants* with $\sigma_0(r, s) = 0$. Then $\rho(d^n t_i) := D_i^n$ defines a representation $\rho : Ph^\infty(A) \rightarrow End_k(V)$, if and only if

$$\sum_{p+q=n \geq 0} \frac{1}{p!q!} [D_i^p, D_j^q] = 0$$

which is exactly when

$$\sum_{p+q=n \geq 0} \frac{1}{p!q!} \left(\alpha_i^p(r, s) r_j^q(s, r) - r_i^p(r, s) \alpha_j^q(s, r) \right) = 0$$

Computing, we find the condition

$$\sum_{p+q=n \geq 0} \frac{1}{p!q!} \sigma_l(r, s) \left(\alpha_i^p(r, s) \alpha_j^{q-l}(s, r) - \alpha_i^{q-l}(r, s) \alpha_j^p(s, r) \right) = 0$$

for $r, s = 1, 2$ and $l \geq 1$. The situation above arises when we consider two (different) points

$$P_1 = (\alpha_1^0(1), \dots, \alpha_d^0(1)), \quad P_2 = (\alpha_1^0(2), \dots, \alpha_d^0(2))$$

in space, with pre-described tangents, $\xi_1 = (\alpha_1^1(1), \dots, \alpha_d^1(1))$ and $\xi_2 = (\alpha_1^1(2), \dots, \alpha_d^1(2))$. For these two points, considered as dimension 1 representations of $Ph(A)$, we saw that there is a

1-dimensional space of *tangents* between the points, i.e the $Ext_{Ph(A)}^1(k(P_1), k(P_2)) = k$. This leads to possibly non-zero elements $r_i^1(1, 2)$, $r_i^1(2, 1)$ in the matrix representation of the non-commutative deformation of the family $\{k(P_1), k(P_2)\}$ of $Ph(A)$ -modules.

We now have a much more complete picture of the situation. The dynamics of the pair of points is described by the Dirac derivation. Assuming that for the time-moment $\tau = 0$ we know the position $\alpha^0(1), \alpha^0(2)$ and the momenta $\alpha^1(1), \alpha^1(2)$ of the two points, then the dynamics is described, in terms of the time τ by the matrices $X_i = \rho(\exp(\tau\delta))(t_i)$. Putting

$$\alpha_i(r, s) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \alpha_i^n(r, s), \quad \sigma(r, s) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sigma_n(r, s)$$

we find the explicit formulas

$$X_i = \begin{pmatrix} \alpha_i(1) & \sigma(1, 2)\alpha_i(1, 2) \\ \sigma(2, 1)\alpha_i(2, 1) & \alpha_i(2) \end{pmatrix}, \quad i = 1, \dots, d$$

The trace, and determinant are

$$\text{tr}(X_i) = (\alpha_i(1) + \alpha_i(2)), \quad \det(X_i) = (\alpha_i(1)\alpha_i(2)) - \sigma(1, 2)\alpha_i(1, 2)\sigma(2, 1)\alpha_i(2, 1)$$

The *spectrum* of X_i , or the eigenvalues, are given as

$$X_i(r) = \frac{1}{2} \left(\text{tr}(X_i) \pm \sqrt{\text{tr}(X_i)^2 - 4\det(X_i)} \right), \quad r = 1, 2$$

From this we see that if all coupling constants vanish, i.e if $\sigma(r, s) = 0$ ($r, s = 1, 2$) then we have undisturbed independent motions of the two points, $X_i(r) = \alpha_i(r)$, $r = 1, 2$. If the coupling constants are nonzero the representation becomes simple, with trivial automorphism group, and so, according to the definition, of maximal entropy.

Moreover, assume $\sigma_l(1, 2) = \sigma_l(2, 1) = 0$ for $l = 0$ and $l \geq 2$, then the conditions above becomes for $r, s = 1, 2$ and for all $n \geq 0$,

$$\sigma_l(r, s) \cdot \sum_{p+q=n} \frac{1}{p!q!} \left(\alpha_i^p(r, s) \alpha_j^{q-1}(s, r) - \alpha_i^{q-1}(r, s) \alpha_j^p(s, r) \right) = 0$$

If $d = 3$ this simplifies to,

$$\sigma_1(1, 2) \cdot \sum_{p+q=n} \frac{1}{p!q!} (\alpha^p(1, 2) \times \alpha^{q-1}(2, 1)) = 0$$

where, of course

$$\alpha^m(r, s) = -\alpha^m(s, r) \quad r, s = 1, 2, \quad m \geq 0$$

In general, considering an *Object*, $\mathcal{P}$ in space, consisting of r points $\{P_p\}_{p=1, \dots, r}$, we find that the dynamics is closely related to the interaction process, involving non-commutative deformation of families of representations, described in [5]. In fact, we easily obtain the following.

Theorem 4.3. *Given an object, $\mathcal{P}$ consisting of r points $\{P_p\}_{p=1, \dots, r}$ in d -space, $\text{Simp}(A)$, where $A = k[t_1, \dots, t_d]$. With the notations above, in particular, $\alpha_i^n(p, q) = \alpha_i^n(p) - \alpha_i^n(q)$, $i = 1, \dots, d$, consider the matrix*

$$D_i^n := \begin{pmatrix} \alpha_i^n(1) & r_i^n(1, 2) & \dots & r_i^n(1, r) \\ r_i^n(2, 1) & \alpha_i^n(2) & \dots & r_i^n(2, r) \\ \vdots & \vdots & \ddots & \vdots \\ r_i^n(r, 1) & r_i^n(r, 2) & \dots & \alpha_i^n(r) \end{pmatrix}$$

with

$$r_i^0(p, q) = 0, \quad r_i^n(p, q) = \sum_{l=0}^n \binom{n}{l} \alpha_i^l(p, q) \sigma_{n-l}(p, q)$$

where $\sigma_m(p, q) \in k$ are arbitrary coupling constants, with $\sigma_0 = 0$. Then these operators define a representation $\rho : Ph^\infty(A) \rightarrow M_r(k)$ with $\rho(d^n t_i) = D_i^n$ if and only if, for all $n \geq 1$,

$$\begin{aligned} \sum_h \binom{n}{h} \sigma_{n-h}(p, q) \left(\alpha_i^h(p, q) \alpha_j^0(p, q) - \alpha_i^0(p, q) \alpha_j^h(p, q) \right) \\ = \sum_{k, l, m, s} \frac{n! \sigma_{n-k-m}(p, s) \sigma_{k-l}(s, q)}{l! m! (k-l)! (n-k-m)!} \left(\alpha_j^m(p, s) \alpha_i^l(s, q) - \alpha_i^l(p, s) \alpha_j^m(s, q) \right) \end{aligned}$$

Proof. As above, consider the matrix

$$X_i = \rho(\exp(\tau \delta))(t_i) = \sum_{n \geq 0} \frac{\tau^n}{n!} D_i^n$$

and put

$$\alpha_i(r) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \alpha_i^n(r), \quad \alpha_i(r, s) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \alpha_i^n(r, s), \quad \sigma(r, s) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sigma_n(r, s)$$

we find the explicit formulas

$$X_i = \begin{pmatrix} \alpha_i(1) & \sigma(1, 2) \alpha_i(1, 2) & \dots & \sigma(1, r) \alpha_i(1, r) \\ \sigma(2, 1) \alpha_i(2, 1) & \alpha_i(2) & \dots & \sigma(2, r) \alpha_i(2, r) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(r, 1) \alpha_i(r, 1) & \sigma(r, 2) \alpha_i(r, 2) & \dots & \alpha_i(r) \end{pmatrix}, \quad i = 1, \dots, d$$

Now compute $[X_i, X_j] = 0$ and see that the condition of the theorem emerges. $\square$

Remark 4.4. We may consider the *space*

$$\mathbf{A}(r) = k[\alpha_i^n(p), \sigma_n(p, q)]/\mathfrak{a}$$

with coordinates

$$\{\alpha_i^n(p), \sigma_n(p, q), i = 1, \dots, d, p, q = 1, \dots, r, n \geq 0, \sigma_0(p, q) = 0\}$$

and where the ideal $\mathfrak{a}$ is generated by the equations above, as the versal base space for the versal family of the non-commutative deformation theory applied to the family of $Ph^\infty(A)$ modules defined by the object $\mathcal{P}$, and with Dirac derivation, δ acting as, $[\delta](\alpha_i^n(p)) = \alpha_i^{n+1}(p)$.

Since $Ph^\infty(A)$ is infinitely generated, there is, strictly speaking, no such thing, but we shall see that in special cases, we can overcome this difficulty by finding clever dynamical structures.

In (4.2) we saw that to overcome another difficulty related to non-existing moduli spaces, we opted for approximating a state $\xi \in \Theta_C$ by a finite dimensional representation, defined for every finite point set object $\mathcal{P} = \{P_p\}$ of $Simp_1(C)$ by $\xi_{\mathcal{P}} : Ph(C) \rightarrow End_k(\mathcal{P})$, where

$$\xi_{\mathcal{P}}(t_i) = \alpha_i^0(p), \quad \xi_{\mathcal{P}}(dt_i) = \alpha_i^1(p)$$

We wanted to look at the object $\mathcal{P}$ as the set of molecules in our observatory (or in the Universe), and at $\xi_{\mathcal{P}}$ as the combined state of these.

Extending the representation $\xi_{\mathcal{P}}$ to a representation $\xi_{\mathcal{P}} : Ph^{\infty}(C) \rightarrow End_k(\mathcal{P})$ or cutting it down to a representation of some dynamical system, $A(\sigma) := Ph^{\infty}(C)/(\sigma)$ $\xi_{\mathcal{P},\sigma} : A(\sigma) \rightarrow End_k(\mathcal{P})$ for some reasonable dynamical structure (σ) we might be able to compute the versal family $\mathbf{M}$ of the representation $\xi_{\mathcal{P},\sigma}$ for some fundamental *state* ξ , of the *Furniture* of the Universe, and also the *Moduli Suite*, $\{\mathbf{M}_{\alpha}\}$.

For any deformation η of $\xi_{\mathcal{P},\sigma}$ represented by an equivalence class $\tilde{\eta}$ in the versal base space $\mathbf{M}$ there will be one α such that $\tilde{\eta} \subset \mathbf{M}_{\alpha}$ and the entropy of η would be $S(\eta) = \dim(\mathbf{M}_{\alpha})$.

From the classification above, it follows that the minimal entropy would correspond to all $\sigma_l(p, q) = 0$, i.e to some *dust-like* furniture of the space. And the maximal entropy would correspond to all $\sigma_l(p, q) \neq 0$, i.e to some *black hole-like* object $\mathcal{P}$ containing a huge number of gravitational and/or other parameters.

The goal is still to show that, for reasonable classical cases, this should come out close to the Boltzmann's definition

$$S(\eta) := \log Vol(M_{\alpha}), \quad \eta \in M_{\alpha}$$

where M_{α} is the substratum of the corresponding *Coarse Graining* of the classical phase space of the situation, containing the *distribution* η , see again [9], chapter 2. In particular, consider Penrose's need (for) some clear-cut way of saying that "the gravitational degrees of freedom were not activated", and his need to identify the mathematical quantity that actually measures "gravitational degrees of freedom". I suggest that the above, coupled with my "Toy Model", see [5], i.e working on $Hilb^{(2)}(\mathbf{A}^3)$ instead of the trivial affine space $\mathbf{A}^d = Spec(k[t_1, \dots, t_d])$, may be of interest for this quest.

We have already looked at the case $r = 2$ and seen that the result makes physical sense. For $r = 3$ and $n = 2$ we find

$$\sigma_1(p, q)(\alpha_i^1(p, q)\alpha_j^0(p, q) - \alpha_i^0(p, q)\alpha_j^1(p, q)) = \sigma_1(p, s)\sigma_1(s, q)(\alpha_j^0(p, s)\alpha_i^0(s, q) - \alpha_i^0(p, s)\alpha_j^0(s, q))$$

In dimension $d = 3$ this has a particularly nice interpretation. Let $\alpha^0(i, j)$ be the vector starting at P_i and ending at P_j , and let ξ_i be a tangent vector at P_i for $i = 1, 2, 3$. Put $\alpha^1(i, j) = \xi_i - \xi_j$, then the condition above reads

$$\sigma_1(p, q)(\alpha^1(p, q) \times \alpha^0(p, q)) = -\sigma_1(p, s)\sigma_1(s, q)(\alpha^0(p, s) \times \alpha^0(s, q)), \quad \forall p, s, q = 1, 2, 3$$

This says that for any two of the three points in space, the relative momentum must sit in the plane defined by the three points, the length being determined by the 3 coupling constants. Moreover, the sum of all three relative momenta must be 0. In fact, there are coefficients, $u, v \in k$ such that

$$\alpha^1(p, q) = u\alpha^0(p, s) + v\alpha^0(s, q)$$

Put this into the left side of the formula above, and find

$$\sigma_1(p, q)(v - u) = \sigma_1(p, s)\sigma_1(s, q)$$

assuming that the three points are not co-linear.

Assume now that the coupling constants are given as

$$\sigma_1(p, q) := m(p, q)|\alpha^0(p, q)|^{-2}$$

Then after some computation we find the following differential equations for the dynamics of the 3 points

$$\begin{aligned}\alpha^1(p, q) &= -m(q, p)|\alpha^0(p, q)|^{-1}\epsilon(p, q) - m(p, s)|\alpha^0(p, s)|^{-1}\epsilon(p, s) + \rho\alpha^0(p, q) \\ \alpha^1(q, s) &= -m(s, q)|\alpha^0(s, q)|^{-1}\epsilon(q, s) - m(q, p)|\alpha^0(q, p)|^{-1}\epsilon(q, p) + \rho\alpha^0(q, s) \\ \alpha^1(s, p) &= -m(p, s)|\alpha^0(p, s)|^{-1}\epsilon(s, p) - m(s, q)|\alpha^0(s, q)|^{-1}\epsilon(s, q) + \rho\alpha^0(s, p)\end{aligned}$$

Here $\epsilon(i, j)$ is the unit vector from the point P_i to the point P_j .

Notice that there is a different set-up, related to Grothendieck's generalized differential algebra. For A commutative, consider an A -module E , and an extension of this representation to $\rho: Ph^\infty(A) \rightarrow \text{End}_k(E)$. We have seen that ρ must be given in terms of operators,

$$D_i^p := \rho(d^p t_i) \in \text{End}_k(E)$$

satisfying the conditions

$$\sum_{p+q=n} \frac{1}{p!q!} [D_i^p D_j^q] = 0, \quad \forall n \geq 0$$

There is an obvious family of solutions of these equations, given by any differential operator $Q \in \text{Diff}(E)$, with

$$D_i^0 = \rho(t_i) = t_i, \quad D_i^p := \text{ad}(Q)^p(t_i) \in \text{Diff}(E), \quad p \geq 1$$

Looking at the case of finite dimensional representations, treated above, one sees the difference between the two set-ups, and the much greater generality obtained by considering the representations of Ph^∞ , the way we do.

References

- [1] Laudal O A 2002 *Homology, Homotopy, Appl.* **4** 357
- [2] Laudal O A 2003 *Revista Matematica Iberoamericana* **9** 1
- [3] Laudal O A 2005 Time-space and space-times *Noncommutative Geometry and Representation Theory in Mathematical Physics* eds J Fuchs et al *Contemp. Math.* **391** p 249
- [4] Laudal O A 2008 *Acta Appl. Math.* **101** 191
- [5] Laudal O A 2011 *Geometry of Time-Spaces* (World Sci.)
- [6] Laudal O A and Pfister G 1988 *Local Moduli and Singularities* Lecture Notes in Math. **1310** (Springer-Verlag)
- [7] Procesi C 1967 *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **8** 239
- [8] Procesi C 1973 *Rings with Polynomial Identities* (Marcel Dekker)
- [9] Penrose R 2011 *What Came Before the Big Bang. Cycles of Time* (London: Vintage Books)